Heat Kernel Estimate for $\Delta + \Delta^{\alpha/2}$ in $C^{1,1}$ open sets

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Abstract

We consider a family of pseudo differential operators $\{\Delta + a^\alpha \Delta^{\alpha/2};\ a \in (0,1]\}$ on $\mathbb{R}^d$ for every $d \geq 1$ that evolves continuously from Δ to $\Delta + \Delta^{\alpha/2}$, where $\alpha \in (0,2)$. It gives rise to a family of Lévy processes $\{X^a, a \in (0,1]\}$ in $\mathbb{R}^d$, where X^a is the sum of a Brownian motion and an independent symmetric α-stable process with weight a. We establish sharp two-sided estimates for the heat kernel of $\Delta + a^\alpha \Delta^{\alpha/2}$ with zero exterior condition in a family of open subsets, including bounded $C^{1,1}$ (possibly disconnected) open sets. This heat kernel is also the transition density of the sum of a Brownian motion and an independent symmetric α-stable process with weight a in such open sets.

Our result is the first sharp two-sided estimates for the transition density of a Markov process with both diffusion and jump components in open sets. Moreover, our result is uniform in a in the sense that the constants in the estimates are independent of $a \in (0,1]$ so that it recovers the Dirichlet heat kernel estimates for Brownian motion by taking $a \to 0$. Integrating the heat kernel estimates in time t, we recover the two-sided sharp uniform Green function estimates of X^a in bounded $C^{1,1}$ open sets in $\mathbb{R}^d$, which were recently established in [14] by using a completely different approach.

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1 Introduction

Many physical and economic systems should be and in fact have been successfully modeled by discontinuous Markov processes; see for example, [27, 26, 28] and the references therein. Discontinuous Markov processes are also very important from a theoretical point of view, since they contain stable processes, relativistic stable processes and jump diffusions as special cases. Due to their importance

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both in theory and in applications, discontinuous Markov processes have been receiving intensive study in recent years.

In general, Markov processes may have both diffusion and jump components. A Markov process having continuous sample paths is called a diffusion. Diffusion processes in $\mathbb{R}^d$ and second order elliptic differential operators on $\mathbb{R}^d$ are closely related in the following sense. For a large class of second order elliptic differential operators $\mathcal{L}$ on $\mathbb{R}^d$, there is a diffusion process X in $\mathbb{R}^d$ associated with it so that $\mathcal{L}$ is the infinitesimal generator of X, and vice versa. The connection between $\mathcal{L}$ and X can also be seen as follows. The fundamental solution $p(t,x,y)$ of $\partial_t u = \mathcal{L}u$ (also called the heat kernel of $\mathcal{L}$) is the transition density of X. Thus obtaining sharp two-sided estimates for $p(t,x,y)$ is a fundamental problem in both analysis and probability theory. In fact, two-sided heat kernel estimates for diffusions in $\mathbb{R}^d$ have a long history and many beautiful results have been established. See [21, 23] and the references therein. But, due to the complication near the boundary, two-sided estimates on the transition density of killed diffusions in a domain D (equivalently, the Dirichlet heat kernel) have been established only recently. See [22, 23, 24] for upper bound estimates and [31] for the lower bound estimate of the Dirichlet heat kernels in bounded $C^{1,1}$ domains.

The infinitesimal generator of a discontinuous Markov process in $\mathbb{R}^d$ is no longer a differential operator but rather a non-local (or integro-differential) operator $\mathcal{L}$. For instance, the infinitesimal generator of a rotationally symmetric α-stable process in $\mathbb{R}^d$ with $\alpha \in (0,2)$ is the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$. Most of the recent studies concentrate on pure jump Markov processes, like the rotationally symmetric α-stable processes, that do not have a diffusion component. For a summary of some of these recent results from the probability literature, one can see [1, 7] and the references therein. We refer the readers to [4, 5, 6] for a sample of recent progresses in the PDE literature.

Recently in [9], we obtained sharp two-sided estimates for the heat kernel of the fractional Laplacian $\Delta^{\alpha/2}$ in D with zero exterior condition (or equivalently, the transition density function of the symmetric α-stable process killed upon exiting D) for any $C^{1,1}$ open set $D \subset \mathbb{R}^d$ with $d \geq 1$. As far as we know, this was the first time sharp two-sided estimates were established for Dirichlet heat kernels of non-local operators. Since then, studies on this topic have been growing rapidly. In [10, 11, 12], the ideas of [9] were adapted to establish two-sided heat kernel estimates of other pure jump Markov processes in open subsets of $\mathbb{R}^d$. In [19], the large time behaviors of heat kernels for symmetric α-stable processes and censored stable processes in unbounded open sets were studied. Very recently in [2, 3], the heat kernel of the fractional Laplacian in non-smooth open set was discussed. We refer the readers to [8] for a survey on the recent progresses in the heat kernel estimates of jump Markov processes.

However until now, two-sided heat kernel estimates of Markov processes with both diffusion and jump components in proper open subsets of $\mathbb{R}^d$ have not been studied. The fact that such a process X has both diffusion and jump components is the source of many difficulties. The main difficulty stems from the fact that such a process X runs on two different scales: on the small scale the diffusion part dominates, while on the large scale the jumps take over. Another difficulty is encountered at the exit of X from an open set: for diffusions, the exit is through the boundary, while for pure jump processes, typically the exit happens by jumping across the boundary. For a process X that has both diffusion and jump components, both cases will occur, which makes the process X much more difficult to study.

In this paper, we consider Lévy processes that are independent sums of Brownian motions and (rotationally) symmetric stable processes in $\mathbb{R}^d$ with $d \geq 1$. We establish two-sided heat kernel estimates for such Lévy processes killed upon exiting a $C^{1,1}$ open set. The processes studied in this paper serve as a test case for more general processes with both diffusion and jump components, just like Brownian motions do for more general diffusions. We hope that our study will help to shed new light on the understanding of the heat kernel behavior of more general Markov processes. Although two-sided heat kernel estimates for Markov processes with both diffusion and jump components in $\mathbb{R}^d$ have been studied recently in [17, 29], as far as we know, this is the first time that sharp two-sided estimates on the Dirichlet heat kernels for Markov processes with both diffusion and jump components in proper open subsets are established.

Let us now describe the main result of this paper and at the same time fix the notations. Throughout this paper, we assume that $d \geq 1$ is an integer and $\alpha \in (0,2)$. Let $X^0 = (X^0_t, t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$ with generator $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$. Let $Y = (Y_t, t \geq 0)$ be a (rotationally) symmetric α-stable process in $\mathbb{R}^d$, that is, a Lévy process such that

$$\mathbb{E}_x \left[e^{i\xi\cdot(Y_t - Y_0)} \right] = e^{-t|\xi|^\alpha} \qquad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d.$$

The infinitesimal generator of a symmetric α-stable process Y in $\mathbb{R}^d$ is the fractional Laplacian $\Delta^{\alpha/2}$, which is a prototype of non-local operators. The fractional Laplacian can be written in the form

$$\Delta^{\alpha/2} u(x) = \lim_{\varepsilon\downarrow 0} \int_{\{y\in\mathbb{R}^d:\, |y-x|>\varepsilon\}} (u(y)-u(x)) \frac{\mathcal{A}(d,\alpha)}{|x-y|^{d+\alpha}}\, dy, \tag{1.1}$$

where $\mathcal{A}(d,\alpha) := \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma(\frac{d+\alpha}{2})\Gamma(1-\frac{\alpha}{2})^{-1}$. Here Γ is the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1}e^{-t}dt$ for every $\lambda > 0$. Assume that X^0 and Y are independent. For any $a > 0$, we define X^a by $X^a_t := X^0_t + aY_t$. We will call the process X^a the independent sum of the Brownian motion X^0 and the symmetric α-stable process Y with weight $a > 0$. The infinitesimal generator of X^a is $\Delta + a^\alpha \Delta^{\alpha/2}$ and

$$\mathbb{E}_x \left[e^{i\xi\cdot(X^a_t - X^a_0)} \right] = e^{-t(|\xi|^2 + a^\alpha|\xi|^\alpha)} \qquad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d.$$

Since

$$a^\alpha|\xi|^\alpha = \int_{\mathbb{R}^d} (1-\cos(\xi\cdot y))\, \frac{a^\alpha \mathcal{A}(d,\alpha)}{|y|^{d+\alpha}} dy,$$

the density of the Lévy measure of X^a with respect to the Lebesgue measure on $\mathbb{R}^d$ is

$$J^a(x,y) = j^a(|x-y|) := a^\alpha \mathcal{A}(d,\alpha)|x-y|^{-(d+\alpha)}.$$

The function $J^a(x,y)$ determines a Lévy system for X^a, which describes the jumps of the process X^a: for any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s,y,y) = 0$ for all $y \in \mathbb{R}^d$, any stopping time T (with respect to the filtration of X^a) and any $x \in \mathbb{R}^d$,

$$\mathbb{E}_x \left[\sum_{s\leq T} f(s, X^a_{s-}, X^a_s) \right] = \mathbb{E}_x \left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X^a_s, y) J^a(X^a_s, y) dy \right) ds \right] \tag{1.2}$$

(see, for example, [15, Proof of Lemma 4.7] and [16, Appendix A]). Let $p^a(t,x,y)$ be the transition density of the process X^a with respect to the Lebesgue measure on $\mathbb{R}^d$, which is known to exist and is jointly continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. For any $\lambda>0$, the process $(\lambda X^a_{\lambda^{-2}t}, t\geq 0)$ has the same distribution as $(X^{a\lambda^{(\alpha-2)/\alpha}}_t, t\geq 0)$ (see the second paragraph of Section 2), so we have

$$p^{a\lambda^{(\alpha-2)/\alpha}}(t,x,y)=\lambda^{-d}p^a(\lambda^{-2}t,\lambda^{-1}x,\lambda^{-1}y) \qquad \text{for } t>0 \text{ and } x,y\in\mathbb{R}^d. \tag{1.3}$$

The following sharp two-sided estimates on $p^a(t,x,y)$ follows from (1.3) and the main results in [17, 29] that give the sharp estimates on $p^1(t,x,y)$.

Theorem 1.1 *There are constants $C_i\geq 1$, $i=1,2$, such that, for all $a\in[0,\infty)$ and $(t,x,y)\in(0,\infty]\times\mathbb{R}^d\times\mathbb{R}^d$*

$$\begin{aligned}&C_1^{-1}\left(t^{-d/2}\wedge(a^\alpha t)^{-d/\alpha}\right)\wedge\left(t^{-d/2}e^{-C_2|x-y|^2/t}+(a^\alpha t)^{-d/\alpha}\wedge\frac{a^\alpha t}{|x-y|^{d+\alpha}}\right)\\&\leq p^a(t,x,y)\leq C_1\left(t^{-d/2}\wedge(a^\alpha t)^{-d/\alpha}\right)\wedge\left(t^{-d/2}e^{-|x-y|^2/C_2t}+(a^\alpha t)^{-d/\alpha}\wedge\frac{a^\alpha t}{|x-y|^{d+\alpha}}\right).\end{aligned}$$

Here for $a,b\in\mathbb{R}$, $a\wedge b:=\min\{a,b\}$ and $a\vee b:=\max\{a,b\}$. In particular, we have

Corollary 1.2 *For any $M>0$ and $T>0$, there is a constant $C_3\geq 1$ depending only on d, α, M and T such that, for all $a\in(0,M]$ and $(t,x,y)\in(0,T]\times\mathbb{R}^d\times\mathbb{R}^d$*

$$\begin{aligned}&C_3^{-1}\left(t^{-d/2}e^{-C_2|x-y|^2/t}+t^{-d/2}\wedge\frac{a^\alpha t}{|x-y|^{d+\alpha}}\right)\\&\leq p^a(t,x,y)\leq C_3\left(t^{-d/2}e^{-|x-y|^2/C_2t}+t^{-d/2}\wedge\frac{a^\alpha t}{|x-y|^{d+\alpha}}\right),\end{aligned}$$

where $C_2\geq 1$ is the constant in Theorem 1.1.

