

L^p -independence of spectral bounds of generalized non-local Feynman-Kac semigroups

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Abstract

Let X be a symmetric strong Markov process on a Luzin space. In this paper, we present criteria of the L^p -independence of spectral bounds for generalized non-local Feynman-Kac semigroups of X that involve continuous additive functionals of X having zero quadratic variations and discontinuous additive functionals of X .

Keywords: Feynman-Kac transform; Girsanov transform; quadratic form; spectral bound

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1 Introduction

Transformation by multiplicative functionals is one of the most important transforms for Markov processes. Feynman-Kac transforms and Girsanov transforms are particular cases. They play an important role in the probabilistic as well as analytic aspect of potential theory. See [6, 7, 12] and the references therein for some of the recent results in the context of symmetric Markov processes.

Suppose that E is a Luzin space (i.e., a space that is homeomorphic to a Borel subset of a compact metric space) and $\mathcal{B}(E)$ denotes the Borel σ -algebra on E . Let m be a Borel σ -finite measure on E with $\text{supp}[m] = E$ and $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}_x, x \in E)$ be an m -symmetric irreducible Borel standard process on E with lifetime ζ (cf. Sharpe [16] for the terminology). For a continuous additive functional A of X having finite variation, one can do Feynman-Kac transform:

$$T_t f(x) = \mathbb{E}_x [e^{A_t} f(X_t)], \quad t \geq 0.$$

It is easy to check (see [1]) that, under suitable Kato class condition on A , $\{T_t; t \geq 0\}$ forms a strongly continuous symmetric semigroup on $L^p(E; m)$ for every $1 \leq p \leq \infty$ and that its L^2 -infinitesimal generator is $\mathcal{L}^\mu := \mathcal{L} + \mu$, where \mathcal{L} is the L^2 -infinitesimal generator of the process X and μ is the (signed) Revuz measure for the continuous additive functional A . To emphasize the correspondence between continuous additive functionals and Revuz measures, let's denote A by A^μ . In fact, the process X has many continuous additive functionals that do not have finite variations. For example, for u in the extended Dirichlet space \mathcal{F}_e of $(\mathcal{E}, \mathcal{F})$, $u(X_t) - u(X_0)$ has the Fukushima's decomposition $M^u + N^u$, where M^u is a square integrable martingale additive functional of X and

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N^u is a continuous additive function that in general is only of zero quadratic variation. It is also natural to consider the following generalized Feynman-Kac transform:

$$T_t f(x) = \mathbb{E}_x [e^{N_t^u} f(X_t)], \quad t \geq 0.$$

Let $\mu_{\langle u \rangle}$ be the Revuz measure for $\langle M^u \rangle$, the quadratic variation process of M^u . We refer the reader to the Introduction of [11] for a brief history of the above transformation by N^u . It is shown in [11] that when $\mu_{\langle u \rangle}$ is in Kato class of X , $\{T_t; t \geq 0\}$ forms a strongly continuous symmetric semigroup on $L^2(E; m)$ and its associate quadratic form is $(Q, \mathcal{D}(Q))$, where $\mathcal{D}(Q)_b \subset \mathcal{F}_b$ and

$$Q(f, g) = \mathcal{E}(f, g) + \mathcal{E}(fg, u) \quad \text{for } f, g \in \mathcal{F}_b.$$

Here for a function space \mathcal{H} , we use \mathcal{H}_b to denote space of bounded functions in \mathcal{H} . When the process X is discontinuous, it has many discontinuous additive functionals. Let F be a bounded symmetric function on $E \times E$ that vanishes along the diagonal d of $E \times E$. We always extend it to be zero off $E \times E$. Then $\sum_{0 < s \leq t} F(X_{s-}, X_s)$, whenever it is summable, is an additive functional of X . Hence one can perform generalized non-local Feynman-Kac transform

$$T_t^{u, \mu, F} f(x) := \mathbb{E}_x \left[\exp \left(N_t^u + A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s) \right) f(X_t) \right], \quad t \geq 0. \quad (1.1)$$

Under some suitable Kato class conditions on the measures $\mu_{\langle u \rangle}$, μ and the function F , it can be shown (see Theorem 3.5 below) that $\{T_t^{u, \mu, F}; t \geq 0\}$ is a strongly continuous symmetric semigroup on $L^p(E; m)$ for every $1 \leq p \leq \infty$. Hence the limit

$$\lambda_p(X; u + \mu + F) := - \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T_t^{u, \mu, F}\|_{p,p}$$

exists, which will be called the L^p -spectral bound of the generalized non-local Feynman-Kac semigroup $\{T_t^{u, \mu, F}; t \geq 0\}$. We will show in this paper that under suitable conditions, $\lambda_p(X; u + \mu + F) = \lambda_2(X; u + \mu + F)$ for all $1 \leq p \leq \infty$ if $\lambda_2(X; u + \mu + F) \leq 0$. If in addition X is conservative, then $\lambda_2(X; u + \mu + F) \leq 0$ becomes a necessary and sufficient condition for the independence of $\lambda_p(X; u + \mu + F)$ in $p \in [1, \infty]$. The L^2 -spectral bound $\lambda_2(X; u + \mu + F)$ has a variational formula in terms of the Dirichlet form of X , μ and F , see (3.10) below.

When $F = 0$ and $u = 0$, the L^p -independence of spectral bounds for continuous Feynman-Kac transforms $\{T_t^{0, \mu, 0}; t \geq 0\}$ was investigated by Takeda in [18, 19] for conservative Feller processes or symmetric Hunt processes satisfying strong Feller property and a tightness assumption, respectively, both using a large deviation approach. The results in [18] were extended to purely discontinuous Feynman-Kac transforms $\{T_t^{0, 0, F}; t \geq 0\}$ (i.e. with $u = 0$ and $\mu = 0$) first in [20] for rotationally symmetric α -stable processes and then in [21] for conservative doubly Feller processes, both papers again using a large deviation approach similar to those in [18, 19]. A stochastic process is said to be doubly Feller if it is a Feller process that has the strong Feller property. The L^p -independence of spectral bounds for continuous generalized Feynman-Kac transforms $\{T_t^{u, \mu, 0}; t \geq 0\}$ (i.e. with $F = 0$) is studied recently in [13] for doubly Feller processes on a locally compact metric space E and for those $u \in \mathcal{F}_e$ that is continuous on E and vanishes at infinity, also using a large deviation approach refined from [18, 19]. In a very recent paper [5] by the author, a completely different

approach is developed to study the L^p -independence of spectral bounds for non-local Feynman-Kac semigroups $\{T_t^{0,\mu,F}, t \geq 0\}$ (i.e. with $u = 0$) for symmetric Markov processes that may not have strong Feller property, using the gaugeability results established in [3] for continuous Feynman-Kac functionals. This new approach yields new criteria for the L^p -independence of spectral bound even for local Feynman-Kac semigroups.

The approach of this paper is different from that of [13]. We do not use large deviation theory. Using the idea from [11], we decompose transformation by multiplicative functional $e^{N_t^u}$ into a combination of a Girsanov transform, a continuous Feynman-Kac transform followed by an h -transform. So essentially, after a Girsanov transform, we can reduce the generalized non-local Feynman-Kac transform into a non-local Feynman-Kac transform for a new process. We can then apply the criteria from [5] to the latter to obtain criteria of the L^p -independence of spectral radius for generalized Feynman-Kac semigroup $\{T_t^{u,\mu,F}, t \geq 0\}$.

To keep the exposition of this paper as transparent as possible, we have not attempted to present the most general conditions on u , μ and F . For example, by applying results from [5] for continuous Feynman-Kac transforms instead of that for non-local Feynman-Kac transforms, conditions on μ can be weakened for $\{T_t^{u,\mu,0}, t \geq 0\}$ in the case of $F = 0$.

The rest of the paper is organized as follows. In Section 2, we give precise setup of this paper, including the definitions of Kato classes and Lévy systems and recalling the main results from [5] that will be used in the sequel. Generalized non-local Feynman-Kac transform and its reduction to non-local Feynman-Kac transform via Girsanov transform are studied in Section 3. The criteria of the L^p -independence of spectral bound for generalized non-local Feynman-Kac semigroups are established in Section 4. Several examples are given in Section 5 to illustrate the main results of this paper.

