# Stability of Dirichlet heat kernel estimates for non-local operators under Feynman-Kac perturbation

**Zhen-Qing Chen**[*] **Panki Kim**[†] and **Renming Song**[‡]

(February 2, 2012)

**Abstract**

In this paper we show that Dirichlet heat kernel estimates for a class of (not necessarily symmetric) Markov processes are stable under non-local Feynman-Kac perturbations. This class of processes includes, among others, (reflected) symmetric stable-like processes on closed $d$-sets in $\mathbb{R}^d$, killed symmetric stable processes, censored stable processes in $C^{1,1}$ open sets as well as stable processes with drifts in bounded $C^{1,1}$ open sets. These two-sided estimates are explicit involving distance functions to the boundary.

**AMS 2010 Mathematics Subject Classification**: Primary 60J35, 47G20, 60J75; Secondary 47D07, 47D08

**Keywords and phrases**: fractional Laplacian, symmetric $\alpha$-stable process, symmetric stable-like process, censored stable process, relativistic symmetric stable process, heat kernel, transition density, Dirichlet heat kernel, Feynman-Kac perturbation, Feynman-Kac transform

## 1 Introduction

Recently, sharp two-sided Dirichlet heat kernel estimates have been obtained for several classes of discontinuous processes (or non-local operators), including symmetric stable processes [7], censored stable processes [8], relativistic stable processes [9], and stable processes with drifts [10]. Although the proofs in these papers share a general road map, there are many distinct difficulties and the actual arguments are specific to the underlying processes. The main purpose of this paper is to establish a stability result for the sharp Dirichlet heat kernel estimates of a family of discontinuous processes under non-local Feynman-Kac perturbations. Here for a discontinuous Hunt process $X$, a non-local Feynman-Kac transform is given by

$$T_t f(x) = \mathbb{E}_x \Big[ \exp \Big( A_t + \sum_{s\le t} F(X_{s-}, X_s) \Big) f(X_t) \Big],$$

[*]Research partially supported by NSF Grants DMS-0906743 and DMR-1035196.

[†]This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(0409-20110087).

[‡]Research supported in part by a grant from the Simons Foundation (208236).

where $A$ is a continuous additive functional of $X$ having finite variations on each compact time interval and $F(x, y)$ is a measurable function that vanishes along the diagonal. The approach of this paper is quite robust that it applies to a class of not necessarily symmetric Markov processes which includes all the four families of processes mentioned above in bounded $C^{1,1}$ open sets.

Transformation by multiplicative functionals is one of the most important transforms for Markov processes (see, for example, [13, 22]). Non-local Feynman-Kac transforms are particular cases. They play an important role in the probabilistic as well as analytic aspect of potential theory, and also in mathematical physics. For example, it is shown in [12] that relativistic stable processes can be obtained from the symmetric $\alpha$-stable processes through Feynman-Kac transformations. We refer the reader to [14, 23] for nice accounts on Feynman-Kac semigroups of Brownian motion. In particular, it is shown in [1, 23] that under a certain Kato class condition, the integral kernel (called the heat kernel) of the Feynman-Kac semigroup of Brownian motion admits two-sided Gaussian bound estimates. In [19], sharp two-sided estimates on the densities of (local) Feynman-Kac semigroups of killed Brownain motions in $C^{1,1}$ domains were established. Non-local Feynman-Kac semigroups for symmetric stable processes and their associated quadratic forms were studied in [24, 25]. By combining some ideas from [28] with results from [11], it was proved in [26] that, under a certain Kato class condition, the heat kernel of the non-local Feynman-Kac semigroup of a symmetric stable-like process $X$ on $\mathbb{R}^d$ is comparable to that of $X$. The symmetry condition on $F(x, y)$ plays an essential role in the argument of [26]. The nonsymmetric pure jump case for stable-like processes is dealt with in [27]. For recent development in the study of non-local Feynman-Kac transforms for general symmetric Markov processes, we refer the reader to [4, 5] and the references therein. We also mention that the stability of Martin boundary under non-local Feynman-Kac perturbation is addressed in [6]. To the best of the authors knowledge, Dirichlet heat kernel estimates for (either local or non-local) Feynman-Kac semigroups of discontinuous processes is studied here for the first time. The main challenge in studying Dirichlet heat kernel estimates of Feynman-Kac semigroups is to get exact boundary decay behavior of the heat kernels. While our main interest is in the Dirichlet heat kernel estimates for Feynman-Kac semigroups, our theorem also covers the whole space case as well as "reflected" stable-like processes on subsets of $\mathbb{R}^d$. In particular, our result recovers and extends the main results of [26, 27] where $D = \mathbb{R}^d$. Even in the whole space case, our approach is different from those in [26, 27].

## 1.1 Setup and main result

In this paper we always assume that $\alpha \in (0, 2)$, $d \geq 1$, $D$ is a Borel set in $\mathbb{R}^d$. For $x \in D$, $\delta_D(x)$ denotes the Euclidean distance between $x$ and $D^c$. We use ":=" to denote a definition, which is read as "is defined to be". For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. The Euclidean distance between $x$ and $y$ is denoted as $|x - y|$.

For $\gamma \geq 0$, let

$$\psi_\gamma(t, x, y) := \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^\gamma, \qquad t > 0,\ x, y \in D.$$

Throughout this paper, $X$ is a Hunt process on $D$ with transition semigroup $\{P_t : t \geq 0\}$ that admits a jointly continuous transition density $p_D(t, x, y)$ with respect to the Lebesgue measure and that there exist $C_0 > 1$ and $\gamma \in [0, \alpha \wedge d)$ such that

$$C_0^{-1}\psi_\gamma(t,x,y)q(t,x,y) \le p_D(t,x,y) \le C_0\psi_\gamma(t,x,y)q(t,x,y) \tag{1.1}$$

for all $(t,x,y) \in (0,1] \times D \times D$, where

$$q(t,x,y) := t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}. \tag{1.2}$$

It is easy to see that under this assumption, $X$ is a Feller process satisfying the strong Feller property. It is easy to see that, by increasing the value of $C_0$ if necessary,

$$C_0^{-1} \le \int_{\mathbb{R}^d} q(t,x,y)dy \le C_0 \qquad \text{for all } (t,x) \in (0,\infty) \times \mathbb{R}^d. \tag{1.3}$$

Thus

$$\int_D p_D(t,x,y)dy \le C_0^2 \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma \qquad \text{for all } (t,x) \in (0,1] \times D. \tag{1.4}$$

Note that $X$ is not necessarily symmetric. We further assume that $X$ has a Lévy system $(N,t)$ where $N = N(x,dy)$ is a kernel given by

$$N(x,dy) = \frac{c(x,y)}{|x-y|^{d+\alpha}}dy,$$

with $c(x,y)$ a measurable function that is bounded between two positive constants on $D \times D$. That is, for any $x \in D$, any stopping time $T$ (with respect to the filtration of $X$) and any non-negative measurable function $f$ on $D \times D$ with $f(y,y) = 0$ for all $y \in D$ that is extended to be zero off $D \times D$,

$$\mathbb{E}_x\left[\sum_{s\le T} f(X_{s-}, X_s)\right] = \mathbb{E}_x\left[\int_0^T \left(\int_D f(X_s,y)\frac{c(X_s,y)}{|X_s-y|^{d+\alpha}}dy\right)ds\right]. \tag{1.5}$$

By increasing the value of $C_0$ if necessary, we may and do assume that

$$1/C_0 \le c(x,y) \le C_0 \qquad \text{for } x,y \in D. \tag{1.6}$$

Recall that an open set $D$ in $\mathbb{R}^d$ (when $d \ge 2$) is said to be a $C^{1,1}$ open set if there exist a localization radius $r_0 > 0$ and a constant $\Lambda_0 > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\phi = \phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0$, $\nabla\phi(0) = (0,\dots,0)$, $\|\nabla\phi\|_\infty \le \Lambda_0$, $|\nabla\phi(x) - \nabla\phi(w)| \le \Lambda_0|x-w|$, and an orthonormal coordinate system $y = (y_1, \cdots, y_{d-1}, y_d) := (\widetilde{y}, y_d)$ such that $B(z,r_0) \cap D = B(z,r_0) \cap \{y : y_d > \phi(\widetilde{y})\}$. We call the pair $(r_0, \Lambda_0)$ the characteristics of the $C^{1,1}$ open set $D$. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be expressed as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive.

It follows from [7, 8, 10, 11] that the following are true:

(i) the (reflected) symmetric stable-like process on any closed $d$-subset $D$ in $\mathbb{R}^d$ (see Subsection 4.1 for the definition of $d$-set) satisfies the conditions (1.1) and (1.5) with $\gamma = 0$ and $c(x,y)$ a symmetric measurable function that is bounded between two positive constants;

(ii) the killed symmetric $\alpha$-stable process on a $C^{1,1}$ open set $D$ satisfies the conditions (1.1) and (1.5) with $\gamma = \alpha/2$ and $c(x, y) = c$;

(iii) when $d \geq 2$ and $\alpha \in (1, 2)$, the killed symmetric $\alpha$-stable process with drift in a bounded $C^{1,1}$ open set $D$ satisfies the conditions (1.1) and (1.5) with $\gamma = \alpha/2$ and $c(x, y) = c$; and

(iv) when $\alpha \in (1, 2)$, the censored $\alpha$-stable process in a $C^{1,1}$ open set $D$ satisfies the conditions (1.1) and (1.5) with $\gamma = \alpha - 1$ and $c(x, y) = c$.

By a signed measure $\mu$ we mean in this paper the difference of two nonnegative $\sigma$-finite measures $\mu_1$ and $\mu_2$ in $D$. We point out that $\mu = \mu_1 - \mu_2$ may not be a signed measure in $D$ in the usual sense as both $\mu_1(D)$ and $\mu_2(D)$ may be infinite. However, there is an increasing sequence of subsets $\{F_k, k \geq 1\}$ whose union is $D$ so that $\mu_1(F_k) + \mu_2(F_k) < \infty$ for every $k \geq 1$. So when restricted to each $F_k$, $\mu$ is a finite signed measure. Consequently, the positive and negative parts of $\mu$ are well defined on each $F_k$ and hence on $D$, which will be denoted as $\mu^+$ and $\mu^-$, respectively. We use $|\mu| = \mu^+ + \mu^-$ to denote the total variation measure of $\mu$. Taking such an extended view of signed measures is desirable when one studies the correspondence between signed measures and continuous functions of finite variations or the correspondence between signed smooth measures and continuous additive functionals of finite variations for a Hunt process. For a signed measure $\mu$ on $D$ and $t > 0$, we define

$$N_\mu^{\alpha,\gamma}(t) = \sup_{x \in D} \int_0^t \int_D \left(1 \wedge \frac{\delta_D(y)}{s^{1/\alpha}}\right)^\gamma q(s, x, y) |\mu|(dy) ds.$$

**Definition 1.1** *A signed measure $\mu$ on $D$ is said to be in the Kato class $\mathbf{K}_{\alpha,\gamma}$ if $\lim_{t \downarrow 0} N_\mu^{\alpha,\gamma}(t) = 0$.*

Note that if $N_\mu^{\alpha,\gamma}(t) < \infty$ for some $t > 0$, then $|\mu|$ is a Radon measure on $D$. We say that a measurable function $g$ belongs to the Kato class $\mathbf{K}_{\alpha,\gamma}$ if $g(x)dx \in \mathbf{K}_{\alpha,\gamma}$ and we denote $N_{g(x)dx}^{\alpha,\gamma}$ by $N_g^{\alpha,\gamma}$. It is well known that any $\mu \in \mathbf{K}_{\alpha,\gamma}$ is a smooth measure in the sense of [16]. Moreover, using the fact that $X$ has a transition density function under each $\mathbb{P}_x$, one can show that the continuous additive functional $A_t^\mu$ of $X$ with Revuz measure $\mu \in \mathbf{K}_{\alpha,\gamma}$ can be defined without exceptional set, see [17, pp. 236–237] for details. Concrete conditions for $\mu \in \mathbf{K}_{\alpha,\gamma}$ are given in Proposition 4.1.