Recall that an open set D in $\mathbb{R}^d$ (when $d\geq 2$) is said to be $C^{1,1}$ if there exist a localization radius $R>0$ and a constant $\Lambda>0$ such that for every $z\in\partial D$, there is a $C^{1,1}$-function $\phi=\phi_z:\mathbb{R}^{d-1}\to\mathbb{R}$ satisfying $\phi(0)=0$, $\nabla\phi(0)=(0,\dots,0)$, $\|\nabla\phi\|_\infty\leq\Lambda$, $|\nabla\phi(x)-\nabla\phi(z)|\leq\Lambda|x-z|$, and an orthonormal coordinate system CS_z: $y=(y_1,\cdots,y_{d-1},y_d):=(\widetilde{y},\,y_d)$ with origin at z such that $B(z,R)\cap D=\{y=(\tilde{y},y_d)\in B(0,R)\text{ in } CS_z: y_d>\phi(\widetilde{y})\}$. The pair (R,Λ) will be called the $C^{1,1}$ characteristics of the open set D. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be written as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive. Note that a $C^{1,1}$ open set can be unbounded and disconnected, and that a bounded $C^{1,1}$ open set has only finitely many connected components.

For an open set $D\subset\mathbb{R}^d$ and $x\in D$, we will use $\delta_D(x)$ to denote the Euclidean distance between x and D^c. For an open set $D\subset\mathbb{R}^d$ and $(r_0,\lambda_0)\in(0,\infty)\times[1,\infty)$, we say *the path distance in each connected component of D is comparable to the Euclidean distance with characteristics* (r_0,λ_0) if the following holds for any $r\in(0,r_0)$: for every x,y in the same component of D with $\delta_D(x)\wedge\delta_D(y)\geq r$, there is a rectifiable curve l in D connecting x to y so that the length of l is no larger than $\lambda_0|x-y|$. Clearly, such a property holds for all bounded $C^{1,1}$ open sets, $C^{1,1}$ open sets with compact complements and domains above graphs of $C^{1,1}$ functions.

For any open subset $D\subset\mathbb{R}^d$, we use τ^a_D to denote the first time the process X^a exits D. We define the process $X^{a,D}$ by $X^{a,D}_t=X^a_t$ for $t<\tau^a_D$ and $X^{a,D}_t=\partial$ for $t\geq\tau^a_D$, where ∂ is a cemetery point. $X^{a,D}$ is called the subprocess of X^a killed upon exiting D. The infinitesimal generator of $X^{a,D}$ is $(\Delta + a^\alpha \Delta^{\alpha/2})|_D$. It follows from [17] that $X^{a,D}$ has a continuous transition density $p^a_D(t,x,y)$ with respect to the Lebesgue measure.

The goal of this paper is to get the following sharp two-sided estimates on $p^a_D(t,x,y)$ for any $C^{1,1}$ open set D in which the path distance in each connected component of D is comparable to the Euclidean distance.

Let

$$h^a_C(t,x,y) := \begin{cases} \left(1\wedge \frac{\delta_D(x)}{\sqrt{t}}\right)\left(1\wedge \frac{\delta_D(y)}{\sqrt{t}}\right)\left(t^{-d/2}e^{-C|x-y|^2/t} + \frac{a^\alpha t}{|x-y|^{d+\alpha}} \wedge t^{-d/2}\right) \\ \qquad \text{when } x,y \text{ are in the same component of } D, \\ \left(1\wedge \frac{\delta_D(x)}{\sqrt{t}}\right)\left(1\wedge \frac{\delta_D(y)}{\sqrt{t}}\right)\left(\frac{a^\alpha t}{|x-y|^{d+\alpha}} \wedge t^{-d/2}\right) \\ \qquad \text{when } x,y \text{ are in different components of } D. \end{cases} \tag{1.4}$$

One can easily show that, when D is bounded, the operator $-(\Delta + a^\alpha \Delta^{\alpha/2})|_D$ has discrete spectrum (see, for instance, the first paragraph of the proof of Theorem 1.3 (ii) and (iii) in Section 4). In this case, we use $\lambda^{a,D}_1 > 0$ to denote the smallest eigenvalue of $-(\Delta + a^\alpha \Delta^{\alpha/2})|_D$. Denote by $D(x)$ the connected component of D that contains x and let $\lambda^{a,D(x)}_1 > 0$ be the smallest eigenvalue of $-(\Delta + a^\alpha \Delta^{\alpha/2})|_{D(x)}$.

Theorem 1.3 *Let $d \geq 1$. Suppose that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristic (R,Λ) such that the path distance in each connected component of D is comparable to the Euclidean distance with characteristics (r_0, λ_0).*

(i) *For every $M > 0$ and $T > 0$, there are positive constants $C_i = C_i(R,\Lambda, r_0, \lambda_0, M, \alpha, T) \geq 1$, $i = 4, 5$, such that for all $a \in (0, M]$ and $(t,x,y) \in (0,T] \times D \times D$,*

$$C_4^{-1}\, h^a_{C_5}(t,x,y) \leq p^a_D(t,x,y) \leq C_4 h^a_{1/C_5}(t,x,y).$$

(ii) *Suppose in addition that D is bounded and connected. For every $M > 0$ and $T > 0$, there is a constant $C_6 = C_6(D, M, \alpha, T) \geq 1$ so that for all $a \in (0, M]$ and $(t,x,y) \in [T,\infty) \times D \times D$,*

$$C_6^{-1}\, e^{-t\,\lambda^{a,D}_1}\, \delta_D(x)\, \delta_D(y) \;\leq\; p^a_D(t,x,y) \;\leq\; C_6\, e^{-t\,\lambda^{a,D}_1}\, \delta_D(x)\, \delta_D(y).$$

(iii) *Suppose that D is bounded but disconnected. Then for every $M > 0$ and $T > 0$, there are constants $C_i = C_i(D, M, \alpha, T) \geq 1$. $i = 7, 8$, such that for all $a \in (0, M]$, $t \in [T, \infty)$, the following hold.*

(a) *If x, y are in the same component $D(x)$ of D,*

$$C_7^{-1}\, e^{-t\,\lambda^{a,D(x)}_1}\, \delta_D(x)\, \delta_D(y) \;\leq\; p^a_D(t,x,y) \;\leq\; C_7\left(e^{-t\,\lambda^{a,D(x)}_1} + \big(1\wedge (a^\alpha\, t)\big) e^{-t\,\lambda^{a,D}_1}\right) \delta_D(x)\, \delta_D(y).$$

(b) *If x, y are in different components of D,*

$$C_8^{-1}\, a^\alpha\, t\, e^{-t\,(\lambda^{a,D(x)}_1 \vee \lambda^{a,D(y)}_1)}\, \delta_D(x)\, \delta_D(y) \;\leq\; p^a_D(t,x,y) \;\leq\; C_8\, (1 \wedge (a^\alpha t))\, e^{-t\,\lambda^{a,D}_1}\, \delta_D(x)\, \delta_D(y).$$

Remark 1.4 (i) Unlike the Brownian motion case, even though D may be disconnected, the process $X^{a,D}$ is always irreducible when $a > 0$ because $X^{a,D}$ can jump from one component of D to another. When $a > 0$ is smaller, the connection between different components of D by X^a becomes weaker. The estimates given in Theorem 1.3 present a precise quantitative description of such a phenomenon. Letting $a \to 0$, Theorem 1.3 recovers the Dirichlet heat kernel estimates for Brownian motion in D (even when D is disconnected); see [20, 31] and the reference therein for the latter. In particular, for x and y in different components of D, we have $\lim_{a\to 0+} p^a_D(t, x, y) = 0$ for all $x, y > 0$, which is the case for Brownian motion.

(ii) In fact, the estimates in Theorem 1.3(i) will be established under a weaker assumption on D: the lower bounded estimate is proved under the uniform interior ball condition and the condition that the path distance in each connected component of D is comparable to the Euclidean distance (see Theorem 2.4), while the upper bound estimate is proved under a weaker version of the uniform exterior ball condition (see Theorem 3.9). Here an open set $D \subset \mathbb{R}^d$ is said to satisfy the *uniform interior ball condition* with radius $R_1 > 0$ if for every $x \in D$ with $\delta_D(x) < R_1$, there is $z_x \in \partial D$ so that $|x - z_x| = \delta_D(x)$ and $B(x_0, R_1) \subset D$ for $x_0 := z_x + R_1(x - z_x)/|x - z_x|$. We say D satisfies *a weaker version of the uniform exterior ball condition* with radius $R_1 > 0$ if for every $z \in \partial D$, there is a ball B^z of radius R_1 such that $B^z \subset (\overline{D})^c$ and $\partial B^z \cap \partial D = \{z\}$.

Integrating the heat kernel estimates in Theorem 1.3 over time t yields the following two-sided sharp estimates of the Green function of X^a in bounded $C^{1,1}$ open sets, which were first obtained in [14] by a different method. We will not give the details in this paper on how these estimates can be obtained by integrating the estimates in Theorem 1.3. Interested readers are referred to the proof of [9, Corollary 1.2], where the sharp estimates for the Green functions of symmetric stable processes in bounded $C^{1,1}$ open sets are obtained from the sharp heat kernel estimates for the heat kernels by integration over time t.

Define for $d \geq 3$ and $a > 0$,

$$g^a_D(x,y) := \begin{cases} \frac{1}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) & \text{when } x, y \text{ are in the same component of } D, \\ \frac{a^\alpha}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) & \text{when } x, y \text{ are in different components of } D; \end{cases}$$

for $d = 2$ and $a > 0$,

$$g^a_D(x,y) := \begin{cases} \log\left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) & \text{when } x, y \text{ are in the same component of } D, \\ a^\alpha \log\left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) & \text{when } x, y \text{ are in different components of } D; \end{cases}$$

and for $d = 1$ and $a > 0$,

$$g^a_D(x,y) := \begin{cases} (\delta_D(x)\delta_D(y))^{1/2} \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|} & \text{when } x, y \text{ are in the same component of } D, \\ a^\alpha\big((\delta_D(x)\delta_D(y))^{1/2} \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|}\big) & \text{when } x, y \text{ are in different components of } D. \end{cases}$$

Corollary 1.5 *Let $M > 0$. Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. There exists $C_9 = C_9(D, M, \alpha) > 1$ such that for all $x, y \in D$ and all $a \in (0, M]$*

$$C_9^{-1}\, g^a_D(x,y) \leq G^a_D(x,y) \leq C_9\, g^a_D(x,y).$$

This paper is a natural continuation of [9], where sharp two-sided heat kernel estimates for symmetric α-stable processes in $C^{1,1}$ open sets are first derived, as well as [13], where the boundary Harnack principle for X^a is established. Some ideas of the approach in this paper can be traced back to [9] but a number of new ideas are needed to handle the combined effects of Brownian motion and discontinuous stable process. A comparison with subordination of killed Brownian motion is used for the lower bound short time heat kernel estimates for $X^{a,D}$. We would like to point out that, unlike [9], the boundary Harnack principle for X^a is not used directly in this paper. Instead we use one of the key lemmas established in [13] to obtain the upper bound of the heat kernel (see Lemma 3.1). Theorem 1.3(i) will be established through Theorem 2.4 and Theorem 3.9, which give the lower bound and upper bound estimates, respectively. In contrast to that in [9, 10, 12], the proof of large time heat kernel estimates in Theorem 1.3(ii)-(iii) does not use intrinsic ultracontractivity of $X^{a,D}$. The proof presented here is more direct, and uses only the continuity of $\lambda_1^{a,D}$ and its corresponding first eigenfunction in $a \in (0, M]$, which is established in [18]. Lastly, we point out that the approach of [3] relies critically on the fact the symmetric stable processes do not have diffusion component and so it is not directly applicable to the processes considered in this paper.