2 Kato classes and non-local Feynman-Kac transform

Let E be a Lusin space and $\mathcal{B}(E)$ be the Borel σ -algebra on E . Let m be a Borel σ -finite measure on E with $\text{supp}[m] = E$ and $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}_x, x \in E)$ be an m -symmetric irreducible transient Borel standard process on E with lifetime ζ . We like to point here that, since we are only concerned with the Schrödinger semigroups of X , the transience assumption on X is just a matter of convenience and is unimportant—we can always consider the 1-subprocess $X^{(1)}$ of X instead of X if necessary. Let $(\mathcal{E}, \mathcal{F})$ denote the Dirichlet form of X ; that is, if we use \mathcal{L} to denote the infinitesimal generator of X , then \mathcal{F} is the domain of the operator $\sqrt{-\mathcal{L}}$ and for $u, v \in \mathcal{F}$,

$$\mathcal{E}(u, v) = (\sqrt{-\mathcal{L}}u, \sqrt{-\mathcal{L}}v)_{L^2(E; m)}.$$

We refer the reader to [8] or [14] for terminology and various properties of Dirichlet forms such as continuous additive functional, martingale additive functional, extended Dirichlet space.

The transition operators P_t , $t \geq 0$, are defined by

$$P_t f(x) := \mathbb{E}_x[f(X_t)] = \mathbb{E}_x[f(X_t); t < \zeta].$$

(Here and in the sequel, unless mentioned otherwise, we use the convention that a function defined on E takes the value 0 at the cemetery point ∂ .) We assume that there is a Borel symmetric

function $G(x, y)$ on $E \times E$ such that

$$\mathbb{E}_x \left[\int_0^\infty f(X_s) ds \right] = \int_E G(x, y) f(y) m(dy)$$

for all measurable $f \geq 0$. $G(x, y)$ is called the Green function of X . The Green function G will always be chosen so that for each fixed $y \in E$, $x \mapsto G(x, y)$ is an excessive function of X . This choice of the Green function is always possible; see [16].

For every $\alpha > 0$, one deduces from the existence of the Green function $G(x, y)$ that there exists a kernel $G_\alpha(x, y)$ so that

$$\mathbb{E}_x \left[\int_0^\infty e^{-\alpha s} f(X_s) ds \right] = \int_E G_\alpha(x, y) f(y) m(dy), \quad x \in E,$$

for all measurable $f \geq 0$. Clearly, $G_\alpha(x, y) \leq G(x, y)$. Note that by [14, Theorem 4.2.4], for every $x \in E$ and $t > 0$, X_t under \mathbb{P}_x has a density function $p(t, x, y)$ with respect to the measure m .

A set B is said to be m -polar if $\mathbb{P}_m(\sigma_B < \infty) = 0$, where $\sigma_B := \inf\{t > 0 : X_t \in B\}$. We call a positive measure μ on E a smooth measure of X if there is a positive continuous additive functional (PCAF in abbreviation) A of X such that

$$\int_E f(x) \mu(dx) = \uparrow \lim_{t \downarrow 0} \mathbb{E}_m \left[\frac{1}{t} \int_0^t f(X_s) dA_s \right]. \quad (2.1)$$

for any Borel $f \geq 0$. Here $\uparrow \lim_{t \downarrow 0}$ means the quantity is increasing as $t \downarrow 0$. The measure μ is called the Revuz measure of A . We refer to [8, 14] for the characterization of smooth measures in terms of nests and capacity.

For any given positive smooth measure μ , define $G\mu(x) = \int_E G(x, y) \mu(dy)$. It is known (see Stollmann and Voigt [17]) that for any positive smooth measure μ of X ,

$$\int_E u(x)^2 \mu(dx) \leq \|G\mu\|_\infty \mathcal{E}(u, u) \quad \text{for } u \in \mathcal{F}. \quad (2.2)$$

Recall that as X is assumed to have a Green function, any m -polar set is polar. Hence by (2.2) a PCAF A in the sense of [14] with an exceptional set that has a bounded potential (that is, $x \mapsto \mathbb{E}_x[A_\zeta] = G\mu$ is bounded almost everywhere on E , where μ is the Revuz measure of A) can be uniquely refined into a PCAF in the strict sense (as defined on p.195 of [14]). This can be proved by using the same argument as that in the proof of Theorem 5.1.6 of [14].

For a signed measure μ , we use μ^+ and μ^- to denote the positive part and negative part of μ appearing in the Hahn-Jordan decomposition of μ . The following definitions are taken from Chen [3].

Definition 2.1 *Suppose that μ is a signed smooth measure. Let A^μ and $A^{|\mu|}$ be the continuous additive functional and positive continuous additive functional of X with Revuz measures μ and $|\mu|$, respectively.*

(i) *We say μ is in the Kato class of X , $\mathbf{K}(X)$ in abbreviation, if*

$$\limsup_{t \rightarrow 0} \sup_{x \in E} \mathbb{E}_x \left[A_t^{|\mu|} \right] = 0.$$

- (ii) μ is said to be in the class $\mathbf{K}_\infty(X)$ if for any $\varepsilon > 0$, there is a Borel set $K = K(\varepsilon)$ of finite $|\mu|$ -measure and a constant $\delta = \delta(\varepsilon) > 0$ such that for all measurable set $B \subset K$ with $|\mu|(B) < \delta$,

$$\|G(\mathbf{1}_{K^c \cup B}|\mu|)\|_\infty < \varepsilon. \quad (2.3)$$

- (iii) μ is said to be in the class $\mathbf{K}_1(X)$ if there is a Borel set K of finite $|\mu|$ -measure and a constant $\delta > 0$ such that

$$\beta_1(\mu) := \sup_{B \subset K: |\mu|(B) < \delta} \|G(\mathbf{1}_{K^c \cup B}|\mu|)\|_\infty < 1. \quad (2.4)$$

- (iv) A function q is said to be in class $\mathbf{K}(X)$, $\mathbf{K}_\infty(X)$ or $\mathbf{K}_1(X)$ if $\mu(dx) := q(x)m(dx)$ is in the corresponding spaces.

According to [3, Proposition 2.3(i)], $\mathbf{K}_\infty(X) \subset \mathbf{K}(X) \cap \mathbf{K}_1(X)$. Suppose that μ is a positive measure in $\mathbf{K}_1(X)$. By Propositions 2.2 in [3], $G\mu(x) = \mathbb{E}_x[A_\infty^\mu]$ is bounded and so (2.2) is satisfied. Therefore the PCAF corresponding to μ can and is always taken to be in the strict sense.

Let (N, H) be a Lévy system for X (cf. Benveniste and Jacod [2] and Theorem 47.10 of Sharpe [16]); that is, $N(x, dy)$ is a kernel from $(E, \mathcal{B}(E))$ to $(E, \mathcal{B}(E))$ satisfying $N(x, \{x\}) = 0$, and H_t is a PCAF of X with bounded 1-potential such that for any nonnegative Borel function f on $E \times E$ vanishing on the diagonal and any $x \in E$,

$$\mathbb{E}_x \left[\sum_{s \leq t} f(X_{s-}, X_s) \mathbf{1}_{\{s < \zeta\}} \right] = \mathbb{E}_x \left[\int_0^t \int_E f(X_s, y) N(X_s, dy) dH_s \right]. \quad (2.5)$$

The Revuz measure for H will be denoted as μ_H .

Definition 2.2 Suppose F is a bounded function on $E \times E$ vanishing on the diagonal d . It is always extended to be zero off $E \times E$. Define $\mu_F(dx) := \left(\int_E F(x, y) N(x, dy) \right) \mu_H(dx)$. We say F belongs to the class $\mathbf{J}(X)$ (respectively, $\mathbf{J}_\infty(X)$) if the measure

$$\mu_{|F|}(dx) := \left(\int_E |F(x, y)| N(x, dy) \right) \mu_H(dx)$$

belongs to $\mathbf{K}(X)$ (respectively, $\mathbf{K}_\infty(X)$).

See [4, Section 2] for concrete examples of $\mu \in \mathbf{K}_\infty(X)$ and $F \in \mathbf{J}_\infty(X)$.