For any measurable function $F$ on $D \times D$ vanishing on the diagonal, we define

$$N_F^{\alpha,\gamma}(t) := \sup_{y \in D} \int_0^t \int_{D \times D} \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma q(s, y, z) \left(1 + \frac{|z - w| \wedge t^{1/\alpha}}{|y - z|}\right)^\gamma \frac{|F|(z, w) + |F|(w, z)}{|z - w|^{d+\alpha}} dw dz ds.$$

**Definition 1.2** *Suppose that $F$ is a measurable function on $D \times D$ vanishing on the diagonal. We say that $F$ belongs to the Kato class $\mathbf{J}_{\alpha,\gamma}$ if $F$ is bounded and $\lim_{t \downarrow 0} N_F^{\alpha,\gamma}(t) = 0$.*

It follows immediately from the two definitions above that if $F \in \mathbf{J}_{\alpha,\gamma}$, then the function

$$z \mapsto \int_D \frac{|F|(z, w) + |F|(w, z)}{|z - w|^{d+\alpha}} dw$$

belongs to $\mathbf{K}_{\alpha,\gamma}$. See Proposition 4.2 for a sufficient condition for $F \in \mathbf{J}_{\alpha,\gamma}$.

It is easy to check that if $F$ and $G$ belong to $\mathbf{J}_{\alpha,\gamma}$ and $c$ is a constant, then the functions $cF, e^F - 1, F + G$ and $FG$ all belong to $\mathbf{J}_{\alpha,\gamma}$. Throughout this paper, we will use the following notation: For any given measurable function $F$ on $D \times D$, $F_1(x, y)$ denotes the function $e^{F(x,y)} - 1$.

For any signed measure $\mu$ on $D$ and any measurable function $F$ on $D \times D$ vanishing on the diagonal, we define

$$N^{\alpha,\gamma}_{\mu,F}(t) := N^{\alpha,\gamma}_{\mu}(t) + N^{\alpha,\gamma}_{F}(t).$$

When $\mu \in \mathbf{K}_{\alpha,\gamma}$ and $F$ is a measurable function with $F_1 \in \mathbf{J}_{\alpha,\gamma}$, we put

$$A^{\mu,F}_t = A^{\mu}_t + \sum_{0<s\le t} F(X_{s-}, X_s).$$

For any nonnegative Borel function $f$ on $D$, we define

$$T^{\mu,F}_t f(x) = \mathbb{E}_x \left[\exp(A^{\mu,F}_t) f(X_t)\right], \quad t \ge 0, x \in D.$$

Then $(T^{\mu,F}_t : t \ge 0)$ is called the Feynman-Kac semigroup of $X$ corresponding to $\mu$ and $F$. The main purpose of this paper is to establish the following result. Recall that $\gamma \ge 0$ and $C_0 \ge 1$ are the constants in (1.1) and (1.6). For any bounded function $F$ on $D \times D$, we use $\|F\|_\infty$ to denote $\|F\|_{L^\infty(D\times D)}$.

**Theorem 1.3** *Let $d \ge 1$, $\alpha \in (0,2)$ and $\gamma \in [0, \alpha \wedge d)$. Suppose $X$ is a Hunt process in a Borel set $D \subset \mathbb{R}^d$ with a jointly continuous transition density $p_D(t,x,y)$ satisfying* (1.1), (1.5) *and* (1.6)*. If $\mu$ is a signed measure in $\mathbf{K}_{\alpha,\gamma}$ and $F$ is a measurable function so that $F_1 := e^F - 1 \in \mathbf{J}_{\alpha,\gamma}$, then the non-local Feynman-Kac semigroup $(T^{\mu,F}_t : t \ge 0)$ has a continuous density $q_D(t,x,y)$, and for any $T > 0$, there exists a constant $C = C(d,\alpha,\gamma,C_0,N^{\alpha,\gamma}_{\mu,F_1},\|F_1\|_\infty,T) > 0$ such that for all $(t,x,y) \in (0,T] \times D \times D$,*

$$q_D(t,x,y) \le C\psi_\gamma(t,x,y)q(t,x,y).$$

*If $\mu \in \mathbf{K}_{\alpha,\gamma}$ and $F \in \mathbf{J}_{\alpha,\gamma}$, then there exists a constant $\widetilde{C} = \widetilde{C}(d,\alpha,\gamma,C_0,N^{\alpha,\gamma}_{\mu,F},\|F\|_\infty,T) > 1$ such that for all $(t,x,y) \in (0,T] \times D \times D$,*

$$\widetilde{C}^{-1}\psi_\gamma(t,x,y)q(t,x,y) \le q_D(t,x,y) \le \widetilde{C}\psi_\gamma(t,x,y)q(t,x,y).$$

Here and in the sequel, the dependence of the constant $C$ on $N^{\alpha,\gamma}_{\mu,F_1}$ and $\|F_1\|_\infty$ means that the value of the constant $C$ depends only on a specific upper bound for the rate of the function $N^{\alpha,\gamma}_{\mu,F_1}(t)$ going to zero as $t \to 0$ and on a specific upper bound for $\|F_1\|_\infty$. When $D = \mathbb{R}^d$ and $\gamma = 0$, Theorem 1.3 in particular recovers and extends the main results of [26, 27].

## 1.2 Approach

To explain our approach, we first recall the definition of the Stieltjes exponential. If $K_t$ is a right continuous function with left limits on $\mathbb{R}_+$ with $K_0 = 1$ and $\Delta K_t := K_t - K_{t-} > -1$ for every $t > 0$, and if $K_t$ is of finite variation on each compact time interval, then the Stieltjes exponential $\mathrm{Exp}(K)_t$ of $K_t$ is the unique solution $Z_t$ of

$$Z_t = 1 + \int_{(0,t]} Z_{s-} dK_s, \quad t > 0.$$

By [22, (A4.17)],

$$\mathrm{Exp}(K)_t = e^{K^c_t} \prod_{0<s\le t} (1 + \Delta K_s), \tag{1.7}$$

where $K_t^c$ denotes the continuous part of $K_t$. Clearly $\exp(K_t) \geq \mathrm{Exp}(K)_t$ with the equality holds if and only if $K_t$ is continuous. The reason of $\mathrm{Exp}(K)_t$ being called the *Stieltjes* exponential of $K_t$ is that by [15] we have

$$\mathrm{Exp}(K)_t = 1 + \sum_{n=1}^{\infty} \int_{(0,t]} dK_{t_n} \int_{(0,t_n]} dK_{t_{n-1}} \cdots \int_{(0,t_2]} dK_{t_1}. \tag{1.8}$$

The advantage of using the Stieltjes exponential $\mathrm{Exp}(K)_t$ over the usual exponential $\exp(K_t)$ is the identity (1.8), which allows one to apply the Markov property of $X$.

Recall that $F_1(x,y) = e^{F(x,y)} - 1$. In view of (1.7), we can express $\exp(A_t^{\mu,F})$ in terms of the Stieltjes exponential:

$$\exp(A_t^{\mu,F}) = \mathrm{Exp}\Big(A^\mu + \sum_{s\leq \cdot} F_1(X_{s-}, X_s)\Big)_t \qquad \text{for } t \geq 0.$$

Applying (1.8) with $K_t := A_t^\mu + \sum_{s\leq t} F_1(X_{s-}, X_s)$ and using the Markov property of $X$, we have for any bounded $f \geq 0$ on $D$,

$$\begin{aligned} T_t^{\mu,F} f(x) &= \mathbb{E}_x\left[\exp(A_t^{\mu,F}) f(X_t)\right] = \mathbb{E}_x\left[f(X_t)\,\mathrm{Exp}\Big(A^\mu + \sum_{s\leq \cdot} F_1(X_{s-}, X_s)\Big)_t\right] \\ &= P_t f(x) + \mathbb{E}_x\left[f(X_t) \sum_{n=1}^{\infty} \int_{(0,t]} dK_{t_n} \int_{(0,t_n]} dK_{t_{n-1}} \cdots \int_{(0,t_2]} dK_{t_1}\right]. \end{aligned} \tag{1.9}$$

It can be shown that, for $\mu \in \mathbf{K}_{\alpha,\gamma}$ and measurable function $F$ with $F_1 \in \mathbf{J}_{\alpha,\gamma}$, there is some constant $T_0 > 0$ so that we can change the order of the expectation and the infinite sum when $t \leq T_0$. Hence we have for every $t \leq T_0$,

$$\begin{aligned} T_t^{\mu,F} f(x) &= P_t f(x) + \sum_{n=1}^{\infty} \mathbb{E}_x\left[f(X_t) \int_{(0,t]} dK_{t_n} \int_{(0,t_n]} dK_{t_{n-1}} \cdots \int_{(0,t_2]} dK_{t_1}\right] \\ &= P_t f(x) + \sum_{n=1}^{\infty} \mathbb{E}_x\left[\int_{(0,t]} P_{t-t_n} f(X_{t_n}) dK_{t_n} \int_{(0,t_n]} dK_{t_{n-1}} \cdots \int_{(0,t_2]} dK_{t_1}\right]. \end{aligned} \tag{1.10}$$

Note that by (1.5), for any bounded function $g$,

$$\begin{aligned} &\mathbb{E}_x\left[\int_{(0,s]} g(X_r) dK_r\right] = \mathbb{E}_x\left[\int_{(0,s]} g(X_r) dA_r^\mu + \sum_{r\leq s} g(X_r) F_1(X_{r-}, X_r)\right] \\ &= \int_0^s \int_D p_D(r,x,y) g(y) \mu(dy) dr + \mathbb{E}_x\left[\int_0^s \left(\int_D F_1(X_r, y) g(y) \frac{c(X_r, y)}{|X_r - y|^{d+\alpha}} dy\right) dr\right] \\ &= \int_0^s \int_D p_D(r,x,y) g(y) \mu(dy) dr + \int_0^s \int_D p_D(r,x,z) \left(\int_D F_1(z,y) g(y) \frac{c(z,y)}{|y-z|^{d+\alpha}} dy\right) dz dr. \end{aligned} \tag{1.11}$$