We will use capital letters $C_1, C_2, \dots$ to denote constants in the statements of results, and their labeling will be fixed. The lower case constants $c_1, c_2, \dots$ will denote generic constants used in proofs, whose exact values are not important and can change from one appearance to another. The labeling of the lower case constants starts anew in each proof. The dependence of the constant c on the dimension d will not be mentioned explicitly. We will use ":=" to denote a definition, which is read as "is defined to be". We will use ∂ to denote a cemetery point and for every function f, we extend its definition to ∂ by setting $f(\partial) = 0$. We will use dx to denote the Lebesgue measure in $\mathbb{R}^d$. The Lebesgue measure of a Borel set $A \subset \mathbb{R}^d$ will be denoted by $|A|$.

2 Lower bound estimate

In this section, we assume that D is an open set in $\mathbb{R}^d$ satisfying the *uniform interior ball condition* with radius $R_1 > 0$ and that the path distance in each connected component of D is comparable to the Euclidean distance with characteristics (r_0, λ_0). Observe that under the uniform interior ball condition, the condition that the path distance in each connected component of D is comparable to the Euclidean distance is equivalent to the following: there exist $r_2, \lambda > 0$ such that for all $r \in (0, r_2]$ and all x, y in the same connected component of D with $\delta_D(x) \wedge \delta_D(y) \geq r$, there is a rectifiable curve l in D connecting x to y so that the length of l is no larger than $\lambda|x-y|$ and $\delta_D(z) \geq r$ for every $z \in l$. The latter is also equivalent to the following, which is called the connected ball condition in [20]: For all $r \in (0, r_2]$ and x, y in the same connected component of D with $\delta_D(x) \wedge \delta_D(y) > r$, there exist m and x_k, $k = 1, 2, \dots, m$ such that $x_0 = x$, $x_m = y$, $x_{k-1} \in B(x_k, \frac{r}{2}) \subset B(x_k, r) \subset D$ and $r \cdot m \leq \lambda_0 |x-y|$.

Observe for all $\lambda, a > 0$ and $\xi, x \in \mathbb{R}^d$,

$$\mathbb{E}_x \left[e^{i\xi\cdot(\lambda(X^a_{t/\lambda^2} - X^a_0))} \right] = e^{-t|\xi|^2} \, \mathbb{E}_x \left[e^{i(a\lambda\xi)\cdot(Y_{t/\lambda^2} - Y_0)} \right] = e^{-t(|\xi|^2 + (a\lambda^{(\alpha-2)/\alpha})^\alpha |\xi|^\alpha)}.$$

It follows that if $\{X^{a,D}_t, t \geq 0\}$ is the subprocess in D of the independent sum of a Brownian motion and a symmetric α-stable process on $\mathbb{R}^d$ with weight a, then $\left\{\lambda X^{a,D}_{\lambda^{-2}t}, t \geq 0\right\}$ is the subprocess in

λD of the independent sum of a Brownian motion and a symmetric α-stable process on $\mathbb{R}^d$ with weight $a\lambda^{(\alpha-2)/\alpha}$. So for any $\lambda > 0$, we have

$$p^{a\lambda^{(\alpha-2)/\alpha}}_{\lambda D}(t,x,y) = \lambda^{-d} p^a_D(\lambda^{-2}t, \lambda^{-1}x, \lambda^{-1}y) \qquad \text{for } t>0 \text{ and } x,y \in \lambda D. \tag{2.1}$$

The above scaling property of X^a will be used throughout this paper. For $t>0$, we define

$$a_t := at^{(2-\alpha)/(2\alpha)}. \tag{2.2}$$

This notation will be used in this paper when we scale an open D by $s^{-1/2}$ to $s^{-1/2}D$.

We first recall the definition of subordinate killed Brownian motion: Assume that U is an open subset in $\mathbb{R}^d$ and T_t is an $\alpha/2$-stable subordinator independent of the killed Brownian motion $X^{0,U}$. For each $a \geq 0$, let T^a be the subordinator defined by $T^a_t := t + a^2 T_t$. Then the process $\{Z^{a,U}_t : t \geq 0\}$ defined by $Z^{a,U}_t = X^{0,U}_{T^a_t}$ is called a subordinate killed Brownian motion in U. Let $q^a_U(t,x,y)$ be the transition density of $Z^{a,U}$. Then it follows from [30, Proposition 3.1] that

$$p^a_U(t,z,w) \geq q^a_U(t,z,w), \quad (t,z,w) \in (0,\infty) \times U \times U. \tag{2.3}$$

We will use this fact in the next result.

Lemma 2.1 *Suppose that M and T are positive constants. Then there exist positive constants $C_i = C_i(R_1, r_0, \lambda_0, \alpha, T, M)$, $i = 10, 11$, such that for all $a \in (0, M]$, $t \in (0,T]$ and x, y in the same connected component of D,*

$$p^a_D(t,x,y) \geq C_{10} t^{-d/2} \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right) e^{-C_{11}|x-y|^2/t}.$$

Proof. Suppose that x and y are in the same component, say U, of D. Let $p_U(t,x,y)$ be the transition density of the killed Brownian motion in U. It follows from [20, Theorem 3.3] (see also [31, Theorem 1.2]) that there exist positive constants $c_1 = c_1(R_1, r_0, \lambda_0, \alpha, T)$ and $c_2 = c_2(R_1, r_0, \lambda_0, \alpha)$ such that for any $(s,x,y) \in (0, 2T] \times U \times U$,

$$p_U(s,x,y) \geq c_1 \left(1 \wedge \frac{\delta_U(x)}{\sqrt{s}}\right)\left(1 \wedge \frac{\delta_U(y)}{\sqrt{s}}\right) s^{-d/2} e^{-c_2|x-y|^2/s}.$$

(Although not explicitly mentioned in [20], a careful examination of the proofs in [20] reveals that the constants c_1 and c_2 in the above lower bound estimate can be chosen to depend only on $(R_1, r_0, \lambda_0, \alpha, T)$ and $(R_1, r_0, \lambda_0, \alpha)$, respectively.) Since $p_{t^{-1/2}U}(u, t^{-1/2}x, t^{-1/2}y) = t^{d/2} p_U(ut, x, y)$, we have for $t \leq T$ and $(u,x,y) \in (0,2] \times U \times U$,

$$p_{t^{-1/2}U}(u, t^{-1/2}x, t^{-1/2}y) \geq c_1 \left(1 \wedge \frac{\delta_U(x)}{\sqrt{tu}}\right)\left(1 \wedge \frac{\delta_U(y)}{\sqrt{tu}}\right) u^{-d/2} e^{-c_2|x-y|^2/(tu)}. \tag{2.4}$$

Let $\mu^{a_t}(u,s)$ be the density of ${a_t}^2 T_u$, where a_t is defined in (2.2). Then it follows from the definition of the subordinate killed Brownian motion (for example, see [1, page 149]) that for every $1/3 \leq b \leq 1$ and $0 < t \leq T$,

$$q^{a_t}_{t^{-1/2}U}(b, t^{-1/2}x, t^{-1/2}y) \;=\; \int_b^\infty p_{t^{-1/2}U}(s, t^{-1/2}x, t^{-1/2}y) \mathbb{P}(b + a_t^2 T_b \in ds)$$

$$
\begin{aligned}
&= \int_b^\infty p_{t^{-1/2}U}(s, t^{-1/2}x, t^{-1/2}y)\mu^{a_t}(b, s-b)ds \\
&= \int_0^\infty p_{t^{-1/2}U}(s+b, t^{-1/2}x, t^{-1/2}y)\mu^{a_t}(b, s)ds.
\end{aligned}
$$

Consequently, by (2.3) and (2.4), for every $1/3 \le b \le 1$ and $0 < t \le T$,

$$
\begin{aligned}
& p^{a_t}_{t^{-1/2}D}(b, t^{-1/2}x, t^{-1/2}y) \\
\ge\ & p^{a_t}_{t^{-1/2}U}(b, t^{-1/2}x, t^{-1/2}y) \\
\ge\ & q^{a_t}_{t^{-1/2}U}(b, t^{-1/2}x, t^{-1/2}y) \\
\ge\ & \int_0^1 p_{t^{-1/2}U}(s+b, t^{-1/2}x, t^{-1/2}y)\mu^{a_t}(b, s)ds \\
\ge\ & \frac{c_1}{2}\left(1 \wedge \frac{\delta_U(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_U(y)}{\sqrt{t}}\right) e^{-3c_2|x-y|^2/t} \int_0^1 \mu^{a_t}(b, s)ds \\
=\ & \frac{c_1}{2}\left(1 \wedge \frac{\delta_U(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_U(y)}{\sqrt{t}}\right) e^{-3c_2|x-y|^2/t}\, \mathbb{P}({a_t}^2 T_b \le 1) \\
\ge\ & \frac{c_1}{2}\left(1 \wedge \frac{\delta_U(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_U(y)}{\sqrt{t}}\right) e^{-3c_2|x-y|^2/t}\, \mathbb{P}(T_{1/3} \le M^{-2} T^{-(2-\alpha)/\alpha}) \\
\ge\ & c_3 \left(1 \wedge \frac{\delta_U(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_U(y)}{\sqrt{t}}\right) e^{-3c_2|x-y|^2/t} \\
=\ & c_3 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right) e^{-3c_2|x-y|^2/t}.
\end{aligned} \tag{2.5}
$$

We now conclude from (2.1), (2.2) and (2.5) with $b = 1$ that

$$
p^a_D(t, x, y) = t^{-d/2} p^{a_t}_{t^{-1/2}D}(1, t^{-1/2}x, t^{-1/2}y) \ge c_3 t^{-d/2}\left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right) e^{-3c_2|x-y|^2/t}.
$$

□

The inequality (2.5) above with $b = 1/3$ will be used later.