For $\alpha > 0$, let $X^{(\alpha)}$ denote the α -subprocess of X ; that is, $X^{(\alpha)}$ is the subprocess of X killed at exponential rate α . Let $G^{(\alpha)}$ be the 0-resolvent (or Green operator) of $X^{(\alpha)}$. Then $G^{(\alpha)} = G_\alpha$, the α -resolvent of X . Thus for $\beta > \alpha > 0$, $\mathbf{K}_1(X) \subset \mathbf{K}_1(X^{(\alpha)}) \subset \mathbf{K}_1(X^{(\beta)})$ and $\mathbf{K}_\infty(X) \subset \mathbf{K}_\infty(X^{(\alpha)}) \subset \mathbf{K}_\infty(X^{(\beta)})$. In fact, it follows from the resolvent equation $G_\alpha = G_\beta + (\beta - \alpha)G_\alpha G_\beta$ that $\mathbf{K}_\infty(X^{(\alpha)}) = \mathbf{K}_\infty(X^{(\beta)})$ for every $\beta > \alpha > 0$. Consequently, $\mathbf{J}_\infty(X^{(\alpha)}) = \mathbf{J}_\infty(X^{(\beta)})$ for every $\beta > \alpha > 0$. Clearly, $\mathbf{K}(X^{(\alpha)}) = \mathbf{K}(X)$ for every $\alpha > 0$.

Assume that μ is a signed smooth measure with $\mu^+ \in \mathbf{K}(X)$ and $G\mu^-$ bounded, and $F \in \mathbf{J}(X)$ symmetric. Define the non-local Feynman-Kac semigroup

$$P_t^{\mu, F} f(x) := \mathbb{E}_x \left[\exp \left(A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s) \right) f(X_t) \right], \quad t \geq 0.$$

It follows from the proof of [10, Proposition 2.3] and Hölder inequality that $\{P_t^{\mu, F}; t \geq 0\}$ is a strongly continuous semigroup on $L^p(E; m)$ for every $1 \leq p \leq \infty$. Moreover, it is easy to verify that $P_t^{\mu, F}$ is a symmetric operator in $L^2(E; m)$ for every $t \geq 0$. The L^p -spectral bound of $\{P_t^{\mu, F}; t \geq 0\}$ is defined to be

$$\lambda_p(X, \mu + F) := - \lim_{t \rightarrow \infty} \frac{1}{t} \log \|P_t^{\mu, F}\|_{p,p}.$$

Necessary and sufficient conditions for $\lambda_p(X, \mu + F)$ to be independent of $1 \leq p \leq \infty$ have been investigated in [5] by using gaugeability results for Schrödinger semigroups obtained in [3]. The following three results are established in [5].

Theorem 2.3 (See [5, Theorem 5.3]) *Assume that $m(E) < \infty$ and that the following condition holds*

$$\text{there is some } t_0 > 0 \text{ so that } P_{t_0} \text{ is a bounded operator from } L^2(E; m) \text{ into } L^\infty(E; m). \quad (2.6)$$

Let μ be a signed smooth measure with $\mu^+ \in \mathbf{K}_\infty(X^{(\alpha)})$ and $G_\alpha \mu^-$ bounded for some $\alpha \geq 0$, and $F \in \mathbf{J}_\infty(X^{(\alpha)})$ symmetric. Then $\lambda_p(X, \mu + F)$ is independent of $p \in [1, \infty]$.

Theorem 2.4 (See [5, Theorem 5.4]) *Suppose that μ is a signed smooth measure with $\mu^+ \in \mathbf{K}_\infty(X^{(1)})$ and $G_1 \mu^-$ bounded, and $F \in \mathbf{J}_\infty(X^{(1)})$ symmetric.*

- (i) $\lambda_\infty(X, \mu + F) \geq \min\{\lambda_2(X, \mu + F), 0\}$. *Consequently, $\lambda_p(X, \mu + F)$ is independent of $p \in [1, \infty]$ if $\lambda_2(X, \mu + F) \leq 0$.*
- (ii) *Assume in addition that X is conservative and that $\mu \in \mathbf{K}_\infty(X^{(1)})$. Then $\lambda_\infty(X, \mu + F) = 0$ if $\lambda_2(X, \mu + F) > 0$. Hence $\lambda_p(X, \mu + F)$ is independent of $p \in [1, \infty]$ if and only if $\lambda_2(X, \mu + F) \leq 0$.*

Theorem 2.5 ([5, Theorem 5.5]) *Suppose that $1 \in \mathbf{K}_\infty(X^{(1)})$, $\mu \in \mathbf{K}_\infty(X^{(1)})$ and $F \in \mathbf{J}_\infty(X^{(1)})$ symmetric. Then $\lambda_p(X, \mu + F)$ is independent of $p \in [1, \infty]$.*

3 Generalized Feynman-Kac semigroup

Denote by \mathcal{F}_e the extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$. Every $u \in \mathcal{F}_e$ admits a quasi-continuous version, which we still denote as u . In this paper, every $u \in \mathcal{F}_e$ is always represented by its quasi-continuous version. For such u , the following Fukushima's decomposition holds (cf. [8, 14]):

$$u(X_t) = u(X_0) + M_t^u + N_t^u, \quad t \geq 0,$$

where M^u is a martingale additive functional of X having finite energy and N^u is a continuous additive functional of X having zero energy. The continuous martingale part of M^u will be denoted as $M^{u,c}$. Let $\langle M^u \rangle$ and $\langle M^{u,c} \rangle$ be the predictable quadratic variation processes of M^u and $M^{u,c}$, respectively. Both of them are positive continuous additive functionals of X , whose Revuz measures will be denoted as $\mu_{\langle u \rangle}$ and $\mu_{\langle u \rangle}^c$, respectively. Note that by [8, Theorem 4.3.11] or [14, Theorem 5.2.3], for bounded u in \mathcal{F}_e , $\mu_{\langle u \rangle}$ can be computed from

$$\int_E f(x) \mu_{\langle u \rangle}(dx) = 2\mathcal{E}(uf, u) - \mathcal{E}(u^2, f) \quad \text{for bounded } f \in \mathcal{F}_e. \quad (3.1)$$

A similar formula holds for $\mu_{\langle u \rangle}^c$ as well; see [8, Exercise 4.3.12].

Let u be a bounded function in \mathcal{F}_e with $\mu_{\langle u \rangle} \in \mathbf{K}_\infty(X^{(1)})$, $\mu \in \mathbf{K}_\infty(X^{(1)})$ and F be a bounded symmetric function in $\mathbf{J}_\infty(X^{(1)})$. Define the Feynman-Kac semigroup $\{T_t^{u,\mu,F}, t \geq 0\}$ by

$$T_t^{u,\mu,F} f(x) = \mathbb{E}_x \left[\exp \left(N^u + A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s) \right) f(X_t) \right].$$

We will show that for every $p \in [1, \infty]$, $\{T_t^{u,\mu,F}, t \geq 0\}$ is a strongly continuous symmetric semigroup on $L^2(E; m)$. This will be achieved by reducing the generalized non-local Feynman-Kac semigroup $\{T_t^{u,\mu,F}, t \geq 0\}$ via a suitable Girsanov transform to a non-local Feynman-Kac semigroup of the new process.

Note that since u is bounded, $v(x) := e^{-u} - 1$ is a bounded function in \mathcal{F}_e . Clearly for every $x, y \in E$,

$$|v(x)| \leq e^{\|u\|_\infty} |u(x)| \quad \text{and so} \quad |v(x) - v(y)| \leq e^{\|u\|_\infty} |u(x) - u(y)|.$$

We thus deduce from [8, (4.3.12) and Theorem 4.3.7] that

$$\mu_{\langle v \rangle}(dx) \leq e^{2\|u\|_\infty} \mu_{\langle u \rangle}(dx). \quad (3.2)$$

Let $Z = \text{Exp}(M)$ be the Doléans-Dade exponential martingale of $M_t := \int_0^t e^{u(X_{s-})} dM_s^v$; that is, Z is the unique solution of

$$Z_t = 1 + \int_0^t Z_{s-} dM_s, \quad t \geq 0.$$

It follows from Doléans-Dade formula (cf. [15, Theorem 9.39]) that

$$\begin{aligned} Z_t &= \exp \left(M_t - \frac{1}{2} \langle M^c \rangle_t \right) \prod_{0 < s \leq t} (1 + M_s - M_{s-}) e^{-(M_s - M_{s-})} \\ &= \exp \left(M_t - \frac{1}{2} \langle M^{u,c} \rangle_t \right) \prod_{0 < s \leq t} \exp \left(u(X_{s-}) - u(X_s) + 1 - e^{u(X_{s-}) - u(X_s)} \right). \end{aligned} \quad (3.3)$$

Note that

$$M_t - M_{t-} = e^{u(X_{t-}) - u(X_t)} - 1 \geq e^{-2\|u\|_\infty} - 1$$

and that by (3.2),

$$\sup_{x \in E} \mathbb{E}_x [M]_\infty = \sup_{x \in E} \mathbb{E}_x \left[\int_0^\infty e^{2u(X_{s-})} d[M^v]_s \right] \leq e^{2\|u\|_\infty} \sup_{x \in E} \mathbb{E}_x [\langle M^v \rangle_\infty] \leq e^{4\|u\|_\infty} \|G\mu_{\langle u \rangle}\|_\infty < \infty,$$

where $[M]$ is the quadratic variation process of the martingale M . Therefore we conclude by the uniform integrability criteria for exponential martingales established in [5, Theorem 3.2] that $Z = \text{Exp}(M)$ is a uniformly integrable martingale under \mathbb{P}_x for every $x \in E$.