This together with (1.9) motives us to define $p^0(t,x,y) := p_D(t,x,y)$ and, for $k \geq 1$

$$p^k(t,x,y) = \int_0^t \left(\int_D p_D(s,x,z) p^{k-1}(t-s,z,y) \mu(dz)\right) ds$$

$$+\int_0^t \left( \int_{D\times D} p_D(s,x,z) \frac{c(z,w)F_1(z,w)}{|z-w|^{d+\alpha}} p^{k-1}(t-s,w,y)dzdw \right) ds. \qquad (1.12)$$

One then concludes from (1.9) that

$$T_t^{\mu,F} f(x) = \int_D q_D(t,x,y)f(y)dy,$$

where

$$q_D(t,x,y) := \sum_{k=0}^{\infty} p^k(t,x,y). \qquad (1.13)$$

We then proceed to establish the following key estimates: there exist constants $T_1 \in (0, T_0]$, $c > 0$ and $0 < \lambda < 1$ such that

$$|p^k(t,x,y)| \le (\lambda^k + ck\lambda^{k-1}) p_D(t,x,y) \qquad \text{on } (0,T_1] \times D \times D \text{ for every } k \ge 1. \qquad (1.14)$$

From this we can deduce that for every $t \in (0, T_1]$,

$$q_D(t,x,y) = \sum_{k=0}^{\infty} p^k(t,x,y) \le \left( \frac{1}{1-\lambda} + \frac{c}{(1-\lambda)^2} \right) p_D(t,x,y), \qquad (1.15)$$

and, under the assumption $F \in \mathbf{J}_{\alpha,\gamma}$, that

$$q_D(t,x,y) \ge 2^{-2(\lambda+c)} p_D(t,x,y),$$

which establish Theorem 1.3 for $t \le T_1$. The general case of $t \le T$ follows from an application of the Chapman-Kolmogorov equation.

The key to establish the estimate (1.14) are two integral forms of the 3P inequality given in Lemma 2.3 and Theorem 2.6 below. For a killed Brownian motion in a smooth domain, the following form of 3P inequality is known (see [18, 20]): for any $0 < c < a \wedge (b-a)$, there exists $M = M(a,b,c) > 0$ such that for every $0 < s < t$ and $x, y, z \in D$

$$\frac{p_a^W(t-s,x,z) p_b^W(s,z,y)}{p_a^W(t,x,y)} \le M \frac{\delta_D(z)}{\delta_D(x)} p_c^W(t-s,x,z) + M \frac{\delta_D(z)}{\delta_D(y)} p_c^W(s,y,z) \qquad (1.16)$$

where $p_c^W(t,x,y) := \psi_1(t,x,y) t^{-d/2} e^{-c|x-y|^2/t}$. For symmetric $\alpha$-stable processes in $\mathbb{R}^d$, one has the following form of 3P inequality (see [3] and (2.11) below):

$$\frac{q(s,x,z)q(t-s,z,y)}{q(t,x,y)} \le c\,(q(s,x,z) + q(t-s,z,y)) \quad \text{ for every } 0 < s < t \text{ and } x,y,z \in \mathbb{R}^d. \quad (1.17)$$

The above 3P type inequalities (1.16) and (1.17) played essential roles in establishing the heat kernel estimates in [3, 18, 20]. It seems that, for the processes we are dealing with in this paper, the above two types of 3P inequalities are not true in general. Moreover, we need a 3P type estimate on $p_D(t-s,x,z)p_D(s,w,y)/p_D(t,x,y)$, where $z \neq w$.

The rest of the paper is organized as follows. In Section 2, we prove some key inequalities, including two forms of the 3P inequality. The main estimates (1.14) and Theorem 1.3 will be established in Section 3. In the last section, we give some applications of our main results.

In this paper, we will use capital letters $\widetilde{C}, C, C_0, C_1, C_2, \ldots$ to denote constants in the statements of results, and their values will be fixed. The lower case letters $c_1, c_2, \ldots$ will denote generic constants used in proofs, whose exact values are not important and can change from one appearance to another. The labeling of the lower case constants starts anew in each proof. For two positive functions $f$ and $g$, we use the notation $f \asymp g$, which means that there are two positive constants $c_1$ and $c_2$ whose values depend only on $d, \alpha$ and $\gamma$ so that $c_1 g \le f \le c_2 g$.

## 2 3P inequalities

In this section we will establish some key inequalities which will be essential in proving Theorem 1.3. The main results of this section are Lemma 2.2, Theorem 2.4, Lemma 2.5 and Theorem 2.6. Throughout this section, $D$ is a Borel set in $\mathbb{R}^d$.

The following elementary facts will be used several times in this section.

**Lemma 2.1** *For any $s, t > 0$ and $(y, z) \in D \times D$, we have*

$$1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}} = \frac{\delta_D(y)}{t^{1/\alpha}} \left( \frac{\delta_D(z) \wedge t^{1/\alpha}}{\delta_D(y)} \right) \tag{2.1}$$

*and*

$$\left(1 \wedge \frac{\delta_D(y)}{s^{1/\alpha}}\right) \left(1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}}\right) \le 2 \left(1 + \frac{|y-z|}{s^{1/\alpha} + \delta_D(y)}\right) \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right). \tag{2.2}$$

**Proof.** The identity (2.1) is clear, so we only need to prove (2.2). Since $\delta_D(z) \le |y-z| + \delta_D(y)$, we see that

$$1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}} \le 1 \wedge \left( \left( \frac{|y-z| + \delta_D(y)}{\delta_D(y)} \right) \frac{\delta_D(y)}{t^{1/\alpha}} \right) \le \left(1 + \frac{|y-z|}{\delta_D(y)}\right) \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right).$$

Thus, applying the elementary inequality

$$\frac{a}{a+b} \le 1 \wedge \frac{a}{b} \le \frac{2a}{a+b}, \quad a, b > 0 \tag{2.3}$$

we get

$$\begin{aligned} \left(1 \wedge \frac{\delta_D(y)}{s^{1/\alpha}}\right) \left(1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}}\right) &\le \left(1 \wedge \frac{\delta_D(y)}{s^{1/\alpha}}\right) \left(1 + \frac{|y-z|}{\delta_D(y)}\right) \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right) \\ &\le 2 \left(1 + \frac{|y-z|}{s^{1/\alpha} + \delta_D(y)}\right) \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right). \end{aligned}$$

□

Using (1.2) and (2.3), we get that

$$\frac{t}{(t^{1/\alpha} + |x-y|)^{d+\alpha}} \le q(t, x, y) \le 2^{d+\alpha} \frac{t}{(t^{1/\alpha} + |x-y|)^{d+\alpha}}. \tag{2.4}$$

**Lemma 2.2** *For any $\gamma \in [0, 2\alpha)$, there exists a constant $C_1 := C_1(d, \alpha, \gamma) > 1$ such that for all $(t, y, z) \in (0, \infty) \times D \times D$,*

$$\left(1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}}\right)^{\gamma} \int_0^{t/2} \psi_\gamma(s, z, y) q(s, z, y) ds \leq C_1 \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\gamma} \int_0^{t/2} \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^{\gamma} q(s, z, y) ds. \tag{2.5}$$

**Proof.** The inequality holds trivially when $\gamma = 0$ with $C_1 = 1$ so for the rest of the proof, we assume $\gamma \in (0, 2\alpha)$. The inequality (2.5) is obvious if $\delta_D(y) \geq t^{1/\alpha}$ or $\delta_D(z) \leq 2\delta_D(y)$. So we will assume $\delta_D(y) < t^{1/\alpha} \wedge (\delta_D(z)/2)$ throughout this proof. Note that in this case,

$$|z - y| \geq \delta_D(z) - \delta_D(y) \geq \frac{\delta_D(z)}{2} \geq \delta_D(y). \tag{2.6}$$

By (2.2), we have

$$\begin{aligned} &\int_{(t/2)\wedge|z-y|^\alpha}^{t/2} \left(1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}}\right)^{\gamma} \psi_\gamma(s, z, y) q(s, z, y) ds \\ \leq\ & 2^{2\gamma} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\gamma} \int_{(t/2)\wedge|z-y|^\alpha}^{t/2} \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^{\gamma} q(s, z, y) ds, \end{aligned} \tag{2.7}$$

while by (2.1)

$$\begin{aligned} &\left(1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}}\right)^{\gamma} \int_0^{(t/2)\wedge|z-y|^\alpha} \psi_\gamma(s, z, y) q(s, z, y) ds \\ \leq & \left(\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\gamma} \left(\frac{\delta_D(z) \wedge t^{1/\alpha}}{\delta_D(y)}\right)^{\gamma} \int_0^{(t/2)\wedge|z-y|^\alpha} \left(1 \wedge \frac{\delta_D(y)}{s^{1/\alpha}}\right)^{\gamma} q(s, z, y) ds. \end{aligned} \tag{2.8}$$

In view of (2.4), (2.6) and (2.8),

$$\begin{aligned} &\int_0^{(t/2)\wedge|z-y|^\alpha} \left(1 \wedge \frac{\delta_D(y)}{s^{1/\alpha}}\right)^{\gamma} q(s, z, y) ds \\ \asymp & \int_0^{(t/2)\wedge\delta_D(y)^\alpha} \frac{s}{|z-y|^{d+\alpha}} ds + \int_{(t/2)\wedge\delta_D(y)^\alpha}^{(t/2)\wedge|z-y|^\alpha} \left(\frac{\delta_D(y)}{s^{1/\alpha}}\right)^{\gamma} \frac{s}{|z-y|^{d+\alpha}} ds \\ \asymp & \frac{1}{|z-y|^{d+\alpha}} \Big( ((t/2) \wedge \delta_D(y)^\alpha)^2 + \delta_D(y)^\gamma \Big( ((t/2) \wedge |z-y|^\alpha)^{2-\gamma/\alpha} - ((t/2) \wedge \delta_D(y)^\alpha)^{2-\gamma/\alpha} \Big) \Big) \\ \asymp & \frac{1}{|z-y|^{d+\alpha}} \Big( ((t/2) \wedge \delta_D(y)^\alpha)^2 \\ & + (\delta_D(y) \wedge (t/2)^{1/\alpha})^\gamma \Big( ((t/2) \wedge |z-y|^\alpha)^{2-\gamma/\alpha} - ((t/2) \wedge \delta_D(y)^\alpha)^{2-\gamma/\alpha} \Big) \Big) \\ \asymp & \frac{(\delta_D(y) \wedge (t/2)^{1/\alpha})^\gamma \ ((t/2) \wedge |z-y|^\alpha)^{2-\gamma/\alpha}}{|z-y|^{d+\alpha}} \asymp \frac{\delta_D(y)^\gamma \ ((t/2) \wedge |z-y|^\alpha)^{2-\gamma/\alpha}}{|z-y|^{d+\alpha}}. \end{aligned} \tag{2.9}$$