Lemma 2.2 *For all $M, r, b > 0$, there exists $C_{12} = C_{12}(M, \alpha, r, b) > 0$ such that*

$$
\mathbb{P}_0(\tau^a_{B(0,r)} > b) \ge C_{12} > 0 \qquad \text{for all } a \in (0, M].
$$

Proof. It follows from Lemma 2.1 that

$$
\begin{aligned}
\inf_{a\in(0,M]} \mathbb{P}_0(\tau^a_{B(0,r)} > b) &= \inf_{a\in(0,M]} \int_{B(0,r)} p^a_{B(0,r)}(b, 0, y)dy \\
&\ge c\, b^{-d/2}\left(\frac{r}{\sqrt{b}} \wedge 1\right) \int_{B(0,r)} \left(\frac{|y|}{\sqrt{b}} \wedge 1\right) e^{-c_1|y|^2/b} dy.
\end{aligned}
$$

□

Lemma 2.3 *Suppose that M and r are positive constants. Then there is a constant $C_{13} = C_{13}(M, \alpha, d, r) \in (0, 1/3]$ such that for all $a \in (0, M]$ and $u, v \in \mathbb{R}^d$,*

$$
p^a_{B(u,r)\cup B(v,r)}(1/3, u, v) \ \ge\ C_{13}(J^a(u, v) \wedge 1).
$$

Proof. If $|u-v|\le r/2$, by Lemma 2.1

$$p^a_{B(u,r)\cup B(v,r)}(1/3,u,v) \ge \inf_{|z|<r/2} p^a_{B(0,r)}(1/3,0,z) \ge c_1\,(r\wedge 1)^2\, e^{-c_2 r^2} \ge c_3 \ge c_3(J^a(u,v)\wedge 1).$$

Let $E=B(u,r)\cup B(v,r)$. If $|u-v|\ge r/2$, with $E_1=B(u,r/8)$ and $E_2=B(v,r/8)$, we have by the strong Markov property and the Lévy system (1.2) of X^a that

$$\begin{aligned}
p^a_E(1/3,u,v) &\ge \mathbb{E}_u\left[p^a_E(1/3-\tau^a_{E_1}, X^a_{\tau^a_{E_1}}, v) : \tau^a_{E_1}<1/3, X^a_{\tau^a_{E_1}}\in E_2\right]\\
&= \int_0^{1/3}\left(\int_{E_1} p^a_{E_1}(s,u,w)\left(\int_{E_2} J^a(w,z)p^a_E(1/3-s,z,v)dz\right)dw\right)ds\\
&\ge \left(\inf_{w\in E_1,\,z\in E_2} J^a(w,z)\right)\int_0^{1/3}\mathbb{P}_u\left(\tau^a_{E_1}>s\right)\left(\int_{E_2} p^a_E(1/3-s,z,v)dz\right)ds\\
&\ge \mathbb{P}_u(\tau^a_{E_1}>1/3)\left(\inf_{w\in E_1,\,z\in E_2} J^a(w,z)\right)\int_0^{1/3}\int_{E_2} p^a_{E_2}(1/3-s,z,v)dzds\\
&= \mathbb{P}_u(\tau^a_{E_1}>1/3)\left(\inf_{w\in E_1,\,z\in E_2} j^a(|w-z|)\right)\int_0^{1/3}\mathbb{P}_v(\tau^a_{E_2}>1/3-s)ds\\
&\ge \frac{1}{3}\mathbb{P}_u(\tau^a_{E_1}>1/3)\left(\inf_{w\in E_1,\,z\in E_2} j^a(|w-z|)\right)\mathbb{P}_v(\tau^a_{E_2}>1/3)\,.
\end{aligned}$$

Thus by Lemma 2.2,

$$\begin{aligned}
p^a_{B(u,r)\cup B(v,r)}(1/3,u,v) &\ge \frac{1}{3}\left(\mathbb{P}_0(\tau^a_{B(0,r/8)}>1/3)\right)^2\left(\inf_{w\in E_1,\,z\in E_2} j^a(|w-z|)\right)\\
&\ge c_4 j^a(|u-v|) \ge c_4(J^a(u,v)\wedge 1)\,.
\end{aligned}$$

□

Recall that the function $h^a_C(t,x,y)$ is defined in (1.4).

Theorem 2.4 *Suppose that M and T are positive constants. There are positive constants $C_i = C_i(M,R_1,r_0,\lambda_0,\lambda,T,\alpha)$, $i=14,15$, such that for all $a\in(0,M]$ and $(t,x,y)\in(0,T]\times D\times D$*

$$p^a_D(t,x,y)\ge C_{14}h^a_{C_{15}}(t,x,y). \tag{2.6}$$

Proof. Since $t^{-1/2}D$ satisfies the uniform interior ball condition with radius $R_1(T)^{-1/2}$ for every $0<t\le T$, there exist $\delta=\delta(R_1,T)\in(0,R_1(T)^{-1/2})$ and $L=L(R_1,T)>1$ such that for all $t\in(0,T]$ and $x,y\in D$, we can choose $\xi_x\in(t^{-1/2}D)\cap B(t^{-1/2}x,L\delta)$ and $\xi_y\in(t^{-1/2}D)\cap B(t^{-1/2}y,L\delta)$ with $B(\xi_x,2\delta)\cap B(\xi_y,2\delta)=\emptyset$ and $B(\xi_x,2\delta)\cup B(\xi_y,2\delta)\subset t^{-1/2}D$.

Let $x_t:=t^{-1/2}x$ and $y_t:=t^{-1/2}y$. Note that by (2.5) with $b=1/3$,

$$\begin{aligned}
\int_{B(\xi_x,\delta)} p^{at}_{t^{-1/2}D}(1/3,x_t,u)du &\ge c_1\left(\frac{\delta_D(x)}{\sqrt{t}}\wedge 1\right)\int_{B(\xi_x,\delta)}(\delta_{t^{-1/2}D}(u)\wedge 1)\,e^{-c_2|x_t-u|^2}du\\
&\ge c_1\left(\frac{\delta_D(x)}{\sqrt{t}}\wedge 1\right)e^{-c_2(L+1)^2\delta^2}\,|B(\xi_x,\delta)|\\
&\ge c_3\left(\frac{\delta_D(x)}{\sqrt{t}}\wedge 1\right).
\end{aligned} \tag{2.7}$$

Similarly

$$\int_{B(\xi_y,\delta)} p^{a_t}_{t^{-1/2}D}(1/3, y_t, u)du \geq c_3 \left(\frac{\delta_D(y)}{\sqrt{t}} \wedge 1 \right). \tag{2.8}$$

Now we deal with the cases $|x_t - y_t| \geq \delta/8$ and $|x_t - y_t| < \delta/8$ separately. Recall the definition of a_t from (2.2).

Case 1: Suppose $|x_t - y_t| \geq \delta/8$. Note that by the semigroup property and Lemma 2.3,

$$\begin{aligned}
& p^{a_t}_{t^{-1/2}D}(1, x_t, y_t) \\
\geq & \int_{B(\xi_y,\delta)} \int_{B(\xi_x,\delta)} p^{a_t}_{t^{-1/2}D}(1/3, x_t, u) p^{a_t}_{t^{-1/2}D}(1/3, u, v) p^{a_t}_{t^{-1/2}D}(1/3, v, y_t) dudv \\
\geq & \int_{B(\xi_y,\delta)} \int_{B(\xi_x,\delta)} p^{a_t}_{t^{-1/2}D}(1/3, x_t, u) p^{a_t}_{B(u,\delta/2)\cup B(v,\delta/2)}(1/3, u, v) p^{a_t}_{t^{-1/2}D}(1/3, v, y_t) dudv \\
\geq & c_4 \int_{B(\xi_y,\delta)} \int_{B(\xi_x,\delta)} p^{a_t}_{t^{-1/2}D}(1/3, x_t, u)(J^{a_t}(u, v) \wedge 1) p^{a_t}_{t^{-1/2}D}(1/3, v, y_t) dudv \\
\geq & c_5 \left(\inf_{(u,v)\in B(\xi_x,\delta)\times B(\xi_y,\delta)} (J^{a_t}(u, v) \wedge 1) \right) \int_{B(\xi_y,\delta)} \int_{B(\xi_x,\delta)} p^{a_t}_{t^{-1/2}D}(1/3, x_t, u) p^{a_t}_{t^{-1/2}D}(1/3, v, y_t) dudv.
\end{aligned}$$

It then follows from (2.7)–(2.8) that

$$p^{a_t}_{t^{-1/2}D}(1, x_t, y_t) \geq c_6 \left(\inf_{(u,v)\in B(\xi_x,\delta)\times B(\xi_y,\delta)} (J^{a_t}(u, v) \wedge 1) \right) \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_D(y)}{\sqrt{t}} \wedge 1 \right). \tag{2.9}$$

Using the fact that

$$J^{a_t}(x_t, y_t) = a^\alpha t^{1+d/2} \mathcal{A}(d, \alpha)|x - y|^{-(d+\alpha)} = t^{1+d/2} J^a(x, y) \tag{2.10}$$

and the assumption $|x_t - y_t| \geq \delta/8$ which implies that $|u - v| \leq 2(1 + L)\delta + |x_t - y_t| \leq (17 + 16L)|x_t - y_t|$, we have

$$\inf_{(u,v)\in B(\xi_x,\delta)\times B(\xi_y,\delta)} (J^{a_t}(u, v) \wedge 1) \geq c_7 \left(J^{a_t}(x_t, y_t) \wedge 1\right) = c_7 \left(t^{1+d/2} J^a(x, y) \wedge 1\right). \tag{2.11}$$

Thus combining (2.9) and (2.11) with (2.1), we conclude that for $|x_t - y_t| \geq \delta/8$

$$\begin{aligned}
p^a_D(t, x, y) &= t^{-d/2} p^{a_t}_{t^{-1/2}D}(1, t^{-1/2}x, t^{-1/2}y) \\
&\geq c_8 t^{-d/2} \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_D(y)}{\sqrt{t}} \wedge 1 \right) (t^{1+d/2} J^a(x, y) \wedge 1) \\
&= c_8 \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_D(y)}{\sqrt{t}} \wedge 1 \right) (t J^a(x, y) \wedge t^{-d/2}).
\end{aligned} \tag{2.12}$$

Case 2: Suppose $|x_t - y_t| < \delta/8$. By the semigroup property,

$$p^{a_t}_{t^{-1/2}D}(1, x_t, y_t) \geq \int_{B(\xi_y,\delta)} \int_{B(\xi_x,\delta)} p^{a_t}_{t^{-1/2}D}(1/3, x_t, u) p^{a_t}_{t^{-1/2}D}(1/3, u, v) p^{a_t}_{t^{-1/2}D}(1/3, v, y_t) dudv. \tag{2.13}$$

By (2.5) with $b = 1/3$, we have for every $(u, v) \in B(\xi_y, \delta) \times B(\xi_x, \delta)$,

$$p^{a_t}_{t^{-1/2}D}(1/3, u, v) \geq c_9 \left(\delta_{t^{-1/2}D}(u) \wedge 1\right) \left(\delta_{t^{-1/2}D}(v) \wedge 1\right) e^{-c_{10}|u-v|^2} \geq c_{11}(\delta \wedge 1)^2.$$

Thus by (2.7)-(2.8) and (2.13),

$$\begin{aligned} p^{a_t}_{t^{-1/2}D}(1, x_t, y_t) &\geq c_{12} \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1\right) \left(\frac{\delta_D(y)}{\sqrt{t}} \wedge 1\right) \\ &\geq c_{12} \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1\right) \left(\frac{\delta_D(y)}{\sqrt{t}} \wedge 1\right) (t^{1+d/2} J^a(x, y) \wedge 1). \end{aligned} \tag{2.14}$$

Combining (2.1), (2.12) and (2.14) with Lemma 2.1, we have proved the theorem. □

3 Upper bound estimate

In this section, we will establish upper bound estimate for X^a in any open set D (not necessarily connected) satisfying a weaker version of the uniform exterior ball condition.

Suppose that U is a $C^{1,1}$ open set with $C^{1,1}$ characteristics (R, Λ). Without loss of generality, we can always assume that $R \leq 1$ and $\Lambda \geq 1$. By definition, for every $Q \in \partial U$, there is a $C^{1,1}$-function $\phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi_Q(0) = 0$, $\nabla\phi_Q(0) = (0, \dots, 0)$, $\|\nabla\phi_Q\|_\infty \leq \Lambda$, $|\nabla\phi_Q(x) - \nabla\phi_Q(z)| \leq \Lambda|x - z|$, and an orthonormal coordinate system CS_Q: $y = (\widetilde{y}, y_d)$ with origin at Q such that $B(Q, R) \cap U = \{y = (\widetilde{y}, y_d) \in B(0, R) \text{ in } CS_Q : y_d > \phi(\widetilde{y})\}$. Define

$$\rho_Q(x) := x_d - \phi_Q(\widetilde{x}),$$

where $(\widetilde{x}, x_d)$ is the coordinates of x in CS_Q. Note that for every $Q \in \partial U$ and $x \in B(Q, R) \cap U$, we have $(1 + \Lambda^2)^{-1/2}\rho_Q(x) \leq \delta_U(x) \leq \rho_Q(x)$. We define for $r_1, r_2 > 0$

$$U_Q(r_1, r_2) := \left\{y \in U : r_1 > \rho_Q(y) > 0, \, |\widetilde{y}| < r_2\right\}.$$

We recall the following key estimates from [13, Lemma 3.5].