Let $\{\tilde{\mathbb{P}}_x, x \in E\}$ be the family of probability measures defined by

$$\frac{d\tilde{\mathbb{P}}_x}{d\mathbb{P}_x} = Z_\infty \quad \text{on } \mathcal{F}_\infty,$$

and for emphasis, let $\tilde{X} = (\tilde{X}_t, \tilde{\mathbb{P}}_x)$ denote the Girsanov transformed process $(X_t, \tilde{\mathbb{P}}_x)$. The following result is proved in [11, Theorem 3.4].

Theorem 3.1 *The process \tilde{X} is a symmetric strong Markov process with symmetrizing measure $e^{-2u(x)}m(dx)$, whose associated Dirichlet form on $L^2(E; e^{-2u(x)}m(dx))$ is $(\tilde{\mathcal{E}}, \mathcal{F})$, where for $f \in \mathcal{F}$,*

$$\begin{aligned}\tilde{\mathcal{E}}(f, f) &= \frac{1}{2} \int_E e^{-2u(x)} \mu_{\langle f \rangle}^c(dx) + \frac{1}{2} \int_{E \times E \setminus d} (f(x) - f(y))^2 e^{-u(x)-u(y)} N(x, dy) \mu_H(dx) \\ &\quad + \int_E f(x)^2 e^{-u(x)} \kappa(dx).\end{aligned}$$

It follows that \tilde{X} has a Lévy system $(\tilde{N}(x, dy), \tilde{H})$ with

$$\tilde{N}(x, dy) = e^{-u(y)} N(x, dy) \quad \text{and} \quad \mu_{\tilde{H}}(dx) = e^{-u(x)} \mu_H(dx).$$

In view of Lemma 3.2 below, the latter is equivalent to $\tilde{H}_t = \int_0^t e^{u(X_s)} dH_s$. The next lemma is established in [11, Theorem 3.3 and Lemma 4.2].

Lemma 3.2 *If A is a positive continuous additive functional of X with Revuz measure μ , then A is a positive continuous additive functional of \tilde{X} with Revuz measure $e^{-2u(x)}\mu(dx)$. Moreover, if $\mu \in \mathbf{K}(X)$, then $e^{-2u(x)}\mu(dx) \in \mathbf{K}(\tilde{X})$.*

Since u is bounded, the second half of the above lemma says that $\mathbf{K}(X) \subset \mathbf{K}(\tilde{X})$.

Lemma 3.3

$$\mathbf{K}_\infty(X^{(1)}) \subset \mathbf{K}_\infty(\tilde{X}^{(1)}) \quad \text{and so} \quad \mathbf{J}_\infty(X^{(1)}) \subset \mathbf{J}_\infty(\tilde{X}^{(1)}).$$

Proof. Let μ be a non-negative measure in $\mathbf{K}_\infty(X^{(1)})$. It suffices to show that $\nu(dx) := e^{-2u(x)}\mu(dx) \in \mathbf{K}_\infty(\tilde{X}^{(1)})$. Let A be the positive continuous additive functional of X having Revuz measure μ . In view of Lemma 3.2, it can also be viewed as the positive continuous additive functional of \tilde{X} with Revuz measure ν . Observe that $\mu \in \mathbf{K}(X^{(1)}) = \mathbf{K}(X)$. By Lemma 3.2, $\nu \in \mathbf{K}(\tilde{X})$ and so there is $\alpha > 0$ such that $\|\tilde{G}_\alpha \nu\|_\infty \leq 1$. For any given $\varepsilon > 0$, choose $t_0 > 0$ so that $e^{-\alpha t_0} < \varepsilon/2$. Then for any $x \in E$ and any $B \in \mathcal{B}(E)$,

$$\begin{aligned}\tilde{G}_\alpha(\mathbf{1}_B \nu)(x) &= \mathbb{E}_x \left[\int_0^\infty e^{-\alpha s} \mathbf{1}_B(\tilde{X}_s) dA_s \right] \leq \tilde{\mathbb{E}}_x \left[\int_0^{t_0} \mathbf{1}_B(\tilde{X}_s) dA_s \right] + \tilde{\mathbb{E}}_x \left[\int_{t_0}^\infty e^{-\alpha s} dA_s \right] \\ &\leq \mathbb{E}_x \left[Z_t \int_0^{t_0} \mathbf{1}_B(X_s) dA_s \right] + e^{-\alpha t_0} \tilde{\mathbb{E}}_x \left[\tilde{G}_\alpha \mu(X_{t_0}) \right] \\ &\leq (\mathbb{E}_x [Z_{t_0}^2])^{1/2} \left(\mathbb{E}_x \left[\left(\int_0^{t_0} \mathbf{1}_B(X_s) dA_s \right)^2 \right] \right)^{1/2} + \varepsilon/2.\end{aligned}\tag{3.4}$$

By [11, Lemma 4.1(ii)], $\sup_{x \in E} \mathbb{E}_x [Z_{t_0}^2] = c(t_0) < \infty$. On the other hand, denoting $f(x) := \mathbb{E}_x \left[\int_0^{t_0} \mathbf{1}_B(X_s) dA_s \right]$, we have by the Markov property of X ,

$$\begin{aligned}\mathbb{E}_x \left[\left(\int_0^{t_0} \mathbf{1}_B(X_s) dA_s \right)^2 \right] &= 2 \mathbb{E}_x \left[\int_0^{t_0} \mathbf{1}_B(X_s) \left(\int_s^{t_0} \mathbf{1}_B(X_r) dA_r \right) dA_s \right] \\ &\leq 2 \mathbb{E}_x \left[\int_0^{t_0} \mathbf{1}_B(X_s) f(X_s) dA_s \right] \leq 2 \|f\|_\infty^2 \leq 2e^{2\alpha t_0} \|G_\alpha(\mathbf{1}_B \mu)\|_\infty^2.\end{aligned}$$

This together with (3.4) yields

$$\|\tilde{G}_\alpha(\mathbf{1}_B\nu)\|_\infty \leq \sqrt{2c(t_0)} e^{\alpha t_0} \|G_\alpha(\mathbf{1}_B\mu)\|_\infty + \varepsilon/2.$$

It follows from the definition of \mathbf{K}_∞ that $\nu \in \mathbf{K}_\infty(\tilde{X}^{(\alpha)}) = \mathbf{K}_\infty(\tilde{X}^{(1)})$. \square

Remark 3.4 (i) By [11, Theorem 3.5], X can be recovered from \tilde{X} through an analogous Girsanov transform. Thus we in fact have $\mathbf{K}(X) = \mathbf{K}(\tilde{X})$, $\mathbf{K}_\infty(X^{(1)}) = \mathbf{K}_\infty(\tilde{X}^{(1)})$ and $\mathbf{J}_\infty(X^{(1)}) = \mathbf{J}_\infty(\tilde{X}^{(1)})$.