On the other hand, using (2.6) we have

$$\int_0^{(t/2)\wedge|z-y|^\alpha} \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^{\gamma} q(s, z, y) ds$$

$$
\begin{aligned}
&\asymp \int_0^{(t/2)\wedge(\delta_D(z)/2)^\alpha} \frac{s}{|z-y|^{d+\alpha}}ds + \int_{(t/2)\wedge(\delta_D(z)/2)^\alpha}^{(t/2)\wedge|z-y|^\alpha} \left(\frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma \frac{s}{|z-y|^{d+\alpha}}ds \\
&\asymp \frac{1}{|z-y|^{d+\alpha}} \left( \left(\frac{t}{2} \wedge \left(\frac{\delta_D(z)}{2}\right)^\alpha\right)^2 + \delta_D(z)^\gamma \left( \left(\frac{t}{2} \wedge |z-y|^\alpha\right)^{2-\gamma/\alpha} - \left(\frac{t}{2} \wedge \left(\frac{\delta_D(z)}{2}\right)^\alpha\right)^{2-\gamma/\alpha} \right) \right) \\
&\geq \frac{1}{|z-y|^{d+\alpha}} \left( \left(\frac{t}{2} \wedge \left(\frac{\delta_D(z)}{2}\right)^\alpha\right)^2 \right. \\
&\quad \left. + \left(\frac{\delta_D(z)}{2} \wedge \left(\frac{t}{2}\right)^{1/\alpha}\right)^\gamma \left( \left(\frac{t}{2} \wedge |z-y|^\alpha\right)^{2-\gamma/\alpha} - \left(\frac{t}{2} \wedge \left(\frac{\delta_D(z)}{2}\right)^\alpha\right)^{2-\gamma/\alpha} \right) \right) \\
&\asymp \frac{(\delta_D(z) \wedge t^{1/\alpha})^\gamma \, ((t/2) \wedge |z-y|^\alpha)^{2-\gamma/\alpha}}{|z-y|^{d+\alpha}}.
\end{aligned} \tag{2.10}
$$

One then deduces from (2.8)–(2.10) and the assumption $\delta_D(y) \leq t^{1/\alpha}$ that

$$
\begin{aligned}
&\left(1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}}\right)^\gamma \int_0^{(t/2)\wedge|z-y|^\alpha} \psi_\gamma(s,z,y) q(s,z,y) ds \\
\leq\ & c_1 \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^\gamma (\delta_D(z) \wedge t^{1/\alpha})^\gamma \frac{((t/2) \wedge |z-y|^\alpha)^{2-\gamma/\alpha}}{|z-y|^{d+\alpha}} \\
\leq\ & c_2 \int_0^{(t/2)\wedge|z-y|^\alpha} \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma q(s,z,y) ds.
\end{aligned}
$$

This combining with (2.7) establishes the inequality (2.5). □

It follows from (2.3) and (2.4) that for every $0 < s < t$, and $x, y, z \in \mathbb{R}^d$,

$$
\begin{aligned}
& \frac{q(s,x,z) q(t-s,z,y)}{q(t,x,y)} \\
\leq\ & 4^{d+\alpha} \frac{s(t-s)}{t} \left( \frac{t^{1/\alpha} + |x-y|}{(s^{1/\alpha} + |x-z|)((t-s)^{1/\alpha} + |y-z|)} \right)^{d+\alpha} \\
\leq\ & 4^{d+\alpha} (s \wedge (t-s)) \left( \frac{(s + (t-s))^{1/\alpha} + |x-z| + |y-z|}{(s^{1/\alpha} + |x-z|)((t-s)^{1/\alpha} + |y-z|)} \right)^{d+\alpha} \\
\leq\ & 2^{(d+\alpha)(3+1/\alpha)} (s \wedge (t-s)) \left( \frac{1}{(s^{1/\alpha} + |x-z|)^{d+\alpha}} + \frac{1}{((t-s)^{1/\alpha} + |y-z|)^{d+\alpha}} \right) \\
\leq\ & 2^{(d+\alpha)(3+1/\alpha)} \left( q(s,x,z) + q(t-s,z,y) \right).
\end{aligned} \tag{2.11}
$$

(See also [3].)

Now we are ready to prove one form of the 3P inequality. Note that the right hand side of the 3P inequality below has the term $q(s,x,z) + q(s,z,y)$ rather than $q(t-s,x,z) + q(s,z,y)$.

**Lemma 2.3 (3P inequality)** *For every $\gamma \in [0,\alpha)$, there exists a constant $C_2 := C_2(d,\alpha,\gamma) > 0$ such that for all $(t,x,y,z) \in (0,\infty) \times D \times D \times D$,*

$$
\int_0^t \frac{\psi_\gamma(t-s,x,z) q(t-s,x,z) \psi_\gamma(s,z,y) q(s,z,y)}{\psi_\gamma(t,x,y) q(t,x,y)} ds \leq C_2 \int_0^t \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma (q(s,x,z) + q(s,z,y)) ds.
$$

**Proof.** When $\gamma = 0$, the desired inequality follows from (2.11) with $C_2 = 2^{(d+\alpha)(3+1/\alpha)}$. So for the rest of the proof, we assume $\gamma \in (0, \alpha)$. Let

$$J(t,x,y,z) := \int_0^t \psi_\gamma(t-s,x,z)q(t-s,x,z)\psi_\gamma(s,z,y)q(s,z,y)ds.$$

Since

$$\begin{aligned} J(t,x,y,z) &\leq c_1 \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma \left(1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}}\right)^\gamma q(t,x,z) \int_0^{t/2} \psi_\gamma(s,z,y)q(s,z,y)ds \\ &\quad + c_1 \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^\gamma \left(1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}}\right)^\gamma q(t,z,y) \int_{t/2}^t \psi_\gamma(t-s,x,z)q(t-s,x,z)ds, \end{aligned}$$

we have by Lemma 2.2 that

$$\begin{aligned} J(t,x,y,z) &\leq c_2\psi_\gamma(t,x,y) \int_0^{t/2} \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma q(t-s,x,z)q(s,z,y)ds \\ &\quad + c_2\psi_\gamma(t,x,y) \int_{t/2}^t \left(1 \wedge \frac{\delta_D(z)}{(t-s)^{1/\alpha}}\right)^\gamma q(t-s,x,z)q(s,z,y)ds. \end{aligned}$$

It then follows from (2.11) that

$$\begin{aligned} J(t,x,y,z) &\leq c_3\psi_\gamma(t,x,y)q(t,x,y) \int_0^{t/2} \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma (q(t-s,x,z)+q(s,z,y))ds \\ &\quad + c_3\psi_\gamma(t,x,y)q(t,x,y) \int_{t/2}^t \left(1 \wedge \frac{\delta_D(z)}{(t-s)^{1/\alpha}}\right)^\gamma (q(t-s,x,z)+q(s,z,y))ds \\ &\leq c_4\psi_\gamma(t,x,y)q(t,x,y) \int_0^t \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma (q(s,z,y)+q(s,x,z))ds. \end{aligned}$$

Here in the last inequality, we used the fact that

$$\int_0^{t/2} \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma q(t-s,x,z)ds \leq c_5 \int_{t/2}^t \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma q(s,x,z)ds$$

and

$$\int_{t/2}^t \left(1 \wedge \frac{\delta_D(z)}{(t-s)^{1/\alpha}}\right)^\gamma q(s,z,y)ds \leq c_5 \int_{t/2}^t \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma q(s,z,y)ds.$$

The above two inequalities can be easily verified by using the facts that $q(s,x,y) \asymp q(t,x,y)$ for $s \in [t/2, t]$ and that

$$\int_0^{t/2} \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma ds \leq \frac{\alpha}{\alpha-\gamma} 2^{\gamma/\alpha-1} t \left(1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}}\right)^\gamma \leq c_6 \int_{t/2}^t \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma ds,$$

which follows easily from the assumption $\gamma \in (0, \alpha)$ by a direct calculation. This completes the proof of the lemma. □

The above 3P inequality immediately implies the following theorem, which will be used later.

**Theorem 2.4** *For every $\gamma \in [0,\alpha)$, there exists a constant $C_3 = C_3(d,\alpha,\gamma) > 0$ such that for any measure $\mu$ on $D$ and any $(t,x,y) \in (0,\infty) \times D \times D$,*

$$\begin{aligned} &\int_0^t \int_D \psi_\gamma(t-s,x,z)\psi_\gamma(s,z,y)q(t-s,x,z)q(s,z,y)\mu(dz)ds \\ \le\;& C_3\, \psi_\gamma(t,x,y)q(t,x,y) \sup_{u\in D} \int_0^t \int_D \left(1 \wedge \frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma q(s,u,z)\mu(dz)ds. \end{aligned}$$

The results of the remainder of this section are geared towards dealing with the discontinuous part of $A^{\mu,F}$.

**Lemma 2.5** *For every $\gamma \in [0,2\alpha)$, there exists a constant $C_4 := C_4(d,\alpha,\gamma) > 1$ such that for all $(t,y,z,w) \in (0,\infty) \times D \times D \times D$,*

$$\begin{aligned} &\left(1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}}\right)^\gamma \int_0^{t/2} \psi_\gamma(s,w,y)q(s,w,y)ds \\ \le& C_4 \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^\gamma \left(1 + \frac{|y-z| \wedge |z-w| \wedge t^{1/\alpha}}{|y-w|}\right)^\gamma \int_0^{t/2} \left(1 \wedge \frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma q(s,w,y)ds. \end{aligned} \tag{2.12}$$

**Proof.** The desired inequality holds trivially for $\gamma = 0$ with $C_4 = 1$ so for the rest of the proof we assume $\gamma \in (0,2\alpha)$. The inequality (2.12) is obvious if $\delta_D(y) \ge t^{1/\alpha}$ or $\delta_D(z) \le 2\delta_D(y)$, so we will assume $\delta_D(y) < t^{1/\alpha} \wedge (\delta_D(z)/2)$ in the remainder of this proof. Note that in this case

$$|y-z| \ge \delta_D(z) - \delta_D(y) \ge \frac{\delta_D(z)}{2} \ge \delta_D(y), \tag{2.13}$$

By (2.1), (2.3) and our assumption $\delta_D(y) < t^{1/\alpha}$, we have that

$$\left(1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}}\right)\left(1 \wedge \frac{\delta_D(y)}{s^{1/\alpha}}\right) \le 2\,\frac{\delta_D(y)}{t^{1/\alpha}}\left(\frac{\delta_D(z) \wedge t^{1/\alpha}}{s^{1/\alpha} + \delta_D(y)}\right) = 2\left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)\left(\frac{\delta_D(z) \wedge t^{1/\alpha}}{s^{1/\alpha} + \delta_D(y)}\right). \tag{2.14}$$