Lemma 3.1 *Suppose $R \in (0, 1]$, $M \in (0, \infty)$ and $\Lambda \in [1, \infty)$ are constants, and let $r_0 := R/(4\sqrt{1 + \Lambda^2})$. There are constants $\delta_0 = \delta_0(R, M, \Lambda, \alpha) \in (0, r_0)$ and $C_{16} = C_{16}(R, M, \Lambda, \alpha) > 0$ such that for all $a \in (0, M]$, $\lambda \geq 1$, $C^{1,1}$ open set U with characteristics (R, Λ), $Q \in \partial U$ and $x \in U_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)$ with $\widetilde{x} = 0$,*

$$\mathbb{P}_x\left(X^a_{\tau^a_{U_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}} \in U\right) \leq C_{16}\lambda\delta_U(x) \tag{3.1}$$

and

$$\mathbb{E}_x\left[\tau^a_{U_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}\right] \leq C_{16}\lambda^{-1}\delta_U(x). \tag{3.2}$$

We note that

$$\begin{aligned} \mathbb{P}_x(\tau^a_U > 1/4) &\leq \mathbb{P}_x\left(\tau^a_{U_Q(\delta_0, r_0)} > 1/4\right) + \mathbb{P}_x\left(X^a_{\tau^a_{U_Q(\delta_0, r_0)}} \in U \text{ and } \tau^a_{U_Q(\delta_0, r_0)} \leq 1/4\right) \\ &\leq 4\,\mathbb{E}_x\left[\tau^a_{U_Q(\delta_0, r_0)}\right] + \mathbb{P}_x\left(X^a_{\tau^a_{U_Q(\delta_0, r_0)}} \in U\right). \end{aligned}$$

Thus, by (3.1)-(3.2) with $\lambda = 1$ and a simple geometric consideration, we obtain the following lemma.

Lemma 3.2 *Suppose that $M > 0$ and U is a $C^{1,1}$ open set with the characteristics (R, Λ). There exists $C_{17} = C_{17}(\Lambda, R, M, \alpha) > 0$ such that for all $a \in (0, M]$ and $x \in U$,*

$$\mathbb{P}_x(\tau^a_U > 1/4) \le C_{17}\delta_U(x).$$

In particular, we have the following.

Corollary 3.3 *Suppose that M and r_1 are positive constants and $E := \{x \in \mathbb{R}^d : |x - x_0| > r_1\}$. There exists $C_{18} = C_{18}(r_1, M, \alpha) > 0$ independent of x_0 such that for all $a \in (0, M]$ and $x \in E$,*

$$\mathbb{P}_x(\tau^a_E > 1/4) \le C_{18}\delta_E(x).$$

The proof of the next lemma is similar to that of [3, Lemma 2], which is a variation of the proof of [9, Lemma 2.2]. We give the proof here for the sake of completeness.

Lemma 3.4 *Suppose that E_1, E_3, E are open subsets of $\mathbb{R}^d$ with $E_1, E_3 \subset E$ and $\mathrm{dist}(E_1, E_3) > 0$. For any $n \ge 1$, let $E_{2,i}$, $i = 1, \dots, n$, be disjoint Borel subsets with $\cup_{i=1}^n E_{2,i} = E \setminus (E_1 \cup E_3)$. If $x \in E_1$ and $y \in E_3$, then for all $a > 0$ and $t > 0$,*

$$p^a_E(t, x, y) \le \sum_{i=1}^n \mathbb{P}_x(X^a_{\tau^a_{E_1}} \in E_{2,i}) \left(\sup_{s<t,\, z\in E_{2,i}} p^a_E(s, z, y) \right) + (t \wedge \mathbb{E}_x[\tau^a_{E_1}]) \left(\sup_{u\in E_1,\, z\in E_3} J^a(u, z) \right). \tag{3.3}$$

Proof. Using the strong Markov property, we have

$$\begin{aligned} p^a_E(t, x, y) &= \mathbb{E}_x\left[p^a_E\big(t - \tau^a_{E_1}, X^a_{\tau^a_{E_1}}, y\big) : \tau^a_{E_1} < t \right] \\ &= \sum_{i=1}^n \mathbb{E}_x\left[p^a_E\big(t - \tau^a_{E_1}, X^a_{\tau^a_{E_1}}, y\big) : \tau^a_{E_1} < t, X^a_{\tau^a_{E_1}} \in E_{2,i} \right] \\ &\quad + \mathbb{E}_x\left[p^a_E\big(t - \tau^a_{E_1}, X^a_{\tau^a_{E_1}}, y\big) : \tau^a_{E_1} < t, X^a_{\tau^a_{E_1}} \in E_3 \right] =: I + II\,. \end{aligned}$$

Clearly

$$\begin{aligned} I &\le \sum_{i=1}^n \mathbb{P}_x\left(\tau^a_{E_1} < t, X^a_{\tau^a_{E_1}} \in E_{2,i} \right) \left(\sup_{s<t,\, z\in E_{2,i}} p^a_E(s, z, y) \right) \\ &\le \sum_{i=1}^n \mathbb{P}_x\left(X^a_{\tau^a_{E_1}} \in E_{2,i} \right) \left(\sup_{s<t,\, z\in E_{2,i}} p^a_E(s, z, y) \right). \end{aligned}$$

On the other hand, by (1.2),

$$\begin{aligned} II &= \int_0^t \left(\int_{E_1} p^a_{E_1}(s, x, u) \left(\int_{E_3} J^a(u, z) p^a_E(t - s, z, y) dz \right) du \right) ds \\ &\le \left(\sup_{u\in E_1,\, z\in E_3} J^a(u, z) \right) \int_0^t \mathbb{P}_x(\tau^a_{E_1} > s) \left(\int_{E_3} p^a_E(t - s, z, y) dz \right) ds \\ &\le \int_0^t \mathbb{P}_x(\tau^a_{E_1} > s) ds \sup_{u\in E_1,\, z\in E_3} J^a(u, z) \le (t \wedge \mathbb{E}_x[\tau^a_{E_1}]) \sup_{u\in E_1,\, z\in E_3} J^a(u, z)\,. \end{aligned}$$

This completes the proof of the lemma. □

Theorem 3.5 *Suppose that $M>0$ is a constant and that D is an open set satisfying a weaker version of the uniform exterior ball condition with radius $R_1>0$. There exists a positive constant $C_{19}=C_{19}(M,\alpha,R_1)$ such that for all $a\in(0,M]$ and $x,y\in D$,*

$$p^a_D(1/2,x,y)\le C_{19}\,(\delta_D(x)\wedge 1)\left(e^{-|x-y|^2/(2C_2)}+(j^a(|x-y|)\wedge 1)\right). \tag{3.4}$$

Proof. First note that for every $x_0\in\mathbb{R}^d$, $\{z\in\mathbb{R}^d:|x-x_0|>R_1\}$ is a $C^{1,1}$ open set with characteristics (R,Λ) depending only on R_1 and d. Let r_0 and δ_0 be the positive constants in Lemma 3.1 for $U=\{z\in\mathbb{R}^d:|x-x_0|>R_1\}$.

It follows from Corollary 1.2 that

$$p^a_D(1/2,x,y)\,\le\,p^a(1/2,x,y)\,\le c_1\left(e^{-|x-y|^2/(2C_2)}+(j^a(|x-y|)\wedge 1)\right),$$

so it suffices to prove the theorem for $x\in D$ with $\delta_D(x)<\delta_0/(32)$.

Now fix $x\in D$ with $\delta_D(x)<\delta_0/(32)$ and let $Q\in\partial D$ be such that $|x-Q|=\delta_D(x)$. Let $B_Q\subset D^c$ be the ball with radius R_1 so that $\partial B_Q\cap\partial D=\{Q\}$ and $E:=(\overline{B_Q})^c$. Observe that $\delta_E(x)=\delta_D(x)=|x-Q|$.

When $|x-y|\le\sqrt{dC_2}\vee((\delta_0+r_0)/2)$, we have from Corollary 1.2 that

$$p^a(1/2,x,y)\ge c_2e^{-c_3|x-y|^2}\ge c_4>0\quad\text{and}\quad\sup_{z\in\mathbb{R}^d}p^a(1/4,z,y)\le c_5.$$

Thus, by the semigroup property and Corollary 3.3,

$$\begin{aligned}
p^a_D(1/2,x,y) &= \int_D p^a_D(1/4,x,z)p^a_D(1/4,z,y)dz\\
&\le \sup_{z\in D}p^a_D(1/4,z,y)\mathbb{P}_x(\tau^a_D>1/4)\\
&\le \sup_{z\in\mathbb{R}^d}p^a(1/4,z,y)\mathbb{P}_x(\tau^a_E>1/4)\\
&\le c_6\delta_E(x)=c_6\delta_D(x)\le c_7\,\delta_D(x)p^a(1/2,x,y)\,.
\end{aligned}\tag{3.5}$$

Finally we consider the case that $|x-y|>\sqrt{dC_2}\vee((\delta_0+r_0)/2)$ (and $\delta_D(x)<\delta_0/(32)$).

There is a $C^{1,1}$-function $\phi:\mathbb{R}^{d-1}\to\mathbb{R}$ satisfying $\phi(0)=0$, $\nabla\phi(0)=(0,\dots,0)$, $\|\nabla\phi\|_\infty\le\Lambda$, $|\nabla\phi(w)-\nabla\phi(z)|\le\Lambda|w-z|$, and an orthonormal coordinate system CS with its origin at Q such that

$$B(Q,R)\cap E=\{z=(\widetilde{z},z_d)\in B(0,R)\text{ in }CS:z_d>\phi(\widetilde{z})\}$$

and that x has coordinate $(\widetilde{0},\delta_D(x))$ in this CS. Let

$$E_1:=\{z=(\widetilde{z},z_d)\text{ in }CS:0<z_d-\phi(\widetilde{z})<\delta_0/8,\ |\widetilde{z}|<r_0/8\},$$

$E_3:=\{z\in E:|z-x|>|x-y|/2\}$ and $E_2:=E\setminus(E_1\cup E_3)$. Note that $|z-x|>(\delta_0+r_0)/4$ for $z\in E_3$. So, if $u\in E_1$ and $z\in E_3$, then

$$|u-z|\ge|z-x|-|x-u|\ge|z-x|-(\delta_0+r_0)/8\ge\frac{1}{2}|z-x|\ge\frac{1}{4}|x-y|. \tag{3.6}$$

Thus

$$\sup_{u\in E_1,\,z\in E_3}J^a(u,z)\quad\le\quad\sup_{(u,z):|u-z|\ge\frac14|x-y|}J^a(u,z)$$

$$\leq \; j^a(|x-y|/4) = \left(j^a(|x-y|/4) \wedge j^M((\delta_0+r_0)/8)\right). \tag{3.7}$$