See [13, Lemma 3.3] for a related result under the assumption that X is a doubly Feller process with no killings inside. \square

In view of (3.3), we can express $e^{N_t^u}$ as follows (see [11, (4.6)]),

$$\exp(N_t^u) = \exp(u(X_t) - u(X_0) - M_t^u) = e^{-u(x)} Z_t e^{-A_t} e^{u(X_t)}, \quad (3.5)$$

where

$$A_t := \int_0^t \left(\int_{E_\partial} \left(u(X_s) - u(y) + 1 - e^{u(X_s) - u(y)} \right) N(X_s, dy) \right) dH_s - \frac{1}{2} \langle M^{u,c} \rangle_t.$$

Let ν be the signed Revuz measure of A , that is,

$$\nu(dx) := \left(\int_{E_\partial} \left(u(x) - u(y) + 1 - e^{u(x) - u(y)} \right) N(x, dy) \right) d\mu_H(dx) - \frac{1}{2} \mu_{\langle u \rangle}^c(dx). \quad (3.6)$$

Since u is bounded,

$$|\nu(dx)| \leq \frac{e^{\|u\|_\infty}}{2} \left(\int_{E_\partial} (u(x) - u(y))^2 N(x, dy) \right) d\mu_H(dx) + \frac{1}{2} \mu_{\langle u \rangle}^c(dx) \leq \frac{e^{\|u\|_\infty}}{2} \mu_{\langle u \rangle}(dx) \quad (3.7)$$

and so $\nu \in \mathbf{K}_\infty(X^{(1)}) \subset \mathbf{K}_\infty(\tilde{X}^{(1)})$.

For convenience, if A^μ is a continuous additive functional of X with (signed) Revuz measure μ , in view of Lemma 3.2, we will denote A^μ by $\tilde{A}^{e^{-2u}\mu}$ when viewed as a continuous additive functional of \tilde{X} .

By (3.5),

$$\begin{aligned} T_t^{u,\mu,F} f(x) &= e^{-u(x)} \mathbb{E}_x \left[Z_\infty \exp \left(-A_t^\nu + A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s) \right) (e^u f)(X_t) \right] \\ &= e^{-u(x)} \tilde{\mathbb{E}}_x \left[\exp \left(\tilde{A}_t^{e^{-2u}(\mu-\nu)} + \sum_{0 < s \leq t} F(\tilde{X}_{s-}, \tilde{X}_s) \right) (e^u f)(\tilde{X}_t) \right] \\ &= e^{-u(x)} \tilde{T}_t^{e^{-2u}(\mu-\nu),F} (e^u f)(x), \end{aligned} \quad (3.8)$$

where $\{\tilde{T}_t^{e^{-2u}(\mu-\nu),F}, t \geq 0\}$ is the non-local Feynman-Kac semigroup of \tilde{X} defined by

$$\tilde{T}_t^{e^{-2u}(\mu-\nu),F} g(x) = \tilde{\mathbb{E}}_x \left[\exp \left(\tilde{A}_t^{e^{-2u}(\mu-\nu)} + \sum_{0 < s \leq t} F(\tilde{X}_{s-}, \tilde{X}_s) \right) g(\tilde{X}_t) \right].$$

Theorem 3.5 *Let u be a bounded function in \mathcal{F}_e with $\mu_{\langle u \rangle} \in \mathbf{K}_\infty(X^{(1)})$, $\mu \in \mathbf{K}_\infty(X^{(1)})$ and F be a bounded symmetric function in $\mathbf{J}_\infty(X^{(1)})$. Then for every $p \in [1, \infty]$, $\{T_t^{u, \mu, F}, t \geq 0\}$ is a strongly continuous symmetric semigroup on $L^p(E; m)$.*

Proof. Since by (3.7) and Lemma 3.3, $\nu \in \mathbf{K}_\infty(\tilde{X}^{(1)})$, $\mu \in \mathbf{K}_\infty(X^{(1)}) \subset \mathbf{K}_\infty(\tilde{X}^{(1)})$ and $F \in \mathbf{J}_\infty(\tilde{X}^{(1)})$, it follows from the proof of [10, Proposition 2.3] applied to the process \tilde{X} that for every $p \in [1, \infty]$, $\{\tilde{T}_t^{e^{-2u}(\mu-\nu), F}, t \geq 0\}$ is a strongly continuous symmetric semigroup on $L^p(E; e^{2u}dm)$. Thus we have by (3.8) that $\{T_t^{u, \mu, F}, t \geq 0\}$ is a strongly continuous symmetric semigroup on $L^p(E; dm)$ for every $p \in [1, \infty]$. \square

Denote the operator norm of $\tilde{T}_t^{e^{-2u}(\mu-\nu), F} : L^p(E; e^{2u}dm) \rightarrow L^p(E; e^{2u}dm)$ by $\|\tilde{T}_t^{e^{-2u}(\mu-\nu), F}\|_{p,p}$, and the operator norm of $T_t^{u, \mu, F} : L^p(E; m) \rightarrow L^p(E; m)$ by $\|T_t^{u, \mu, F}\|_{p,p}$. For $1 \leq p \leq \infty$, the L^p -spectral bound of semigroup $T_t^{u, \mu, F}$ is defined as

$$\lambda_p(X, u + \mu + F) := - \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T_t^{u, \mu, F}\|_{p,p}.$$

Clearly, in view of (3.8),

$$\|T_t^{u, \mu, F}\|_{2,2} = \|\tilde{T}_t^{e^{-2u}(\mu-\nu), F}\|_{2,2},$$

while

$$e^{-2\|u\|_\infty} \|T_t^{u, \mu, F}\|_{\infty, \infty} \leq \|\tilde{T}_t^{e^{-2u}(\mu-\nu), F}\|_{\infty, \infty} \leq e^{2\|u\|_\infty} \|T_t^{u, \mu, F}\|_{\infty, \infty}.$$

It follows that

$$\lambda_2(X, u + \mu + F) = \lambda_2(\tilde{X}, e^{-2u}(\mu - \nu) + F) \quad \text{and} \quad \lambda_\infty(X, u + \mu + F) = \lambda_\infty(\tilde{X}, e^{-2u}(\mu - \nu) + F). \quad (3.9)$$

Note that by [5, (5.7)]

$$\begin{aligned} & \lambda_2(X, u + \mu + F) = \lambda_2(\tilde{X}, e^{-2u}(\mu - \nu), F) \\ &= \inf \left\{ \tilde{\mathcal{E}}(g, g) - \int_{E \times E} g(x)g(y) \left(e^{F(x,y)} - 1 \right) e^{-u(x)-u(y)} N(x, dy) \mu_H(dx) \right. \\ & \quad \left. - \int_E g(x)^2 e^{-2u(x)} (\nu - \mu)(dx); g \in \mathcal{F} \text{ with } \int_E g(x)^2 e^{-2u(x)} m(dx) = 1 \right\} \\ &= \inf \left\{ \tilde{\mathcal{E}}(g, g) - \int_{E \times E} g(x)g(y) e^{-u(x)-u(y)} \left(e^{F(x,y)} - 1 \right) N(x, dy) \mu_H(dx) \right. \\ & \quad \left. - \int_E g(x)^2 e^{-2u(x)} (\nu - \mu)(dx); g \in \mathcal{F}_b \text{ with } \int_E g(x)^2 e^{-2u(x)} m(dx) = 1 \right\} \\ &= \inf \left\{ \tilde{\mathcal{E}}(fe^u, fe^u) - \int_{E \times E} f(x)f(y) \left(e^{F(x,y)} - 1 \right) N(x, dy) \mu_H(dx) \right. \\ & \quad \left. - \int_E f(x)^2 (\nu - \mu)(dx); f \in \mathcal{F}_b \text{ with } \int_E f(x)^2 m(dx) = 1 \right\} \\ &= \inf \left\{ \mathcal{E}(f, f) + \mathcal{E}(u, f^2) + \int_E f(x)^2 \mu(dx) - \int_{E \times E} f(x)f(y) \left(e^{F(x,y)} - 1 \right) N(x, dy) \mu_H(dx); \right. \\ & \quad \left. f \in \mathcal{F}_b \text{ with } \int_E f(x)^2 m(dx) = 1 \right\}. \end{aligned} \quad (3.10)$$

In the last equality, we used the fact that

$$\mathcal{E}(fe^u, fe^u) - \int_E f(x)^2 \nu(dx) = \mathcal{E}(f, f) + \mathcal{E}(u, f^2) \quad \text{for } f \in \mathcal{F}_b,$$

whose proof can be found in the paragraph following (4.8) in the proof of [11, Theorem 1.2] for bounded u .