When $s \ge |y-w|^\alpha$, by (2.13),

$$\frac{\delta_D(z) \wedge t^{1/\alpha}}{s^{1/\alpha} + \delta_D(y)} \le 2\,\frac{|y-z| \wedge t^{1/\alpha}}{|y-w|} \le 2\left(1 + \frac{|y-z| \wedge |z-w| \wedge t^{1/\alpha}}{|y-w|}\right),$$

where the last inequality is due to the fact $|y-z| \le |y-w| + (|y-z| \wedge |z-w|)$. This together with (2.14) implies that

$$\begin{aligned} &\left(1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}}\right)^\gamma \int_{(t/2)\wedge|y-w|^\alpha}^{t/2} \left(1 \wedge \frac{\delta_D(y)}{s^{1/\alpha}}\right)^\gamma \left(1 \wedge \frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma q(s,w,y)ds \\ \le& 4^\gamma \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^\gamma \left(1 + \frac{|y-z| \wedge |z-w| \wedge t^{1/\alpha}}{|y-w|}\right)^\gamma \int_{(t/2)\wedge|y-w|^\alpha}^{t/2} \left(1 \wedge \frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma q(s,w,y)ds. \end{aligned} \tag{2.15}$$

On the other hand, by (2.13),

$$\int_0^{(t/2)\wedge|y-w|^\alpha} \left(\frac{\delta_D(z) \wedge t^{1/\alpha}}{s^{1/\alpha} + \delta_D(y)}\right)^\gamma \left(1 \wedge \frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma q(s,w,y)ds$$

$$\begin{aligned}
&\le 2^\gamma \int_0^{(t/2)\wedge|y-w|^\alpha} \left(\frac{|y-z|\wedge t^{1/\alpha}}{s^{1/\alpha}}\right)^\gamma \left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma \frac{s}{|y-w|^{d+\alpha}} ds \\
&= c_1 \frac{\left(|y-z|\wedge t^{1/\alpha}\right)^\gamma}{|y-w|^{d+\alpha}} \int_0^{(t/2)\wedge|y-w|^\alpha} s^{1-\gamma/\alpha}\left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma ds. \qquad (2.16)
\end{aligned}$$

We claim that

$$\int_0^{(t/2)\wedge|y-w|^\alpha} s^{1-\gamma/\alpha}\left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma ds \asymp \left(\frac{t}{2}\wedge|y-w|^\alpha\right)^{-\gamma/\alpha} \int_0^{(t/2)\wedge|y-w|^\alpha} s\left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma ds. \qquad (2.17)$$

The case $\delta_D(w) > (t/2)^{1/\alpha}$ is clear. If $\delta_D(w) \le |y-w|\wedge(t/2)^{1/\alpha}$,

$$\begin{aligned}
&\int_0^{(t/2)\wedge|y-w|^\alpha} s^{1-\gamma/\alpha}\left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma ds \\
&= \int_0^{\delta_D(w)^\alpha} s^{1-\gamma/\alpha} ds + \delta_D(w)^\gamma \int_{\delta_D(w)^\alpha}^{(t/2)\wedge|y-w|^\alpha} s^{1-2\gamma/\alpha} ds \\
&\asymp \delta_D(w)^{2\alpha-\gamma} + \delta_D(w)^\gamma\left(\left(\frac{t}{2}\wedge|y-w|^\alpha\right)^{2-2\gamma/\alpha} - \delta_D(w)^{2(\alpha-\gamma)}\right) \\
&\asymp \delta_D(w)^\gamma \left(\frac{t}{2}\wedge|y-w|^\alpha\right)^{2-2\gamma/\alpha} \\
&= \left(\frac{t}{2}\wedge|y-w|^\alpha\right)^{-\gamma/\alpha} (\delta_D(w))^\gamma \left(\frac{t}{2}\wedge|y-w|^\alpha\right)^{2-\gamma/\alpha} \\
&\asymp \left(\frac{t}{2}\wedge|y-w|^\alpha\right)^{-\gamma/\alpha} \int_0^{(t/2)\wedge|y-w|^\alpha} s\left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma ds.
\end{aligned}$$

The remaining case $|y-w| < \delta_D(w) \le (t/2)^{1/\alpha}$ is simpler. Thus we have proved the claim (2.17). Now by (2.16) and (2.17),

$$\begin{aligned}
&\int_0^{(t/2)\wedge|y-w|^\alpha} \left(\frac{\delta_D(z)\wedge t^{1/\alpha}}{s^{1/\alpha}+\delta_D(y)}\right)^\gamma \left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma q(s,w,y) ds \\
&\le c_2 \left(\frac{|y-z|\wedge t^{1/\alpha}}{|y-w|\wedge t^{1/\alpha}}\right)^\gamma \int_0^{(t/2)\wedge|y-w|^\alpha} \left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma \frac{s}{|y-w|^{d+\alpha}} ds \\
&\le c_2 \left(1+\frac{|y-z|\wedge t^{1/\alpha}}{|y-w|}\right)^\gamma \int_0^{(t/2)\wedge|y-w|^\alpha} \left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma q(s,w,y) ds \\
&\le 2c_2 \left(1+\frac{|y-z|\wedge|z-w|\wedge t^{1/\alpha}}{|y-w|}\right)^\gamma \int_0^{(t/2)\wedge|y-w|^\alpha} \left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma q(s,w,y) ds.
\end{aligned}$$

Here again the last inequality is due to the fact that $|y-z| \le |y-w| + (|y-z|\wedge|z-w|)$. This together with (2.14) and (2.15) establishes the inequality (2.12). □

In the remainder of this section. we use the following notation: For any $(x,y)\in D\times D$,

$$V_{x,y} := \{(z,w)\in D\times D : |x-y| \ge 4(|y-w|\wedge|x-z|)\},$$

$$U_{x,y} \quad := \quad (D\times D)\setminus V_{x,y}.$$

Recall that, for any bounded function $F$ on $D\times D$ we use $\|F\|_\infty$ to denote $\|F\|_{L^\infty(D\times D)}$.

Now we are ready to prove the following generalized 3P inequality.

**Theorem 2.6 (Generalized 3P inequality)** *For every $\gamma\in[0,\alpha\wedge d)$, there exists a constant $C_5:=C_5(\alpha,\gamma,d)>0$ such that for any nonnegative bounded function $F(x,y)$ on $D\times D$, the following are true for $(t,x,y)\in(0,\infty)\times D\times D$.*
*(a) If $|x-y|\le t^{1/\alpha}$, then*

$$\begin{aligned}
&\int_0^t\int_{D\times D}\frac{\psi_\gamma(t-s,x,z)q(t-s,x,z)\psi_\gamma(s,w,y)q(s,w,y)}{\psi_\gamma(t,x,y)q(t,x,y)}\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdwds\\
\le\ & C_5\int_0^t\int_{D\times D}\left(1\wedge\frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma q(s,x,z)\left(1+\frac{|z-w|\wedge t^{1/\alpha}}{|x-z|}\right)^\gamma\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdwds\\
&+C_5\int_0^t\int_{D\times D}\left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma q(s,y,w)\left(1+\frac{|z-w|\wedge t^{1/\alpha}}{|y-w|}\right)^\gamma\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdwds.
\end{aligned}$$

*(b) If $|x-y|>t^{1/\alpha}$, then*

$$\begin{aligned}
&\int_0^t\int_{U_{x,y}}\frac{\psi_\gamma(t-s,x,z)q(t-s,x,z)\psi_\gamma(s,w,y)q(s,w,y)}{\psi_\gamma(t,x,y)q(t,x,y)}\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdwds\\
\le\ & C_5\int_0^t\int_{U_{x,y}}\left(1\wedge\frac{\delta_D(z)}{s^{1/\alpha}}\right)^\gamma q(s,x,z)\left(1+\frac{|z-w|\wedge t^{1/\alpha}}{|x-z|}\right)^\gamma\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdwds\\
&+C_5\int_0^t\int_{U_{x,y}}\left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma q(s,y,w)\left(1+\frac{|z-w|\wedge t^{1/\alpha}}{|y-w|}\right)^\gamma\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdwds.
\end{aligned}$$

*(c) If $|x-y|>t^{1/\alpha}$, then*

$$\int_0^t\int_{V_{x,y}}\frac{\psi_\gamma(t-s,x,z)q(t-s,x,z)\psi_\gamma(s,w,y)q(s,w,y)}{\psi_\gamma(t,x,y)q(t,x,y)}\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdwds\ \le\ C_5\|F\|_\infty.$$

**Proof.** By Lemma 2.5, we get that

$$\begin{aligned}
&\int_0^t\int_{D\times D}\frac{\psi_\gamma(t-s,x,z)q(t-s,x,z)\psi_\gamma(s,w,y)q(s,w,y)}{\psi_\gamma(t,x,y)}\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdwds\\
\le& c_1\int_{D\times D}\int_0^{t/2}\left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma q(s,w,y)q(t-s,x,z)\left(1+\frac{|z-w|\wedge t^{1/\alpha}}{|y-w|}\right)^\gamma ds\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdw\\
&+c_1\int_{D\times D}\int_{t/2}^t\left(1\wedge\frac{\delta_D(z)}{(t-s)^{1/\alpha}}\right)^\gamma q(s,w,y)q(t-s,x,z)\left(1+\frac{|z-w|\wedge t^{1/\alpha}}{|x-z|}\right)^\gamma ds\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdw.
\end{aligned}\tag{2.18}$$

If $|x-y|\le t^{1/\alpha}$ and $s\in(0,t/2]$, we have $q(t-s,x,z)\le 2^{d/\alpha}q(t,x,y)$, and if $|x-y|\le t^{1/\alpha}$ and $s\in(t/2,t]$, we have $q(s,w,y)\le 2^{d/\alpha}q(t,x,y)$. Thus (a) follows immediately from (2.18).

In the remainder of this proof, we fix $(t,x,y)\in(0,\infty)\times D\times D$ with $|x-y|>t^{1/\alpha}$. Let

$$\begin{aligned}U_1 &:= \{(z,w)\in D\times D: |y-w|>4^{-1}|x-y|, |y-w|\ge|x-z|\},\\ U_2 &:= \{(z,w)\in D\times D: |x-z|>4^{-1}|x-y|\}.\end{aligned}$$

Since $q(t-s,x,z)\le 4^{d+\alpha}q(t,x,y)$ for $(s,z,w)\in(0,t)\times U_2$, by Lemma 2.5, we have

$$\begin{aligned}&\int_0^{t/2}\int_{U_2}\frac{\psi_\gamma(t-s,x,z)q(t-s,x,z)\psi_\gamma(s,w,y)q(s,w,y)}{\psi_\gamma(t,x,y)}\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdwds\\ \le& c_2\int_{U_2}\int_0^{t/2}\left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma q(s,w,y)q(t-s,x,z)\left(1+\frac{|z-w|\wedge t^{1/\alpha}}{|y-w|}\right)^\gamma ds\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdw\\ \le& c_3 q(t,x,y)\int_{U_2}\int_0^{t/2}\left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma q(s,w,y)\left(1+\frac{|z-w|\wedge t^{1/\alpha}}{|y-w|}\right)^\gamma ds\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdw \quad (2.19)\end{aligned}$$

and, similarly

$$\begin{aligned}&\int_{t/2}^{t}\int_{U_1}\frac{\psi_\gamma(t-s,x,z)q(t-s,x,z)\psi_\gamma(s,w,y)q(s,w,y)}{\psi_\gamma(t,x,y)}\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdwds\\ \le& c_4\int_{U_1}\int_{t/2}^{t}\left(1\wedge\frac{\delta_D(z)}{(t-s)^{1/\alpha}}\right)^\gamma q(s,w,y)q(t-s,x,z)\left(1+\frac{|z-w|\wedge t^{1/\alpha}}{|x-z|}\right)^\gamma ds\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdw\\ \le& c_5 q(t,x,y)\int_{U_1}\int_{t/2}^{t}\left(1\wedge\frac{\delta_D(z)}{(t-s)^{1/\alpha}}\right)^\gamma q(t-s,x,z)\left(1+\frac{|z-w|\wedge t^{1/\alpha}}{|x-z|}\right)^\gamma ds\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdw.\end{aligned}\quad (2.20)$$