If $z \in E_2$, then $|z-y| \geq |x-y| - |x-z| \geq |x-y|/2$. We also observe that for every $\beta \geq d/4$, $\sup_{s<1/2} s^{-d/2} e^{-\beta/s} = 2^{d/2} e^{-2\beta}$. By Corollary 1.2 and these observations,

$$\begin{aligned}
\sup_{s<1/2,\, z\in E_2} p^a(s,z,y) &\leq C_3 \sup_{s<1/2,\, z\in E_2} \left(s^{-d/2} e^{-|z-y|^2/(C_2 s)} + \left(s^{-d/2} \wedge s J^a(z,y)\right)\right) \\
&\leq C_3 2^{d/2} e^{-|x-y|^2/C_2} + \frac{C_3}{2} j^a(|x-y|/2) \\
&= C_3 2^{d/2} e^{-|x-y|^2/C_2} + \frac{C_3}{2}\left(j^a(|x-y|/2) \wedge j^M((\delta_0+r_0)/4)\right) \\
&\leq c_8 \left(e^{-|x-y|^2/C_2} + \left(j^a(|x-y|) \wedge 1\right)\right) \qquad (3.8)
\end{aligned}$$

for some $c_8 > 0$. Applying Lemmas 3.1 and 3.4, we obtain,

$$\begin{aligned}
p^a_E(1/2,x,y) &\leq c_9 \left(e^{-|x-y|^2/(2C_2)} + \left(j^a(|x-y|) \wedge 1\right)\right)\left(\mathbb{P}_x(X^a_{\tau^a_{E_1}} \in E) + \mathbb{E}_x[\tau^a_{E_1}]\right) \\
&\leq c_{10}\, \delta_E(x) \left(e^{-|x-y|^2/(2C_2)} + \left(j^a(|x-y|) \wedge 1\right)\right) \\
&= c_{10}\, \delta_D(x) \left(e^{-|x-y|^2/(2C_2)} + \left(j^a(|x-y|) \wedge 1\right)\right).
\end{aligned}$$

Therefore

$$p^a_D(1/2,x,y) \leq p^a_E(1/2,x,y) \leq c_{10}\delta_D(x) \left(e^{-|x-y|^2/(2C_2)} + \left(j^a(|x-y|) \wedge 1\right)\right).$$

□

Theorem 3.6 *Assume that $M > 0$ is a constant and that D is an open set satisfying a weaker version of the uniform exterior ball condition with radius $R_1 > 0$. For every $T > 0$, there exists a positive constant $C_{20} = C_{20}(T, R_1, \alpha, M)$ such that for all $a \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,*

$$p^a_D(t,x,y) \leq C_{20} \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right)\left(t^{-d/2} e^{-|x-y|^2/(4C_2^3 t)} + \left(t^{-d/2} \wedge t J^a(x,y)\right)\right). \tag{3.9}$$

Proof. Fix $T, M > 0$ and recall that $a_t := a t^{(2-\alpha)/(2\alpha)} \leq M T^{(2-\alpha)/(2\alpha)}$. Note that $t^{-1/2}D$ is an open set satisfying a weaker version of the uniform exterior ball condition with radius $T^{-1/2}R_1 > 0$ for every $t \in (0, T]$. Thus, by Theorem 3.5, there exists a positive constant $c_1 = c_1(T, R_1, \alpha, M)$ such that for all $t \in (0, T]$ and $a \in (0, M]$,

$$p^{a_t}_{t^{-1/2}D}(1/2,x,y) \leq c_1 (e^{-|x-y|^2/(2C_2)} + (j^{a_t}(|x-y|) \wedge 1))\delta_{t^{-1/2}D}(x). \tag{3.10}$$

Thus by (2.1), (2.10) and (3.10), for every $t \leq T$,

$$\begin{aligned}
p^a_D(t/2,x,y) &= t^{-d/2} p^{a_t}_{t^{-1/2}D}(1/2, t^{-1/2}x, t^{-1/2}y) \\
&\leq c_1\, t^{-d/2} \left(e^{-|x-y|^2/(2C_2 t)} + (j^{a_t}(|x-y|/t^{1/2}) \wedge 1)\right) \delta_{t^{-1/2}D}(t^{-1/2}x) \\
&= c_1 \left(t^{-d/2} e^{-|x-y|^2/(2C_2 t)} + (t^{-d/2} \wedge t J^a(x,y))\right) \frac{\delta_D(x)}{\sqrt{t}}.
\end{aligned}$$

By symmetry, the above inequality holds with the roles of x and y interchanged. Using the semigroup property and Corollary 1.2 (twice), for $t \le T$,

$$\begin{aligned} p_D^a(t,x,y) &= \int_D p_D^a(t/2,x,z)p_D^a(t/2,z,y)dz \\ &\le c_3\,\frac{\delta_D(x)\delta_D(y)}{t}\int_D p^a(2C_2^2t,x,z)p^a(2C_2^2t,z,y)dz \\ &\le c_3\,\frac{\delta_D(x)\delta_D(y)}{t}p^a(4C_2^2t,x,y) \\ &\le c_4\,\frac{\delta_D(x)\delta_D(y)}{t}\left(t^{-d/2}e^{-|x-y|^2/(4C_2^3t)}+(t^{-d/2}\wedge tJ^a(x,y))\right). \end{aligned}$$

This with Corollary 1.2 proves the upper bound (3.9) by noting that

$$(1\wedge u)(1\wedge v)=\min\{1,u,v,uv\} \qquad \text{for } u,v>0.$$

□

We point out that, in view of Theorem 2.4, the above upper bound estimate (3.9) is sharp when x and y are in the same component of D. However it is not sharp when x and y are in different components of D, since in this case when $a \to 0$, it does not go to zero and thus does not give the sharp upper bound for the Dirichlet heat kernel $p_D^0(t,x,y)$ of Brownian motion in D. Next we improve the above estimate to get the sharp estimate stated in Theorem 3.9 below.

For the remainder of this section, we continue to assume D is an open set satisfying a weaker version of the uniform exterior ball condition with radius $R_1>0$. It is easy to see that the distance between any two distinct connected components of D is at least R^* for some $R^*>0$ that depends only on R_1. Without loss of generality, we assume that $R^*=R_1$. Observe that for $c_0>0$, $r\ge r_0$ and $t>0$,

$$t^{-d/2}e^{-c_0r^2/t}\le c_1t^{-d/2}(t/r^2)^{d/2+1}=c_1\frac{t}{r^{d+2}}\le c_1r_0^{\alpha-2}\,\frac{t}{r^{d+\alpha}}, \tag{3.11}$$

where $c_1>0$ depends only on c_0, r_0 and and d. This implies that for x and y in different components of D, the jumping kernel component $tJ^1(x,y)$ dominates the Gaussian component $t^{-d/2}e^{-|x-y|^2/C_2t}$. This fact will be used several times in the rest of this section.

By Theorem 3.6, we only need to consider the case when x and y are in different components of D. Recall that , for any $x\in D$, $D(x)$ denotes the connected component of D that contains x.

First we give an interior upper bound of $p_D^a(t,x,y)$ when x and y are in different components of D.

Lemma 3.7 *Assume that $M>0$ is a constant and that D is an open set satisfying a weaker version of the uniform exterior ball condition with radius $R_1>0$. For every $T>0$, there exists a positive constant $C_{21}=C_{21}(T,R_1,\alpha,M)$ such that for all $a\in(0,M]$, $t\in(0,T]$ and x,y in different components of D,*

$$p_D^a(t,x,y)\le C_{21}a^\alpha t|x-y|^{-d-\alpha}.$$

Proof. Using the strong Markov property and (1.2), we have for $t\le T$,

$$p_D^a(t,x,y)$$

$$
\begin{aligned}
&= \mathbb{E}_x\left[p^a_D\big(t-\tau^a_{D(x)}, X^a_{\tau^a_{D(x)}}, y\big) : \tau^a_{D(x)} < t, X^a_{\tau^a_{D(x)}} \in D\setminus D(x)\right] \\
&= \int_0^t \left(\int_{D(x)} p^a_{D(x)}(s,x,u)\left(\int_{D\setminus D(x)} J^a(u,z)p^a_D(t-s,z,y)dz\right)du\right)ds \\
&\le c_1\frac{a^\alpha}{tM^\alpha}\int_0^t \left(\int_{D(x)} p^a(s,x,u)\left(\int_{D\setminus D(x)} \left(t^{-d/2}\wedge (tJ^M(u,z))\right)p^a(t-s,z,y)dz\right)du\right)ds \\
&\le c_1\frac{a^\alpha}{tM^\alpha}\int_0^t \left(\int_{\mathbb{R}^d} p^a(s,x,u)\left(\int_{\mathbb{R}^d} \left(t^{-d/2}\wedge (tJ^M(u,z))\right)p^a(t-s,z,y)dz\right)du\right)ds. \qquad (3.12)
\end{aligned}
$$

In the second to the last inequality above, we have used the facts that $J^M(u,z)\le j^M(R_1)$ and $t\le T$. By Corollary 1.2, $p^a(s,x,u)\le c_2 p^M(C_2^2 s,x,u)$, $p^a(t-s,z,y)\le c_2 p^M(C_2^2(t-s),z,y)$ and $t^{-d/2}\wedge (tJ^M(u,z))\le c_2 p^M(t,u,z)$. Thus, using the semigroup property and Corollary 1.2, from (3.12) we obtain that

$$
\begin{aligned}
p^a_D(t,x,y) &\le c_3\frac{a^\alpha}{tM^\alpha}\int_0^t \left(\int_{\mathbb{R}^d} p^M(C_2^2 s,x,u)\left(\int_{\mathbb{R}^d} p^M(t,u,z)p^M(C_2^2(t-s),z,y)dz\right)du\right)ds \\
&= c_3\frac{a^\alpha}{tM^\alpha}\int_0^t p^M((C_2^2+1)t,x,y)ds \\
&\le c_4 a^\alpha\left(t^{-d/2}e^{-|x-y|^2/(C_2(C_2^2+1)t)} + t^{-d/2}\wedge\frac{t}{|x-y|^{d+\alpha}}\right) \\
&\le c_5\, a^\alpha\, t\,|x-y|^{-d-\alpha}.
\end{aligned}
$$

In the last inequality above, we have used the fact that $|x-y|\ge R_1$. □

Theorem 3.8 *Assume that $M>0$ is a constant and that D is an open set satisfying a weaker version of the uniform exterior ball condition with radius $R_1>0$. There exists a positive constant $C_{22}=C_{22}(M,\alpha,R_1)$ such that for all $a\in(0,M]$ and x,y in different components of D.*

$$
p^a_D(1,x,y)\le C_{22}\,(\delta_D(x)\wedge 1)(\delta_D(y)\wedge 1)\,(j^a(|x-y|)\wedge 1)\,.
$$

Proof. We first claim that

$$
p^a_D(1/2,x,y)\le c_1 a^\alpha(\delta_D(x)\wedge 1)\left(|x-y|^{-d-\alpha}\wedge 1\right). \qquad (3.13)
$$

Recall that for every $x_0\in\mathbb{R}^d$, $\{z\in\mathbb{R}^d : |x-x_0|>R_1/4\}$ is a $C^{1,1}$ open set with characteristics (R,Λ) depending only on R_1 and d. Let r_0 and δ_0 be the positive constants in Lemma 3.1 for $U=\{z\in\mathbb{R}^d : |x-x_0|>R_1/4\}$. It follows from Lemma 3.7 that

$$
p^a_D(1/2,x,y)\le c_1\,(j^a(|x-y|)\wedge 1)\,.
$$

So it suffices to prove (3.13) for $x\in D$ with $\delta_D(x)<\delta_0/(32)$.

Now fix $x\in D$ with $\delta_D(x)<\delta_0/(32)$ and let $Q\in\partial D$ be such that $|x-Q|=\delta_D(x)$. Let B_Q be the ball with radius $R_1/4$ so that $B_Q\subset D^c$ and $\partial B_Q\cap\partial D=\{Q\}$.