Clearly

$$\|T_t^{u, \mu, F}\|_{\infty, \infty} = \|T_t^{u, \mu, F} 1\|_{\infty} = \sup_{x \in E} \mathbb{E}_x \left[\exp \left(N^u + A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s) \right); t < \zeta \right].$$

By duality, we have $\|T_t^{u, \mu, F}\|_{1,1} = \|T_t^{u, \mu, F}\|_{\infty, \infty}$. Consequently, it follows from the Cauchy-Schwarz inequality that

$$\|T_t^{u, \mu, F} f\|_2^2 \leq \|T_t^{u, \mu, F} 1\|_{\infty} \|T_t^{u, \mu, F} (f^2)\|_1 \leq \|T_t^{u, \mu, F}\|_{\infty, \infty}^2 \|f\|_2^2 \quad \text{for } f \in L^2(E; m).$$

Thus we have $\|T_t^{u, \mu, F}\|_{2,2} \leq \|T_t^{u, \mu, F}\|_{\infty, \infty}$. We now deduce by interpolation that

$$\|T_t^{u, \mu, F}\|_{2,2} \leq \|T_t^{u, \mu, F}\|_{p,p} \leq \|T_t^{u, \mu, F}\|_{\infty, \infty} \quad \text{for } 1 < p < \infty.$$

Hence

$$\lambda_{\infty}(X, u + \mu + F) \leq \lambda_p(X, u + \mu + F) \leq \lambda_2(X, u + \mu + F) \quad \text{for } 1 < p < \infty. \quad (3.11)$$

4 L^p -independence of spectral bounds

We can now present results on the L^p -independence of the spectral bounds of generalized non-local Feynman-Kac semigroups.

Theorem 4.1 *Assume that $m(E) < \infty$, and that (2.6) holds. Let u be a bounded function in \mathcal{F}_e with $\mu_{\langle u \rangle} \in \mathbf{K}_{\infty}(X^{(1)})$, $\mu \in \mathbf{K}_{\infty}(X^{(1)})$ and F a symmetric function in $\mathbf{J}_{\infty}(X^{(1)})$. Then $\lambda_p(X, u + \mu + F)$ is independent of $p \in [1, \infty]$.*

Proof. Since P_{t_0} is a bounded linear operator from $L^2(E; m)$ to $L^{\infty}(E; m)$, by duality, P_{t_0} is a bounded linear operator from $L^1(E; m)$ to $L^2(E; m)$. Hence $P_{2t_0} : L^1(E; m) \rightarrow L^{\infty}(E; m)$ is bounded. Let $M_t := \int_0^t e^{u(X_{s-})} dM_s^{e^{-u}-1}$ and $Z_t = \text{Exp}(M)_t$ be its Doléans-Dade exponential martingale, which admits expression (3.3). Since $\mu_{\langle u \rangle} \in \mathbf{K}_{\infty}(X^{(1)}) \subset \mathbf{K}(X^{(1)}) = \mathbf{K}(X)$, we have by [11, Lemma 4.1(ii)] that $\sup_{x \in E} \mathbb{E} [Z_{2t_0}^2] < \infty$.

Denote by Y the Girsanov transformed process of X via Z . Then for every $f \in L^2(E; m)$,

$$|P_{2t_0}^Y f(x)| := |\mathbb{E}_x[f(Y_{2t_0})]| = |\mathbb{E}_x[Z_{2t_0} f(X_t)]| \leq (\mathbb{E}_x[Z_{2t_0}^2] \mathbb{E}_x[f(X_{2t_0})^2])^{1/2} \leq c \|f\|_{L^2(E; m)}.$$

This proves that condition (2.6) holds for Y with $2t_0$ in place of t_0 . Let ν be the measure defined by (3.6). Note that in view of (3.7), $\nu \in \mathbf{K}_{\infty}(X^{(1)}) \subset \mathbf{K}_{\infty}(\tilde{X}^{(1)})$. Thus we can apply Theorem 2.3 to conclude that

$$\lambda_2(Y, e^{-2u}(\mu - \nu) + F) = \lambda_{\infty}(Y, e^{-2u}(\mu - \nu) + F).$$

We deduce from this and (3.9) that $\lambda_2(X, u + \mu + F) = \lambda_{\infty}(u + \mu + F)$. Consequently, we have from (3.11) that $\lambda_p(X, u + \mu + F)$ is independent of $p \in [1, \infty]$. \square

Theorem 4.2 *Suppose that u is a bounded function in \mathcal{F}_e with $\mu_{\langle u \rangle} \in \mathbf{K}_\infty(X^{(1)})$, $\mu \in \mathbf{K}_\infty(X^{(1)})$ and $F \in \mathbf{J}_\infty(X^{(1)})$ is bounded and symmetric.*

- (i) $\lambda_\infty(X, u + \mu + F) \geq \min\{\lambda_2(X, u + \mu + F), 0\}$ and so $\lambda_p(X, u + \mu + F)$ is independent of $p \in [1, \infty]$ if $\lambda_2(X, u + \mu + F) \leq 0$.
- (ii) *Assume in addition that X is conservative. If $\lambda_2(X, u + \mu + F) > 0$, then $\lambda_\infty(X, u + \mu + F) = 0$. Hence $\lambda_p(X, u + \mu + F)$ is independent of $p \in [1, \infty]$ if and only if $\lambda_2(X, u + \mu + F) \leq 0$.*

Proof. Let $Z_t = \text{Exp}(M)_t$ be the exponential martingale in the proof of Theorem 4.1. As we saw in Section 3, $\{Z_t, t \geq 0\}$ is a uniformly integrable martingale under each \mathbb{P}_x . It follows that the Girsanov transformed process \tilde{X} of X by Z is transient and has a Green function G^Y . Furthermore, \tilde{X} is conservative if so is X . It is clear that \tilde{X} is $e^{-2u}m$ -irreducible since $\text{Exp}(M)_t > 0$ a.s.. Let ν be the measure defined by (3.6). Note that in view of (3.7), $\nu \in \mathbf{K}_\infty(X^{(1)}) \subset \mathbf{K}_\infty(\tilde{X}^{(1)})$. Since $e^{-2u}(\mu - \nu) \in \mathbf{K}_\infty(X^{(1)}) \subset \mathbf{K}_\infty(\tilde{X}^{(1)})$, the conclusion of the theorem now follows from Theorem 2.4 applied to $(\tilde{X}, e^{-2u}(\mu - \nu) + F)$ and (3.9). \square

Theorem 4.3 *Suppose that $1 \in \mathbf{K}_\infty(X^{(1)})$, u is a bounded function in \mathcal{F}_e with $\mu_{\langle u \rangle} \in \mathbf{K}_\infty(X^{(1)})$, $\mu \in \mathbf{K}_\infty(X^{(1)})$ and $F \in \mathbf{J}_\infty(X^{(1)})$ symmetric. Then $\lambda_p(X, u + \mu + F)$ is independent of $p \in [1, \infty]$.*

Proof. As above, let $Z_t = \text{Exp}(M)_t$ be the exponential martingale in the proof of Theorem 4.1 and \tilde{X} the Girsanov transformed process of X by Z . Let ν be the measure defined by (3.6), which in view of (3.7) is in $\mathbf{K}_\infty(X^{(1)}) \subset \mathbf{K}_\infty(\tilde{X}^{(1)})$. The conclusion of this theorem follows from Theorem 2.5 applied to $(\tilde{X}, e^{-2u}(\mu - \nu) + F)$ and (3.9). \square

5 Examples

In this section, we give several concrete examples for functions to be in Kato class $\mathbf{K}_\infty(X^{(1)})$, $\mathbf{J}_\infty(X^{(1)})$ and for bounded $u \in \mathcal{F}_e$ with $\mu_{\langle u \rangle} \in \mathbf{K}_\infty(X^{(1)})$ so that the main results of this paper apply.

Two real-valued functions f and g are said to be comparable if there is a constant $c > 1$ so that $g/c \leq f \leq cg$, and we denote it by $f \asymp g$.

Example 5.1 (Stable-like process on d -sets) Let $n \geq 1$ and $0 < d \leq n$. A Borel subset E in \mathbb{R}^n is said to be a global d -set if there exist a measure m on E and constants $C_2 > C_1 > 0$ so that

$$C_1 r^d \leq m(B(x, r)) \leq C_2 r^d \quad \text{for all } x \in E \text{ and } r > 0. \quad (5.1)$$

Here $B(x, r) := \{y \in E : |x - y| < r\}$ and $|\cdot|$ is the Euclidean metric in \mathbb{R}^n .