On the other hand, we observe that, since $q(s,w,y)\le 4^{d+\alpha}q(t,x,y)$ for $(s,z,w)\in(0,t/2]\times U_1$,

$$\begin{aligned}&\int_0^{t/2}\int_{U_1}\psi_\gamma(t-s,x,z)q(t-s,x,z)\psi_\gamma(s,w,y)q(s,w,y)\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdwds\\ \le& c_6\psi_\gamma(t,x,z)q(t,x,y)\int_{U_1}q(t,x,z)\int_0^{t/2}\psi_\gamma(s,w,y)ds\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdw.\end{aligned}$$

Now, applying the inequality

$$\int_0^{t/2}\psi_\gamma(s,w,y)ds\le\int_0^{t/2}\left(1\wedge\frac{\delta_D(y)}{s^{1/\alpha}}\right)^\gamma ds\le\frac{\alpha}{\alpha-\gamma}2^{\gamma/\alpha-1}t\left(1\wedge\frac{\delta_D(y)}{t^{1/\alpha}}\right)^\gamma,$$

we get

$$\begin{aligned}&\int_0^{t/2}\int_{U_1}\psi_\gamma(t-s,x,z)q(t-s,x,z)\psi_\gamma(s,w,y)q(s,w,y)\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdwds\\ \le& c_7\psi_\gamma(t,x,y)q(t,x,y)\int_{U_1}q(t,x,z)t\left(1\wedge\frac{\delta_D(z)}{t^{1/\alpha}}\right)^\gamma\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdw\\ \le& c_8\psi_\gamma(t,x,y)q(t,x,y)\int_{U_1}\int_0^{t/2}q(t-s,x,z)\left(1\wedge\frac{\delta_D(z)}{(t-s)^{1/\alpha}}\right)^\gamma\frac{F(z,w)}{|z-w|^{d+\alpha}}dsdzdw.\end{aligned}\quad (2.21)$$

Similarly

$$\begin{aligned}
&\int_{t/2}^{t}\int_{U_2}\psi_\gamma(t-s,x,z)q(t-s,x,z)\psi_\gamma(s,w,y)q(s,w,y)\frac{F(z,w)}{|z-w|^{d+\alpha}}dzdwds\\
\le& c_9\psi_\gamma(t,x,y)q(t,x,y)\int_{U_2}\int_{t/2}^{t}\left(1\wedge\frac{\delta_D(w)}{s^{1/\alpha}}\right)^\gamma q(s,w,y)\frac{F(z,w)}{|z-w|^{d+\alpha}}dsdzdw. \qquad (2.22)
\end{aligned}$$

Since $U_{x,y}=U_1\cup U_2$, from (2.19)–(2.22), we know that (b) is true.

Note that for $(z,w)\in V_{x,y}$, we have $|z-w|\ge|x-y|-(|x-z|+|y-w|)\ge 2^{-1}|x-y|$. Thus, by Lemma 2.5 and (1.3), it is easy to see that

$$\begin{aligned}
&\int_0^t\int_{V_{x,y}}\frac{\psi_\gamma(t-s,x,z)q(t-s,x,z)\psi_\gamma(s,w,y)q(s,w,y)}{\psi_\gamma(t,x,y)}\frac{F(z,w)}{q(t,x,y)|z-w|^{d+\alpha}}dzdwds\\
\le& c_{10}\|F\|_\infty t^{-1}\int_{V_{x,y}}\int_0^{t/2}q(s,w,y)q(t-s,x,z)\left(1+\frac{t^{1/\alpha}}{|y-w|}\right)^\gamma dsdzdw\\
&+c_{10}\|F\|_\infty t^{-1}\int_{V_{x,y}}\int_{t/2}^{t}q(s,w,y)q(t-s,x,z)\left(1+\frac{t^{1/\alpha}}{|x-z|}\right)^\gamma dsdzdw\\
\le& c_{11}\|F\|_\infty t^{-1}\int_0^t\left(\int_D q(s,w,y)\left(1+\frac{t^{1/\alpha}}{|y-w|}\right)^\gamma dw+\int_D q(s,x,z)\left(1+\frac{t^{1/\alpha}}{|x-z|}\right)^\gamma dz\right)ds.
\end{aligned}$$

Since, using $\gamma\in(0,\alpha\wedge d)$,

$$\begin{aligned}
&\int_0^t\left(\int_D q(s,w,y)\left(1+\frac{t^{1/\alpha}}{|y-w|}\right)^\gamma dw+\int_D q(s,x,z)\left(1+\frac{t^{1/\alpha}}{|x-z|}\right)^\gamma dz\right)ds\\
\le\ & 2^{d+\alpha+1}\int_0^t\int_{\mathbb{R}^d}\frac{s}{(s^{1/\alpha}+|w|)^{d+\alpha}}\left(\frac{t^{1/\alpha}}{|w|}\right)^\gamma dwds\\
=\ & c_{12}\left(\int_0^\infty\frac{u^{d-1-\gamma}du}{(1+u)^{d+\alpha}}\right)t^{\gamma/\alpha}\int_0^t s^{-\gamma/\alpha}ds\le c_{13}t,
\end{aligned}$$

(c) follows immediately. □

# 3 Heat kernel estimates

In this section we give the proof of our main result, Theorem 1.3. Throughout this section, we fix $\gamma\in[0,\alpha\wedge d)$. Recall the definition of $p^k(t,x,y)$ given by (1.12).

Using (1.1), (1.6), Theorems 2.4 and 2.6, we can choose a constant

$$M=M(\alpha,\gamma,d,C_0)>\frac{\alpha}{\alpha-\gamma}2^{2\gamma/\alpha+d+\alpha+1}C_0^4(C_1\vee C_4) \qquad (3.1)$$

such that for any $\mu$ in $\mathbf{K}_{\alpha,\gamma}$, any measurable function $F$ with $F_1=e^F-1\in\mathbf{J}_{\alpha,\gamma}$ and any $(t,x,y)\in(0,1]\times D\times D$,

$$\int_0^t\int_D\psi_\gamma(t-s,x,z)\psi_\gamma(s,z,y)q(t-s,x,z)q(s,z,y)|\mu|(dz)ds$$

$$\leq \quad M\, p_D(t,x,y) N^{\alpha,\gamma}_{\mu}(t), \tag{3.2}$$

$$\int_0^t \int_{D\times D} \psi_\gamma(t-s,x,z) q(t-s,x,z) \psi_\gamma(s,w,y) q(s,w,y) \frac{c(z,w)|F_1|(z,w)}{|z-w|^{d+\alpha}} dzdwds$$

$$\leq \quad M\, p_D(t,x,y) (N^{\alpha,\gamma}_{F_1}(t) + \|F_1\|_\infty \mathbf{1}_{\{|x-y|>t^{1/\alpha}\}}) \tag{3.3}$$

and

$$\int_0^t \int_{U_{x,y}} \psi_\gamma(t-s,x,z) q(t-s,x,z) \psi_\gamma(s,w,y) q(s,w,y) \frac{c(z,w)|F_1|(z,w)}{|z-w|^{d+\alpha}} dzdwds$$

$$\leq \quad M\, p_D(t,x,y) N^{\alpha,\gamma}_{F_1}(t). \tag{3.4}$$

In the remainder of this section, we fix a signed measure $\mu \in \mathbf{K}_{\alpha,\gamma}$, a measurable function $F$ with $F_1 = e^F - 1 \in \mathbf{J}_{\alpha,\gamma}$ and the constant $M > 0$ in (3.1).

**Lemma 3.1** *For every $k \geq 0$ and $(t,x) \in (0,1] \times D$,*

$$\int_D |p^k(t,x,y)| dy \leq C_0^2 M^k \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma \left(N^{\alpha,\gamma}_{\mu,F_1}(t)\right)^k. \tag{3.5}$$

**Proof.** We use induction on $k \geq 0$. By (1.4), (3.5) is clear when $k = 0$. Suppose (3.5) is true for $k-1 \geq 0$. Then by (1.12) we have

$$\begin{aligned}
\int_D p^k(t,x,y)dy = & \int_0^{t/2} \left( \int_D p^0(t-s,x,z) \Big( \int_D p^{k-1}(s,z,y)dy \Big) \mu(dz) \right) ds \\
& + \int_0^{t/2} \left( \int_D \int_D p^0(t-s,x,z) \frac{c(z,w)F_1(z,w)}{|z-w|^{d+\alpha}} \Big( \int_D p^{k-1}(s,w,y)dy \Big) dzdw \right) ds \\
& + \int_{t/2}^{t} \left( \int_D p^0(t-s,x,z) \Big( \int_D p^{k-1}(s,z,y)dy \Big) \mu(dz) \right) ds \\
& + \int_{t/2}^{t} \left( \int_D \int_D p^0(t-s,x,z) \frac{c(z,w)F_1(z,w)}{|z-w|^{d+\alpha}} \Big( \int_D p^{k-1}(s,w,y)dy \Big) dzdw \right) ds.
\end{aligned}$$

Thus using (1.1) and our induction hypothesis, we have

$$\begin{aligned}
& \int_D |p^k(t,x,y)| dy \\
\leq & 2^{\gamma/\alpha} C_0^3 M^{k-1} \big(N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-1} \left( \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma \int_0^{t/2} \left( \int_D \left(1 \wedge \frac{\delta_D(z)}{(t-s)^{1/\alpha}}\right)^\gamma q(t-s,x,z)|\mu|(dz) \right) ds \right. \\
& + C_0 \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma \int_0^{t/2} \left( \int_D \int_D \left(1 \wedge \frac{\delta_D(z)}{(t-s)^{1/\alpha}}\right)^\gamma q(t-s,x,z) \frac{|F_1|(z,w)}{|z-w|^{d+\alpha}} dzdw \right) ds \\
& + \int_D \left(1 \wedge \frac{\delta_D(z)}{t^{1/\alpha}}\right)^\gamma \int_{t/2}^t \psi_\gamma(t-s,x,z) q(t-s,x,z) ds |\mu|(dz) \\
& \left. + C_0 \int_D \int_D \left(1 \wedge \frac{\delta_D(w)}{t^{1/\alpha}}\right)^\gamma \int_{t/2}^t \psi_\gamma(t-s,x,z) q(t-s,x,z) ds \frac{|F_1|(z,w)}{|z-w|^{d+\alpha}} dzdw \right).
\end{aligned}$$