There is a $C^{1,1}$-function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0$, $\nabla\phi(0) = (0,\dots,0)$, $\|\nabla\phi\|_\infty \le \Lambda$, $|\nabla\phi(w) - \nabla\phi(z)| \le \Lambda|w-z|$, and an orthonormal coordinate system CS with its origin at Q such that

$$B(Q,R) \cap (\overline{B_Q})^c = \{z = (\widetilde{z}, z_d) \in B(0,R) \text{ in } CS : z_d > \phi(\widetilde{z})\}$$

and that x has coordinate $(\widetilde{0}, \delta_D(x))$ in this CS. Let

$$E := D \cup (B(Q, R_1/2) \setminus \overline{B_Q}),$$

$$E_1 := \{z = (\widetilde{z}, z_d) \text{ in } CS : 0 < z_d - \phi(\widetilde{z}) < \delta_0/8,\ |\widetilde{z}| < r_0/8\},$$

$E_3 := \{z \in E : |z-x| > |x-y|/2\}$, $E_{2,1} := (E \setminus (E_1 \cup E_3)) \cap D(y)$ and $E_{2,2} := E \setminus (E_1 \cup E_3 \cup D(y))$. Observe that $\delta_{(\overline{B_Q})^c}(x) = \delta_E(x) = \delta_D(x) = |x-Q|$ and $|x-y| \ge R_1 > R > ((\delta_0 + r_0)/2)$. So, by (3.6)–(3.7), $\mathrm{dist}(E_1, E_3) \ge R_1/4$ and

$$\sup_{u \in E_1,\, z \in E_3} J^a(u,z) \le c_2 \left(j^a(|x-y|) \wedge 1\right). \tag{3.14}$$

If $z \in E_{2,i}, i = 1,2$, then $|z-y| \ge |x-y| - |x-z| \ge |x-y|/2$. Thus by the same argument as the one in (3.8), if $|x-y| > \sqrt{dC_2}$, we have by (3.11)

$$\sup_{s<1/2,\, z \in E_{2,1}} p^a(s,z,y) \le c_3 \left(e^{-|x-y|^2/C_2} + \left(j^a(|x-y|) \wedge 1\right)\right) \le c_4 \left(j^M(|x-y|) \wedge 1\right). \tag{3.15}$$

If $|x-y| \le \sqrt{dC_2}$, since $R_1 \le |x-y|$, we also have

$$\begin{aligned} \sup_{s<1/2,\, z \in E_{2,1}} p^a(s,z,y) &\le C_3 \sup_{s<1/2,\, z \in E_{2,1}} \left(s^{-d/2} e^{-|z-y|^2/(C_2 s)} + \left(s^{-d/2} \wedge s J^a(z,y)\right)\right) \\ &\le c_5 \sup_{s<1/2} \left(s^{-d/2} e^{-R_1^2/(4C_2 s)} + \left(s^{-d/2} \wedge s j^a(R_1)\right)\right) \\ &\le c_6 \le c_7\, j^M(|x-y|). \end{aligned} \tag{3.16}$$

On the other hand, since $D(y) \subset E_{2,2}^c$, by Lemma 3.7,

$$\sup_{s<1/2,\, z \in E_{2,2}} p_E^a(s,z,y) \le C_{21} \sup_{s<1/2,\, z \in E_{2,2}} s j^a(|z-y|) \le c_8 \left(j^a(|x-y|) \wedge 1\right). \tag{3.17}$$

Furthermore, since $\mathrm{dist}(E_1, D(y)) \ge R_1/2$, by the Lévy system (1.2),

$$\begin{aligned} \mathbb{P}_x(X^a_{\tau^a_{E_1}} \in E_{2,1}) &\le \mathbb{P}_x(X^a_{\tau^a_{E_1}} \in D(y)) \\ &= \int_0^\infty \left(\int_{E_1} p^a_{E_1}(s,x,u) \left(\int_{D(y)} J^a(u,z) dz \right) du \right) ds \\ &= a^\alpha \int_0^\infty \left(\int_{E_1} p^a_{E_1}(s,x,u) \left(\int_{D(y)} J^1(u,z) dz \right) du \right) ds \\ &\le a^\alpha \left(\int_{\{|z| > R_1/2\}} j^1(|z|) dz \right) \int_0^\infty \left(\int_{E_1} p^a_{E_1}(s,x,u) du \right) ds \\ &\le c_9 a^\alpha \mathbb{E}_x[\tau^a_{E_1}]. \end{aligned} \tag{3.18}$$

Applying Lemmas 3.1 and 3.4, and combining (3.14)–(3.18), we obtain,

$$
\begin{aligned}
& p^a_E(1/2,x,y) \\
\le\ & \mathbb{P}_x(X^a_{\tau^a_{E_1}} \in E_{2,1}) \left(\sup_{s<1/2,\, z\in E_{2,1}} p^a_E(s,z,y) \right) + \mathbb{P}_x(X^a_{\tau^a_{E_1}} \in E_{2,2}) \left(\sup_{s<1/2,\, z\in E_{2,2}} p^a_E(s,z,y) \right) \\
& \quad + \mathbb{E}_x[\tau^a_{E_1}] \left(\sup_{u\in E_1,\, z\in E_3} J^a(u,z) \right) \\
\le\ & c_9 a^\alpha \mathbb{E}_x[\tau^a_{E_1}] \left(\sup_{s<1/2,\, z\in E_{2,1}} p^a(s,z,y) \right) + \mathbb{P}_x(X^a_{\tau^a_{E_1}} \in (\overline{B_Q})^c) \left(\sup_{s<1/2,\, z\in E_{2,2}} p^a_E(s,z,y) \right) \\
& \quad + \mathbb{E}_x[\tau^a_{E_1}] \left(\sup_{u\in E_1,\, z\in E_3} J^a(u,z) \right) \\
\le\ & c_{10}\, a^\alpha (\delta_D(x)\wedge 1) \left(|x-y|^{-d-\alpha} \wedge 1 \right).
\end{aligned}
$$

Therefore we have proved the claim (3.13). In particular, we have

$$
p^a_D(1/2,x,y) \le c_{10} a^\alpha (\delta_D(x)\wedge 1) \left(e^{-|x-y|^2/(2C_2)} + \left(|x-y|^{-d-\alpha} \wedge 1 \right) \right). \tag{3.19}
$$

By symmetry, the above inequality holds with the roles of x and y interchanged. It follows from the semigroup property that

$$
\begin{aligned}
& p^a_D(1,x,y) \\
=\ & \int_D p^a_D(1/2,x,z) p^a_D(1/2,z,y)\,dz \\
=\ & \int_{D(x)} p^a_D(1/2,x,z) p^a_D(1/2,z,y)\,dz + \int_{D\setminus D(x)} p^a_D(1/2,x,z) p^a_D(1/2,z,y)\,dz.
\end{aligned}
$$

By applying Theorem 3.5 and (3.19), and then applying Corollary 1.2 (twice), we have

$$
\begin{aligned}
p^a_D(1,x,y) &\le c_{11}\, a^\alpha (\delta_D(x)\wedge 1)(\delta_D(y)\wedge 1) \int_D p^1(2C_2^2,x,z) p^1(2C_2^2,z,y)\,dz \\
&\le c_{11}\, a^\alpha (\delta_D(x)\wedge 1)(\delta_D(y)\wedge 1) p^1(4C_2^2,x,y) \\
&\le c_{12}\, a^\alpha (\delta_D(x)\wedge 1)(\delta_D(y)\wedge 1) \left(e^{-|x-y|^2/(4C_2^3)} + \left(|x-y|^{-d-\alpha} \wedge 1 \right) \right) \\
&\le c_{13}\, a^\alpha (\delta_D(x)\wedge 1)(\delta_D(y)\wedge 1) |x-y|^{-d-\alpha} \\
&\le c_{14}\, (\delta_D(x)\wedge 1)(\delta_D(y)\wedge 1) \left(J^a(x,y) \wedge 1 \right).
\end{aligned}
$$

□

Recall that $h^a_C(t,x,y)$ is defined in (1.4).

Theorem 3.9 *Assume that $M>0$ is a constant and that D is an open set satisfying a weaker version of the uniform exterior ball condition with radius $R_1>0$. For every $T>0$, there exists a positive constant $C_{23}=C_{23}(T,R_1,\alpha,M)$ such that for all $a\in(0,M]$ and $(t,x,y)\in(0,T]\times D\times D$,*

$$
p^a_D(t,x,y) \le C_{23}\, h^a_{(4C_2^3)^{-1}}(t,x,y).
$$

Proof. Fix $T, M > 0$ and recall that $a_t = at^{(2-\alpha)/(2\alpha)} \le MT^{(2-\alpha)/(2\alpha)}$. Note that $t^{-1/2}D$ is an open set satisfying a weaker version of the uniform exterior ball condition with radius $T^{-1/2}R_1 > 0$ for every $t \in (0, T]$. Thus, by Theorem 3.8, there exists a positive constant $c_1 = c_1(T, R_1, \alpha, M)$ such that for all $t \in (0, T]$, $a \in (0, M]$ and x, y in different components of D,

$$p^{a_t}_{t^{-1/2}D}(1, x, y) \le c_1 \left(j^{a_t}(|x-y|) \wedge 1\right) \left(\delta_{t^{-1/2}D}(x) \wedge 1\right) \left(\delta_{t^{-1/2}D}(y) \wedge 1\right). \tag{3.20}$$

Thus by (2.1), (2.10) and (3.20), for every $t \le T$, $a \in (0, M]$ and x, y in different components of D,

$$\begin{aligned} p^a_D(t, x, y) &= t^{-d/2} p^{a_t}_{t^{-1/2}D}(1, t^{-1/2}x, t^{-1/2}y) \\ &\le c_1 t^{-d/2} \left(j^{a_t}(|x-y|/t^{1/2}) \wedge 1\right) \left(\delta_{t^{-1/2}D}(t^{-1/2}x) \wedge 1\right) \left(\delta_{t^{-1/2}D}(t^{-1/2}y) \wedge 1\right) \\ &= c_1 \left(t^{-d/2} \wedge tJ^a(x, y)\right) \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1\right) \left(\frac{\delta_D(y)}{\sqrt{t}} \wedge 1\right). \end{aligned}$$

Combining this result with Theorem 3.6, we have proved the theorem. □

4 Large time estimates

In this section we assume that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ which may be disconnected, and we give the proof of Theorem 1.3 (ii) and (iii).

Proof of Theorem 1.3 (ii) and (iii). Let D be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with $d \ge 1$. For each $a \ge 0$, the semigroup of $X^{a,D}$ is Hilbert-Schmidt as, by Theorem 1.1

$$\int_{D\times D} p^a_D(t, x, y)^2 dxdy = \int_D p^a_D(2t, x, x)dx \le C_1 \left((2t)^{-d/2} \wedge (a^\alpha 2t)^{-d/\alpha}\right) |D| < \infty,$$

and hence is compact. For $a \ge 0$, let $\{\lambda^{a,D}_k : k = 1, 2 \dots\}$ be the eigenvalues of $-(\Delta + a^\alpha \Delta^{\alpha/2})|_D$, arranged in increasing order and repeated according to multiplicity, and $\{\phi^{a,D}_k : k = 1, 2, \dots\}$ be the corresponding eigenfunctions normalized to have unit L^2-norm on D. Note that $\{\phi^{a,D}_k : k = 1, 2, \dots\}$ forms an orthonormal basis of $L^2(D; dx)$. It is well known that when $a > 0$, $\lambda^{a,D}_1$ is strictly positive and simple, and that $\phi^{a,D}_1$ can be chosen to be strictly positive on D. It follows from [18, Theorem 1.1(ii)] that the function $a \mapsto \lambda^{a,D}_1$ is continuous on $(0, M]$, and $\lim_{a\to 0+} \lambda^{a,D}_1 = \lambda^{0,D}_1 := \min_{1\le j\le k} \lambda^{0,D_j}_1$, where $D_1, \cdots, D_k$ are the connected components of D and λ^{0,D_j}_1 is the first Dirichlet eigenvalue of $-\Delta|_{D_j}$. Hence there is a constant $c_1 = c_1(D, \alpha, M) \ge 1$ so that

$$1/c_1 \le \lambda^{a,D}_1 \le c_1 \qquad \text{for every } a \in (0, M]. \tag{4.1}$$