For a closed global d -set $E \subset \mathbb{R}^n$ and $0 < \alpha < 2$, define

$$\mathcal{F} = \left\{ u \in L^2(E, m) : \int_{E \times E} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} m(dx)m(dy) < \infty \right\} \quad (5.2)$$

$$\mathcal{E}(u, v) = \frac{1}{2} \int_{E \times E} (u(x) - u(y))(v(x) - v(y)) \frac{c(x, y)}{|x - y|^{d+\alpha}} m(dx)m(dy) \quad (5.3)$$

for $u, v \in \mathcal{F}$, where $c(x, y)$ is a symmetric function on $E \times E$ that is bounded between two strictly positive constants $C_4 > C_3 > 0$, that is,

$$C_3 \leq c(x, y) \leq C_4 \quad \text{for } m\text{-a.e. } x, y \in E. \quad (5.4)$$

It is shown in [9] that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E; m)$ and there is an associated m -symmetric Hunt process X on E starting from every point in E . Moreover, X admits a jointly Hölder continuous transition density function $p(t, x, y)$ with respect to the measure m , which satisfies the following two-sided estimates

$$p(t, x, y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \quad \text{on } (0, \infty) \times E \times E, \quad (5.5)$$

where the comparison constants in (5.5) depend only on C_k , $k = 1, 2, 3, 4$. We call such kind of process a α -stable-like process on E . Note that when $E = \mathbb{R}^n$ and $c(x, y)$ is a constant function, then X is nothing but a rotationally symmetric α -stable process on \mathbb{R}^n . The process X has a Lévy system $(N(x, dy), H)$, with $N(x, dy) = N(x, y)dy = c(x, y)|x - y|^{-(n+\alpha)}dy$ and $H_t = t$ so $\mu_H(dx) = m(dx)$. When $\alpha < d$, the process X is transient and its Green function is given by

$$G(x, y) = \int_0^\infty p(t, x, y)dt \asymp |x - y|^{\alpha-d}, \quad x, y \in E.$$

By the same argument as that for Theorem 2.1 of Chen [3], we can show that when $0 < \alpha < d \wedge 2$,

(a) a signed measure μ is in $\mathbf{K}(X)$ if and only if

$$\limsup_{r \rightarrow 0} \sup_{x \in E} \int_{B(x, r)} |x - y|^{\alpha-d} |\mu|(dy) = 0; \quad (5.6)$$

(b) a finite signed measure μ is in $\mathbf{K}_\infty(X)$ if and only if it is in $\mathbf{K}(X)$;

(c) a signed measure μ is in $\mathbf{K}_\infty(X)$ if and only if both (5.6) and the following condition

$$\limsup_{R \rightarrow \infty} \sup_{x \in E} \int_{B(0, R)^c} |x - y|^{\alpha-d} |\mu|(dy) = 0 \quad (5.7)$$

are satisfied.

It is easy to see that condition (5.6) is satisfied for $\mu(dx) = f(x)m(dx)$ if $f \in L^p(E; m)$ for some $p > d/\alpha$. We next show the following.

Lemma 5.1 *Let $0 < \alpha < 2$ and X be a symmetric α -stable-like process on the d -set E .*

(i) *When $\alpha \leq d$, $L^p(E; m) \subset \mathbf{K}_\infty(X^{(1)})$ for every $p > d/\alpha$. When $0 < d < \alpha$, $L^p(E; m) \subset \mathbf{K}_\infty(X^{(1)})$ for every $p \geq 1$.*

(ii) *For bounded $u \in \mathcal{F}_e$, $\mu_{\langle u \rangle} \in \mathbf{K}_\infty(X^{(1)})$ if $f_u(x) := \int_E (u(x) - u(y))^2 \frac{c(x, y)}{|x - y|^{d+\alpha}} m(dy)$ is in $L^p(E; m)$ for some $p > d/\alpha$. In particular, if $u \in C_c^1(E)$, then $\mu_{\langle u \rangle} \in \mathbf{K}_\infty(X^{(1)})$.*

(iii) If F is a bounded function on $E \times E$ with

$$|F(x, y)| \leq c|x - y|^\gamma \quad \text{for } x, y \in E \quad \text{and} \quad F(x, y) = 0 \quad \text{for } (x, y) \in E \times K^c, \quad (5.8)$$

where K is a compact subset of E , c and γ are two positive constants such that $\gamma > \alpha$, then $F \in \mathbf{J}_\infty(X^{(1)})$.

Proof. (i) In view of (5.5),

$$G_1(x, y) = \int_0^\infty e^{-t} p(t, x, y) dt \asymp \int_0^{|x-y|^\alpha} e^{-t} \frac{t}{|x-y|^{d+\alpha}} dt + \int_{|x-y|^\alpha}^\infty e^{-t} t^{-d/\alpha} dt. \quad (5.9)$$

Observe that

$$\int_0^{|x-y|^\alpha} e^{-t} \frac{t}{|x-y|^{d+\alpha}} dt = \frac{1 - (1 + |x-y|^\alpha) e^{-|x-y|^\alpha}}{|x-y|^{d+\alpha}} \leq c_1 \frac{e^{-|x-y|^\alpha}}{|x-y|^{d-\alpha}}, \quad (5.10)$$

while

$$\int_{|x-y|^\alpha}^\infty e^{-t} t^{-d/\alpha} dt \leq \frac{\alpha}{d-\alpha} \frac{e^{-|x-y|^\alpha}}{|x-y|^{d-\alpha}} \quad \text{when } d > \alpha. \quad (5.11)$$

When $d \leq \alpha$ and $|x - y| < 1$,

$$\begin{aligned} \int_{|x-y|^\alpha}^\infty e^{-t} t^{-d/\alpha} dt &\leq \int_{|x-y|^\alpha}^1 t^{-d/\alpha} dt + \int_1^\infty e^{-t} dt \\ &= \begin{cases} \alpha \log(1/|x-y|) + 1 & \text{if } d = \alpha, \\ \frac{\alpha}{\alpha-d} (1 - |x-y|^{\alpha-d}) + 1 & \text{if } d < \alpha, \end{cases} \end{aligned} \quad (5.12)$$

while for $|x - y| \geq 1$,

$$\int_{|x-y|^\alpha}^\infty e^{-t} t^{-d/\alpha} dt \leq |x-y|^{-d} \int_{|x-y|^\alpha}^\infty e^{-t} dt = |x-y|^{-d} e^{-|x-y|^\alpha}. \quad (5.13)$$

Thus we have by (5.9)-(5.13)

$$G_1(x, y) \leq \begin{cases} c_2 \frac{e^{-|x-y|^\alpha}}{|x-y|^{d-\alpha}} & \text{when } d > \alpha, \\ c_2 \left(\log(1/|x-y|) \mathbf{1}_{\{|x-y| < 1/2\}} + \frac{e^{-|x-y|^\alpha}}{|x-y|^\alpha} \mathbf{1}_{\{|x-y| \geq 1/2\}} \right) & \text{when } d = \alpha, \\ c_2 \frac{e^{-|x-y|^\alpha}}{1+|x-y|^\alpha} & \text{when } d < \alpha. \end{cases} \quad (5.14)$$

Suppose that $0 < \alpha \leq d$. For $f \in L^p(E; m)$ with $p > d/\alpha$, let $q > 1$ be the conjugate of p , that is, $q = p/(p-1)$. Note that $q < \frac{d}{d-\alpha}$. We have by Hölder's inequality that

$$\begin{aligned} \sup_{x \in E} \int_{B(0, R)^c} G_1(x, y) |f(y)| m(dy) &\leq \left(\sup_{x \in E} \int_E G_1(x, y)^q m(dy) \right)^{1/q} \left(\int_{B(0, R)^c} |f(y)|^p m(dy) \right)^{1/p} \\ &\leq c \left(\int_{B(0, R)^c} |f(y)|^p m(dy) \right)^{1/p}. \end{aligned}$$