Applying (3.1), Lemmas 2.2 and 2.5, the above is less than

$$4^{-1} C_0 M^k \big(N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-1} \left( \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma \int_0^{t/2} \left( \int_D \left(1 \wedge \frac{\delta_D(z)}{(t-s)^{1/\alpha}}\right)^\gamma q(t-s,x,z)|\mu|(dz) \right) ds \right.$$

$$
\begin{aligned}
&+C_0\left(1\wedge\frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma\int_0^{t/2}\left(\int_D\int_D\left(1\wedge\frac{\delta_D(z)}{(t-s)^{1/\alpha}}\right)^\gamma q(t-s,x,z)\frac{|F_1|(z,w)}{|z-w|^{d+\alpha}}dzdw\right)ds\\
&+\int_D\left(1\wedge\frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma\int_{t/2}^t\left(1\wedge\frac{\delta_D(z)}{(t-s)^{1/\alpha}}\right)^\gamma q(t-s,x,z)ds|\mu|(dz)\\
&+C_0\int_D\int_D\left(1\wedge\frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma\left(1+\frac{|x-w|\wedge|z-w|\wedge t^{1/\alpha}}{|x-z|}\right)^\gamma\\
&\quad\times\int_{t/2}^t\left(1\wedge\frac{\delta_D(z)}{(t-s)^{1/\alpha}}\right)^\gamma q(t-s,x,z)\frac{|F_1|(z,w)}{|z-w|^{d+\alpha}}dzdw\Bigg)\\
\le& C_0^2M^k\left(1\wedge\frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma(N^{\alpha,\gamma}_{\mu,F_1}(t))^k.
\end{aligned}
$$

□

**Lemma 3.2** *For every $k\ge 0$ and $(t,x,y)\in(0,1]\times D\times D$,*

$$
\int_0^t\int_D p_D(t-s,x,z)dz\int_D|p^k(s,w,y)|dwds\;\le\;t\,\frac{\alpha}{\alpha-\gamma}2^{2\gamma/\alpha}\,C_0^4\,M^k\,\psi_\gamma(t,x,y)\big(N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^k.
$$

**Proof.** By (1.1) and Lemma 3.1,

$$
\begin{aligned}
&\int_0^t\int_D p_D(t-s,x,z)dz\int_D|p^k(s,w,y)|dwds\\
=&\int_0^{t/2}\int_D p_D(t-s,x,z)dz\int_D|p^k(s,w,y)|dwds+\int_{t/2}^t\int_D p_D(t-s,x,z)dz\int_D|p^k(s,w,y)|dwds\\
\le& C_0\int_0^{t/2}\int_D\left(1\wedge\frac{\delta_D(x)}{(t-s)^{1/\alpha}}\right)^\gamma q(t-s,x,z)dz\int_D|p^k(s,w,y)|dwds\\
&+C_0\int_{t/2}^t\int_D\left(1\wedge\frac{\delta_D(x)}{(t-s)^{1/\alpha}}\right)^\gamma q(t-s,x,z)dz\int_D|p^k(s,w,y)|dwds\\
\le& C_0^3M^k\int_0^{t/2}\int_D\left(1\wedge\frac{\delta_D(x)}{(t-s)^{1/\alpha}}\right)^\gamma q(t-s,x,z)dz\left(1\wedge\frac{\delta_D(y)}{s^{1/\alpha}}\right)^\gamma\big(N_{\mu,F_1}(s)\big)^kds\\
&+C_0^3M^k\int_{t/2}^t\int_D\left(1\wedge\frac{\delta_D(x)}{(t-s)^{1/\alpha}}\right)^\gamma q(t-s,x,z)dz\left(1\wedge\frac{\delta_D(y)}{s^{1/\alpha}}\right)^\gamma\big(N_{\mu,F_1}(s)\big)^kds\\
\le& 2^{\gamma/\alpha}C_0^3M^k\big(N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^k\left(1\wedge\frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma\left(\int_0^{t/2}\left(1\wedge\frac{\delta_D(y)}{s^{1/\alpha}}\right)^\gamma ds\right)\int_D q(t-s,x,z)dz\\
&+2^{\gamma/\alpha}C_0^3M^k\big(N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^k\left(1\wedge\frac{\delta_D(y)}{t^{1/\alpha}}\right)^\gamma\int_D\left(\int_{t/2}^t\left(1\wedge\frac{\delta_D(x)}{(t-s)^{1/\alpha}}\right)^\gamma ds\right)q(t-s,x,z)dz.
\end{aligned}
$$

Using

$$
\int_0^{t/2}\left(1\wedge\frac{\delta_D(x)}{s^{1/\alpha}}\right)^\gamma ds\le\frac{\alpha}{\alpha-\gamma}2^{\gamma/\alpha-1}t\left(1\wedge\frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma,
$$

we get that

$$
\int_0^t\int_D p_D(t-s,x,z)dz\int_D p^k(s,w,y)dwds
$$

$$\leq \frac{\alpha}{\alpha-\gamma} 2^{2\gamma/\alpha} C_0^3 M^k \big(N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^k \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^\gamma \int_D q(t-s,x,z)dz.$$

Applying (1.3), we have proved the lemma. $\square$

**Lemma 3.3** *For $k \geq 0$ and $(t,x,y) \in (0,1] \times D \times D$ we have*

$$|p^k(t,x,y)| \leq p^0(t,x,y) \left( \big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^k + k\|F_1\|_\infty C_0^2 M \big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-1} \right). \tag{3.6}$$

**Proof.** We use induction on $k \geq 0$. The $k=0$ case is obvious. Suppose that (3.6) is true for $k-1 \geq 0$. Recall that

$$V_{x,y} = \{(z,w) \in D \times D : |x-y| \geq 4(|y-w| \wedge |x-z|)\}, \quad U_{x,y} = (D \times D) \setminus V_{x,y}.$$

Applying (1.12), (1.1), (3.2) and (3.4), we have by our induction hypothesis

$$\begin{aligned}
|p^k(t,x,y)| \leq& \int_0^t \left( \int_D p^0(t-s,x,z) |p^{k-1}(s,z,y)| |\mu|(dz) \right) ds \\
&+ \int_0^t \left( \int_{U_{x,y}} p^0(t-s,x,z) \frac{c(z,w)|F_1(z,w)|}{|z-w|^{d+\alpha}} |p^{k-1}(s,w,y)| dz dw \right) ds \\
&+ \int_0^t \left( \int_{V_{x,y}} p^0(t-s,x,z) \frac{c(z,w)|F_1(z,w)|}{|z-w|^{d+\alpha}} |p^{k-1}(s,w,y)| dz dw \right) ds \\
\leq& \left( \big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-1} + (k-1)\|F_1\|_\infty C_0^2 M \big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-2} \right) \\
&\quad \times \int_0^t \left( \int_D p^0(t-s,x,z) p^0(s,z,y) |\mu|(dz) \right) ds \\
&+ \left( \big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-1} + (k-1)\|F_1\|_\infty C_0^2 M \big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-2} \right) \\
&\quad \times \int_0^t \left( \int_{U_{x,y}} p^0(t-s,x,z) \frac{c(z,w)|F_1|(z,w)}{|z-w|^{d+\alpha}} p^0(s,w,y) dz dw \right) ds \\
&+ \int_0^t \left( \int_{V_{x,y}} p^0(t-s,x,z) \frac{c(z,w)|F_1|(z,w)}{|z-w|^{d+\alpha}} |p^{k-1}(s,w,y)| dz dw \right) ds \\
\leq& \; p^0(t,x,y) \left( \big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-1} + (k-1)\|F_1\|_\infty C_0^2 M \big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-2} \right) C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t) \\
&+ C_0 \frac{2^{d+\alpha}\|F_1\|_\infty}{|x-y|^{d+\alpha}} \int_0^t \left( \int_{D\times D} p^0(t-s,x,z) |p^{k-1}(s,w,y)| dz dw \right) ds.
\end{aligned}$$

Applying Lemma 3.2 and using (3.1), we get that if $|x-y|^\alpha \geq t$,

$$\begin{aligned}
& C_0 \frac{2^{d+\alpha}\|F_1\|_\infty}{|x-y|^{d+\alpha}} \int_0^t \left( \int_{D\times D} p^0(t-s,x,z) |p^{k-1}(s,w,y)| dz dw \right) ds \\
\leq& \; \psi_\gamma(t,x,y) \frac{t}{|x-y|^{d+\alpha}} \|F_1\|_\infty C_0^5 \frac{\alpha}{\alpha-\gamma} 2^{d+\alpha+\gamma/\alpha} M^{k-1} \big(N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-1} \\
\leq& \; p^0(t,x,y) \|F_1\|_\infty C_0^6 \frac{\alpha}{\alpha-\gamma} 2^{d+\alpha+\gamma/\alpha} M^{k-1} \big(N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-1}
\end{aligned}$$

$$
\begin{aligned}
&\le\ p^0(t,x,y)\|F_1\|_\infty C_0^2 M^k \big(N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-1}\\
&\le\ p^0(t,x,y)\|F_1\|_\infty C_0^2 M \big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-1}.
\end{aligned}
$$

Thus (3.6) is true for $k$ when $|x-y|^\alpha \ge t$.

If $|x-y|^\alpha \le t$, using (1.1), (1.12), (3.2) and (3.3), we have by our induction hypothesis

$$
\begin{aligned}
|p^k(t,x,y)| \ &\le\ \int_0^t \left(\int_D p^0(t-s,x,z)|p^{k-1}(s,z,y)||\mu|(dz)\right) ds\\
&+\int_0^t \left(\int_{D\times D} p^0(t-s,x,z)\frac{c(z,w)|F_1(z,w)|}{|z-w|^{d+\alpha}}|p^{k-1}(s,w,y)|dzdw\right) ds\\
\le\ &\left(\big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-1} + (k-1)\|F_1\|_\infty C_0^2 M \big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-2}\right)\\
&\quad\times \int_0^t \left(\int_D p^0(t-s,x,z)p^0(s,z,y)|\mu|(dz)\right) ds\\
&+\left(\big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-1} + (k-1)\|F_1\|_\infty C_0^2 M \big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-2}\right)\\
&\quad\times \int_0^t \left(\int_{D\times D} p^0(t-s,x,z)\frac{c(z,w)|F_1|(z,w)}{|z-w|^{d+\alpha}}p^0(s,w,y)dzdw\right) ds\\
\le\ & p^0(t,x,y)\left(\big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-1} + (k-1)\|F_1\|_\infty C_0^2 M \big(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t)\big)^{k-2}\right) C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t).
\end{aligned}
$$

The proof is now complete. □

Since $F_1 \in \mathbf{J}_{\alpha,\gamma}$, there is $t_1 := t_1(d,\alpha,\gamma,C_0,M,N^{\alpha,\gamma}_{\mu,F_1},\|F_1\|_\infty) \in (0,1)$ so that

$$
N^{\alpha,\gamma}_{\mu,F_1}(t_1) \le \big(3C_0^2 M\big)^{-1} \wedge \big(9(C_0^2M)^2\|F_1\|_\infty\big)^{-1}.
$$