Using Sobolev embedding (see [18, Example 5.1]), it can be shown that $\{\phi^{a,D}_1, a \in (0, M]\}$ is relatively compact in $L^2(D; dx)$. Hence by [18, Theorem 1.1(ii)] and the fact that each $\lambda^{a,D}_1$ is simple for $a > 0$, $a \to \phi^{a,D}_1$ is continuous in $L^2(D; dx)$ in $a \in (0, M]$. Furthermore, as $a \to 0+$, any weak limit ϕ of $\phi^{a,D}_1$ is a unit non-negative eigenfunction of $-\Delta|_D$ with eigenvalue $\lambda^{0,D}_1$. Note that such ϕ may not be strictly positive everywhere on D when D is disconnected; it is strictly positive on least one component D_j of D where $\lambda^{0,D_j}_1 = \lambda^{0,D}_1$. It follows that there is a constant $c_2 = c_2(D, \alpha, M) > 0$ so that

$$\sup_{a\in(0,M]} \int_D \delta_D(x)\phi^{a,D}_1(x)dx \le c_2. \tag{4.2}$$

Recall that $p^a_D(t,x,y)$ admits the following eigenfunction expansion

$$p^a_D(t,x,y)=\sum_{k=1}^{\infty} e^{-t\lambda_k^{a,D}}\phi_k^{a,D}(x)\phi_k^{a,D}(y) \qquad \text{for } t>0 \text{ and } x,y\in D.$$

This implies that

$$\int_{D\times D}\delta_D(x)p^a_D(t,x,y)\delta_D(y)\,dxdy=\sum_{k=1}^{\infty} e^{-t\lambda_k^{a,D}}\left(\int_D \delta_D(x)\phi_k^{a,D}(x)dx\right)^2. \tag{4.3}$$

Consequently, we have

$$\int_{D\times D}\delta_D(x)p^a_D(t,x,y)\delta_D(y)\,dxdy\le e^{-t\lambda_1^{a,D}}\int_D \delta_D(x)^2dx \tag{4.4}$$

for all $a>0$ and $t>0$. On the other hand, since

$$\phi_1^{a,D}(x)=e^{\lambda_1^{a,D}}\int_D p^a_D(1,x,y)\phi_1^{a,D}(y)dy,$$

by the upper bound estimate in Theorem 1.3(i) and (4.1)-(4.2) that there is a constant $c_3=c_3(D,\alpha,M)>0$ so that for every $a\in(0,M]$ and $x\in D$,

$$\phi_1^{a,D}(x)\le c_3\delta_D(x)\int_D\delta_D(y)\phi_1^{a,D}(y)dy\le c_2c_3\,\delta_D(x).$$

It now follows from (4.3) that for every that for every $a\in(0,M]$ and $t>0$

$$\begin{aligned}&\int_{D\times D}\delta_D(x)p^a_D(t,x,y)\delta_D(y)\,dxdy\ge e^{-t\lambda_1^{a,D}}\left(\int_D\delta_D(x)\phi_1^{a,D}(x)dx\right)^2\\ &\ge e^{-t\lambda_1^{a,D}}\left(\int_D(c_2c_3)^{-1}\phi_1^{a,D}(x)^2dx\right)^2=(c_2c_3)^{-2}\,e^{-t\lambda_1^{a,D}}.\end{aligned} \tag{4.5}$$

It suffices to prove (ii)-(iii) of Theorem 1.3 for $T\ge 3$. For $t\ge T$ and $x,y\in D$, observe that

$$p^a_D(t,x,y)=\int_{D\times D}p^a_D(1,x,z)p^a_D(t-2,z,w)p^a_D(1,w,y)dzdw. \tag{4.6}$$

Since D is bounded, we have by the upper bound estimate in Theorem 1.3(i) and (4.4) that there are constants $c_i=c_i(D,\alpha,M)>0$, $i=4,5$ so that for every $a\in(0,M]$, $t\ge T$ and $x,y\in D$,

$$p^a_D(t,x,y)\le c_4\delta_D(x)\delta_D(y)\int_{D\times D}\delta_D(z)p^a_D(t-2,z,w)\delta_D(w)dzdw\le c_5\,\delta_D(x)\delta_D(y)e^{-t\lambda_1^{a,D}}. \tag{4.7}$$

(ii) Assume first that D is connected. Since D is bounded and connected, we have by the lower bound estimate in Theorem 1.3(i) and (4.5) that there are constants $c_i=c_i(D,\alpha,M)>0$, $i=6,7$, so that for every $a\in(0,M]$, $t\ge T$ and $x,y\in D$,

$$p^a_D(t,x,y)\ge c_6\,\delta_D(x)\delta_D(y)\int_{D\times D}\delta_D(z)p^a_D(t-2,z,w)\delta_D(w)dzdw\ge c_7\,\delta_D(x)\delta_D(y)e^{-t\lambda_1^{a,D}}. \tag{4.8}$$

This combined with (4.7) proves Theorem 1.3(ii).

(iii) Now let consider the case that D is disconnected. Note that it follows from (ii) that for every $t \geq 1$, $x \in D$ and $y \in D(x)$,

$$p_D^a(t,x,y) \geq p_{D(x)}^a(t,x,y) \geq c_8 e^{-t\lambda_1^{a,D(x)}} \delta_D(x)\delta_D(y). \tag{4.9}$$

Moreover, the above inequality, (4.7) and the two-sided estimate in Theorem 1.3(i) yield that there are a constant $c_9 := c_9(D,\alpha,M) \geq 1$ such that for every $a \in (0,M]$, $t > 0$ and $x \in D$,

$$c_9^{-1}\, e^{-t\lambda_1^{a,D(x)}} \delta_D(x) \leq \mathbb{P}_x\left(\tau_{D(x)}^a > t\right) \leq c_9\, e^{-t\lambda_1^{a,D(x)}} \delta_D(x) \tag{4.10}$$

and

$$\mathbb{P}_x(\tau_D^a > t) \leq c_9\, e^{-t\lambda_1^{a,D}} \delta_D(x). \tag{4.11}$$

For $t \geq T$, $x \in D$ and $y \in D \setminus D(x)$, we have by the boundedness of D, (4.10) and the lower bound estimate in Theorem 1.3(i) that

$$\begin{aligned}
p_D^a(t,x,y) &= \mathbb{E}_x\left[p_D^a\big(t-\tau_{D(x)}^a, X_{\tau_{D(x)}^a}^a, y\big);\ \tau_{D(x)}^a < t\right] \\
&= \int_0^t \left(\int_{D(x)} p_{D(x)}^a(s,x,z)\left(\int_{D\setminus D(x)} J^a(z,w)p_D^a(t-s,w,y)dw\right)dz\right)ds \qquad (4.12)\\
&\geq c_{10}\, a^\alpha \int_0^t \left(\int_{D(x)} p_{D(x)}^a(s,x,z)dz\right)\left(\int_{D(y)} p_{D(y)}^a(t-s,w,y)dw\right)ds \\
&= c_{10}\, a^\alpha \int_0^t \mathbb{P}_x\big(\tau_{D(x)}^a > s\big)\,\mathbb{P}_y\big(\tau_{D(y)}^a > t-s\big)\,ds \\
&\geq c_{10}c_9^{-2}\, a^\alpha \int_0^t e^{-s\,\lambda_1^{a,D(x)}}\, \delta_D(x)\, e^{-(t-s)\lambda_1^{a,D(y)}}\, \delta_D(y)\, ds \qquad (4.13)\\
&\geq c_{10}c_9^{-2}\, a^\alpha t\, e^{-t(\lambda_1^{a,D(x)} \vee \lambda_1^{a,D(y)})}\, \delta_D(x)\delta_D(y). \qquad (4.14)
\end{aligned}$$

On the other hand, using (4.10)-(4.11), we have from (4.12) that for $t \geq T$, $x \in D$ and $y \in D\setminus D(x)$,

$$\begin{aligned}
p_D^a(t,x,y) &\leq c_{11}\, a^\alpha \int_0^t \left(\int_{D(x)} p_{D(x)}^a(s,x,z)dz\right)\left(\int_{D\setminus D(x)} p_D^a(t-s,w,y)dw\right)ds \\
&\leq c_{11}\, a^\alpha \int_0^t \mathbb{P}_x(\tau_{D(x)}^a > s)\,\mathbb{P}_y(\tau_D^a > t-s)ds \\
&\leq c_{11}c_9^2\, a^\alpha \int_0^t e^{-s\,\lambda_1^{a,D(x)}}\, \delta_D(x)\, e^{-(t-s)\lambda_1^{a,D}}\, \delta_D(y)ds \\
&= c_{11}c_9^2\, a^\alpha \delta_D(x)\delta_D(y) e^{-t\,\lambda_1^{a,D}} \int_0^t e^{-s\left(\lambda_1^{a,D(x)} - \lambda_1^{a,D}\right)} ds \qquad (4.15)\\
&\leq c_{11}c_9^2\, a^\alpha t\, e^{-t\lambda_1^{a,D}}\, \delta_D(x)\delta_D(y). \qquad (4.16)
\end{aligned}$$

Finally, by (4.7) and the same argument that leads to (4.15), we have that for $t \geq T$, $x, y \in D(x)$,

$$\begin{aligned}
& p_D^a(t,x,y) \\
= \ & p_{D(x)}^a(t,x,y) + \mathbb{E}_x\left[p_D^a\big(t-\tau_{D(x)}^a, X_{\tau_{D(x)}^a}^a, y\big);\ \tau_{D(x)}^a < t\right]
\end{aligned}$$

$$\leq \; c_{12}\delta_D(x)\delta_D(y)e^{-t\lambda_1^{a,D(x)}} + c_{12}\, a^\alpha \delta_D(x)\delta_D(y)e^{-t\,\lambda_1^{a,D}} \int_0^t e^{-s\left(\lambda_1^{a,D(x)}-\lambda_1^{a,D}\right)} ds \qquad (4.17)$$

$$\leq \; c_{12}\delta_D(x)\delta_D(y)e^{-t\lambda_1^{a,D(x)}} + c_{12}\, a^\alpha\, t\, \delta_D(x)\delta_D(y)e^{-t\,\lambda_1^{a,D}}. \qquad (4.18)$$

Combining this with (4.7)–(4.9), (4.14) and (4.16) completes the proof of Theorem 1.3(iii). □

Remark 4.1 In general, when passing from (4.13) to (4.14), from (4.15) to (4.16) and from (4.17) to (4.18), a factor t is needed in order to have the lower estimate and the upper estimate that is uniform in $a \in (0, M]$. Note that for D having at least two connected components, $\lambda_1^{a,D(x)} > \lambda_1^{a,D}$ for every $a > 0$. Since D is a bounded $C^{1,1}$ open set, it has only finite many connected components $D_1, \cdots, D_k$. According to [18, Theorem 1.1], as $a \to 0$, $\lambda_1^{a,D}$ converges to $\min_{1\leq j\leq k} \lambda_1^{0,D_j}$, where λ_1^{0,D_j} is the first Dirichlet eigenvalue of $-\Delta|_{D_j}$ on domain D_j. Let j_0 be such that $\lambda_1^{0,D_{j_0}} = \min_{1\leq j\leq k} \lambda_1^{0,D_j}$. Then for every $x \in D_{j_0}$, we have $\inf_{a\in(0,M]} \left(\lambda_1^{a,D(x)} - \lambda_1^{a,D}\right) = \lim_{a\to 0+} \lambda_1^{a,D(x)} - \lambda_1^{a,D} = 0$. Moreover, if D has two connected components D_1 and D_2 that are isometric to each other, then by [18, Theorem 1.1], for $x \in D_1$ and $y \in D_2$,

$$\lim_{a\to 0+} \lambda_1^{a,D(x)} = \lambda_1^{0,D_1} = \lambda_1^{0,D_2} = \lim_{a\to 0+} \lambda_1^{a,D(y)}.$$

□

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