Hence for every $\varepsilon > 0$, there is some $R > 0$ so that $\sup_{x \in E} \int_{B(0,R)^c} G_1(x,y) |f(y)| m(dy) < \varepsilon/2$. On the other hand, there is $\delta > 0$ so that for every Borel set B with $m(B) < \delta$, $(\int_B |f(y)|^p m(dy))^{1/p} < \frac{\varepsilon}{2c}$ and so by the same argument as above,

$$\sup_{x \in E} \int_B G_1(x,y) |f(y)| m(dy) < \varepsilon/2.$$

This shows that $f \in \mathbf{K}_\infty(X^{(1)})$. Now assume that $0 < d < \alpha < 2$. Using Hölder's inequality, we can deduce from above that $L^p(E; m) \subset \mathbf{K}_\infty(X^{(1)})$ for every $p > 1$. We next show that $L^1(E; m) \subset \mathbf{K}_\infty(X^{(1)})$. Note that since $0 < d < \alpha < 2$, we have by (5.14) that $G_1(x,y)$ is bounded. This in particular implies that for $f \in L^1(E; m)$,

$$\lim_{R \rightarrow \infty} \sup_{x \in E} \int_{B(0,R)^c} G_1(x,y) |f(y)| m(dy) \leq \lim_{R \rightarrow \infty} \int_{B(0,R)^c} |f(y)| m(dy) = 0,$$

and

$$\lim_{\delta \rightarrow 0} \sup_{x \in E} \sup_{B: m(B) < \delta} \int_B G_1(x,y) |f(y)| m(dy) \leq \lim_{\delta \rightarrow 0} \sup_{B: m(B) < \delta} \int_{B(0,R)^c} |f(y)| m(dy) = 0.$$

Therefore we have $L^1(E; m) \subset \mathbf{K}_\infty(X^{(1)})$.

(ii) For bounded $u \in \mathcal{F}_e$, we deduce from (3.1) that

$$\mu_{\langle u \rangle}(dx) = \left(\int_E (u(x) - u(y))^2 \frac{c(x,y)}{|x-y|^{d+\alpha}} m(dy) \right) m(dx) = f_u(x) m(dx).$$

Then by (i), $\mu_{\langle u \rangle} \in \mathbf{K}_\infty(X^{(1)})$ if $f_u \in L^p(E; m)$ for some $p > d/\alpha$. Note that $C_c^1(E) \subset \mathcal{F}$. We next show that for $u \in C_c^1(E)$, $f_u \in L^p(E; m)$ for every $p \geq 1$. Clearly, by the mean value theorem,

$$f_u(x) \leq \int_{\{y \in E: |y-x| < 1\}} \frac{c}{|x-y|^{d+\alpha-2}} m(dy) + \int_{\{y \in E: |y-x| \geq 1\}} \frac{c}{|x-y|^{d+\alpha}} m(dy)$$

and so f_u is bounded. Let $K = \text{supp}[u]$. Then for $x \in K^c$,

$$f_u(x) = \int_K u(y)^2 \frac{c(x,y)}{|x-y|^{d+\alpha}} m(dy) \leq \frac{c}{1+|x|^{d+\alpha}}.$$

Since (E, m) is a d -set, it follows that $f_u \in L^p(E; m)$ for every $p \geq 1$. In particular, we have $\mu_{\langle u \rangle} \in \mathbf{K}_\infty(X^{(1)})$.

(iii) Suppose that F is a bounded Borel function on $E \times E$ satisfying (5.8). Let $f(x) = \int_E |F(x,y)| N(x,y) m(dy)$. Then for every $x \in E$,

$$\begin{aligned} f(x) &\leq \int_{\{y \in K: |y-x| < 1\}} \frac{c}{|x-y|^{d+\alpha-\gamma}} m(dy) + \int_{\{y \in K: |y-x| \geq 1\}} \frac{c}{|x-y|^{d+\alpha}} m(dy) \\ &\leq \frac{c}{d(x,K)^{d+\alpha}}, \end{aligned}$$

where $d(x, K)$ denotes the Euclidean distance between x and K . It follows that $f \in L^p(E; m)$ for every $p \geq 1$. In particular, this implies that $F \in \mathbf{J}_\infty(X^{(1)})$. \square

Example 5.2. (Symmetric diffusions) Let X be a symmetric diffusion in \mathbb{R}^n , $n \geq 1$, with infinitesimal generator $\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right)$, where matrix $(a_{ij}(x))_{1 \leq i,j \leq n}$ is uniformly elliptic and bounded, that is, there is $\lambda > 1$ such that for m -a.e. $x \in \mathbb{R}^n$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$,

$$\lambda^{-1} \|\xi\|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \lambda \|\xi\|^2.$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(\mathbb{R}^n, dx)$ for X is: $\mathcal{F} = W^{1,2}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n, dx) : \nabla f \in L^2(\mathbb{R}^n, dx)\}$ and

$$\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} dx, \quad f, g \in W^{1,2}(\mathbb{R}^n).$$

Lemma 5.2 (i) *If $n \geq 3$, then $L^p(\mathbb{R}^n; dx) \subset \mathbf{K}_\infty(X^{(1)})$ for every $p > n/2$. When $n = 1$ or 2 , then $L^p(\mathbb{R}^n; dx) \subset \mathbf{K}_\infty(X^{(1)})$ for every $p \geq 1$.*

(ii) *Suppose that $u \in L^2_{\text{loc}}(\mathbb{R}^n; dx)$ is bounded with $\nabla u \in L^2(\mathbb{R}^n; dx) \cap L^p(\mathbb{R}^n; dx)$ for some $p > n$. Then $u \in \mathcal{F}_e$ with $\mu_{\langle u \rangle} \in \mathbf{K}_\infty(X^{(1)})$.*

Proof. (i) It is well known that the symmetric diffusion process X has a jointly Hölder continuous transition density function $p(t, x, y)$ with respect to the Lebesgue measure $m(dx) := dx$ on \mathbb{R}^n . Moreover, $p(t, x, y)$ enjoys the following celebrated Anroson's estimate: there are constants $c_1, c_2 \geq 1$ so that for every $t > 0$ and $x, y \in \mathbb{R}^n$,

$$c_1^{-1} t^{-n/2} e^{-c_2|x-y|^2/t} \leq p(t, x, y) \leq c_1 t^{-n/2} e^{-|x-y|^2/(c_2 t)}. \quad (5.15)$$

Note that

$$\begin{aligned} \int_0^\infty e^{-t} t^{-n/2} e^{-r^2/(c_2)t} dt &\stackrel{t=r^2u}{=} r^{2-n} \int_0^\infty u^{-n/2} e^{-1/(c_2u)} e^{-r^2u} du \\ &\leq c_3 r^{2-n} \int_0^\infty (u^{-n/2-1} \wedge 1) e^{-1/(2c_2u) - r^2u/2} du \\ &\leq c_3 r^{2-n} e^{-c_4 r} \int_0^\infty (u^{-n/2-1} \wedge 1) du = c_5 r^{2-n} e^{-c_4 r}. \end{aligned}$$

Thus we have by (5.15) and the above that

$$G_1(x, y) = \int_0^\infty e^{-t} p(t, x, y) dt \leq c_1 c_5 |x - y|^{2-n} e^{-c_4|x-y|}. \quad (5.16)$$

Just as in the proof of Lemma 5.1, using Hölder inequality and (5.16), one can show that when $n \geq 3$, $L^p(\mathbb{R}^n; dx) \subset \mathbf{K}_\infty(X^{(1)})$ for every $p > n/2$. When $n = 1$ or 2 , $L^p(\mathbb{R}^n; dx) \subset \mathbf{K}_\infty(X^{(1)})$ for every $p \geq 1$.

(ii) It is known (see, e.g., Example 1.5.2 of [14]) that the extended Dirichlet space

$$\mathcal{F}_e = \{f \in L^2_{\text{loc}}(\mathbb{R}^n; dx) : \nabla f \in L^2(\mathbb{R}^n; dx)\}.$$

By (3.1), for bounded $u \in \mathcal{F}_e$, its energy measure is $\mu_{\langle u \rangle}(dx) = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx$. Thus by (i) above, a bounded locally L^2 -integrable function u with $\nabla u \in L^2(\mathbb{R}^n; dx) \cap L^p(\mathbb{R}^n; dx)$ for some $p > n$ is a function in \mathcal{F}_e with $\mu_{\langle u \rangle}$ in the Kato class of X . \square

We refer the reader to [4, Examples 2.2 and 2.3] for examples of Kato classes $\mathbf{K}_\infty(X)$ and $\mathbf{J}_\infty(X)$ when X is a symmetric α -stable process, respectively, a censored α -stable process, in a bounded $C^{1,1}$ -open set.

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