It follows from Lemma 3.3 that for every $(t,x,y) \in (0,t_1]\times D\times D$,

$$
\begin{aligned}
\sum_{k=0}^\infty |p^k(t,x,y)| \ &=\ p^0(t,x,y) + \sum_{k=1}^\infty |p^k(t,x,y)|\\
\le\ & p^0(t,x,y) + p^0(t,x,y)\left(\sum_{k=1}^\infty (C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t))^k + \|F_1\|_\infty C_0^2 M \sum_{k=1}^\infty k(C_0^2 M N^{\alpha,\gamma}_{\mu,F_1}(t))^{k-1}\right)\\
\le\ & p^0(t,x,y) + p^0(t,x,y)\left(\frac{1}{2}+\frac{9}{4}\|F_1\|_\infty C_0^2 M\right)\\
\le\ & p^0(t,x,y)\left(\frac{3}{2}+\frac{9}{4}\|F_1\|_\infty C_0^2 M\right).
\end{aligned}
\tag{3.7}
$$

Hence, Fubini's Theorem, (1.9) and (1.11) yield (1.10) and (1.13). Thus we conclude from (3.7) and (1.13) that

**Theorem 3.4** *There exist $t_1 := t_1(d,\alpha,\gamma,C_0,M,N^{\alpha,\gamma}_{\mu,F_1},\|F_1\|_\infty) \in (0,1)$ and a positive constant $C_6 := C_6(d,\alpha,\gamma,C_0,M,\|F_1\|_\infty)$ such that the Feynman-Kac semigroup $T_t^{\mu,F}$ corresponding to $\mu$ and $F$ has a continuous density $q_D(t,x,y)$ for $t \le t_1$ and*

$$
q_D(t,x,y) \le C_6 \psi_\gamma(t,x,y) q(t,x,y) \tag{3.8}
$$

*for every $(t,x,y) \in (0,t_1]\times D\times D$.*

For the lower bound estimate, we need to assume that $F$ is a function in $\mathbf{J}_{\alpha,\gamma}$.

**Theorem 3.5** *Suppose that $\mu \in \mathbf{K}_{\alpha,\gamma}$ and $F$ is a function in $\mathbf{J}_{\alpha,\gamma}$. Then there exist constants $t_2 := t_2(\alpha,\gamma,C_0,M,N^{\alpha,\gamma}_{\mu,F},\|F\|_\infty) \in (0,1)$ and $C_7 := C_7(\alpha,\gamma,C_0,M,N^{\alpha,\gamma}_{\mu,F},\|F\|_\infty) > 1$ such that*

$$C_7^{-1}\psi_\gamma(t,x,y)q(t,x,y) \le q_D(t,x,y) \le C_7\psi_\gamma(t,x,y)q(t,x,y) \tag{3.9}$$

*for every $(t,x,y) \in (0,t_2] \times D \times D$.*

**Proof.** Since $F$ is a bounded function in $\mathbf{J}_{\alpha,\gamma}$, so is $F_1 := e^F - 1$ with $|F_1(x,y)| \le e^{\|F\|_\infty}|F|(x,y)$ and $N^{\alpha,\gamma}_{F_1} \le e^{\|F\|_\infty}N^{\alpha,\gamma}_F$. Thus the upper bound estimate in (3.9) follows directly from Theorem 3.4. To establish the lower bound, we define for $(t,x,y) \in (0,\infty)\times D\times D$,

$$\begin{aligned}\tilde p^1(t,x,y) \;&=\; \int_0^t \left(\int_D p^0(t-s,x,z)p^0(s,z,y)|\mu|(dz)\right)ds \\ &\quad + \int_0^t \left(\int_D\int_D p^0(t-s,x,z)\frac{c(z,w)|F|(z,w)}{|z-w|^{d+\alpha}}p^0(s,w,y)dzdw\right)ds.\end{aligned}$$

Then for any bounded Borel function $f$ on $D$ and any $(t,x) \in (0,\infty)\times D$, we have

$$\mathbb{E}_x\left[A_t^{|\mu|,|F|}f(X_t)\right] = \int_D \tilde p^1(t,x,y)f(y)dy.$$

Applying Lemma 3.3 with $|\mu|$ and $|F|$ in place of $\mu$ and $F_1$, we have

$$\tilde p^1(t,x,y) \;\le\; (C_0^2MN^{\alpha,\gamma}_{\mu,F}(1) + C_0^2M\|F\|_\infty)p^0(t,x,y) \;=:\; (k/2)p^0(t,x,y)$$

for all $(t,x,y) \in (0,1]\times D\times D$. Hence we have for all $(t,x,y) \in (0,1]\times D\times D$,

$$p^0(t,x,y) - \frac{1}{k}\tilde p^1(t,x,y) \ge \frac{1}{2}p^0(t,x,y). \tag{3.10}$$

Using the elementary fact that

$$1 - A_t^{|\mu|,|F|}/k \le \exp\left(-A_t^{|\mu|,|F|}/k\right) \le \exp\left(A_t^{\mu,F}/k\right),$$

we get that for any $B(x,r)\subset D$ and any $(t,y)\in(0,1]\times D$,

$$\frac{1}{|B(x,r)|}\mathbb{E}_y\left[\left(1 - A_t^{|\mu|,|F|}/k\right)\mathbf{1}_{B(x,r)}(X_t)\right] \le \frac{1}{|B(x,r)|}\mathbb{E}_y\left[\exp(A_t^{\mu,F}/k)\mathbf{1}_{B(x,r)}(X_t)\right].$$

Thus, by (3.10) and Hölder's inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{1}{|B(x,r)|}\mathbb{E}_y\left[\mathbf{1}_{B(x,r)}(X_t)\right] \;&\le\; \frac{1}{|B(x,r)|}\mathbb{E}_y\left[\exp(A_t^{\mu,F}/k)\mathbf{1}_{B(x,r)}(X_t)\right] \\ \le \left(\frac{1}{|B(x,r)|}\mathbb{E}_y\left[\exp(A_t^{\mu,F})\mathbf{1}_{B(x,r)}(X_t)\right]\right)^{1/k}&\left(\frac{1}{|B(x,r)|}\mathbb{E}_y\left[\mathbf{1}_{B(x,r)}(X_t)\right]\right)^{1-1/k}.\end{aligned}$$

Therefore

$$\frac{1}{2^k}\frac{1}{|B(x,r)|}\mathbb{E}_y\left[\mathbf{1}_{B(x,r)}(X_t)\right] \le \frac{1}{|B(x,r)|}\mathbb{E}_y\left[\exp(A_t^{\mu,F})\mathbf{1}_{B(x,r)}(X_t)\right].$$

We conclude by sending $r\downarrow 0$ that for every $(t,x,y)\in(0,1]\times D\times D$, $2^{-k}p^0(t,x,y) \le q_D(t,x,y)$. $\square$

Combining the two theorems above with the semigroup property, we immediately get the main result of this paper, Theorem 1.3.

## 4 Applications

In this section, we will apply our main result to (reflected) symmetric stable-like processes, killed symmetric $\alpha$-stable processes, censored $\alpha$-stable processes and stable processes with drifts. We first record the following two facts.

Suppose that $d \geq 2$ and $\alpha \in (0, 2)$. A signed measure $\mu$ on $\mathbb{R}^d$ is said to be in Kato class $\mathbb{K}_{d,\alpha}$ if

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} \frac{1}{|x-y|^{d-\alpha}} |\mu|(dy) = 0.$$

A function $g$ on $\mathbb{R}^d$ is said to be in $\mathbb{K}_{d,\alpha}$ if $g(x)dx \in \mathbb{K}_{d,\alpha}$.

**Proposition 4.1** *Suppose that $d \geq 2$ and $\alpha \in (0, 2)$.*

(i) *Let $D$ be an arbitrary Borel subset of $\mathbb{R}^d$. $\mu \in \mathbf{K}_{\alpha,0}$ if and only if $\mathbf{1}_D \mu \in \mathbb{K}_{d,\alpha}$. Hence for every $\mu \in \mathbb{K}_{d,\alpha}$, $\mu|_D \in \mathbf{K}_{\alpha,\gamma}$ for every $\gamma \geq 0$. In particular, $L^\infty(D; dx) \subset \mathbf{K}_{\alpha,\gamma}$ and $L^p(D; dx) \subset \mathbf{K}_{\alpha,\gamma}$ for every $p > d/\alpha$ and $\gamma \geq 0$.*

(ii) *Suppose that $D$ is a bounded Lipschitz open set in $\mathbb{R}^d$ and $\gamma \in (0, \alpha)$. Let $g$ be a function defined on $D$. If there exist constants $c > 0$, $\beta \in (0, \gamma + (\alpha - \gamma)/d)$ and a compact subset $K$ of $D$ such that $\mathbf{1}_K(x) g(x) \in \mathbb{K}_{d,\alpha}$ and*

$$|g(x)| \leq c\delta_D(x)^{-\beta} \quad \textit{for } x \in D \setminus K,$$

*then $g \in \mathbf{K}_{\alpha,\gamma}$.*

**Proof.** (i) By the proof of [29, Theorem 2], we have that $\mu \in \mathbb{K}_{d,\alpha}$ if and only if

$$\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} q(s, x, y) \mu(dy) ds = 0.$$

This implies that $\mu \in \mathbf{K}_{\alpha,0}$ if and only if $\mathbf{1}_D \mu \in \mathbb{K}_{d,\alpha}$. In particular we have for every $\mu \in \mathbb{K}_{d,\alpha}$, $\mu|_D \in \mathbf{K}_{\alpha,\gamma}$ for every $\gamma \geq 0$. Clearly $L^\infty(D; dx) \subset \mathbb{K}_{d,\alpha}$. Using Hölder's inequality, one concludes that $L^p(\mathbb{R}^d; dx) \subset \mathbb{K}_{d,\alpha}$ for every $p > d/\alpha$.

(ii) Let $g$ be a function defined on $D$ such that there exist constants $c_1 > 0$, $\beta \in (0, \gamma + (\alpha - \gamma)/d)$ and a compact subset $K$ of $D$ so that $\mathbf{1}_K(x) g(x) \in \mathbb{K}_{d,\alpha}$ and $|g(x)| \leq c_1 \delta_D(x)^{-\beta}$ for $x \in D \setminus K$. In view of (i), it suffices to show that $\mathbf{1}_{D \setminus K} g \in \mathbf{K}_{\alpha,\gamma}$. Note that

$$\begin{aligned}
& \sup_{x \in D} \int_0^t \int_{D \setminus K} \left(1 \wedge \frac{\delta_D(y)}{s^{1/\alpha}}\right)^\gamma q(s, x, y) |g(y)| dy ds \\
\leq\ & c_1 \sup_{x \in D} \int_0^t \int_{D \setminus K} \left(1 \wedge \frac{\delta_D(y)}{s^{1/\alpha}}\right)^\gamma \delta_D(y)^{-\beta} q(s, x, y) dy ds \\
\leq\ & c_1 \sup_{x \in D} \int_{D \setminus K} \left( \int_0^{\delta_D(y)^\alpha \wedge t} \left( s^{-d/\alpha} \wedge \frac{s}{|x-y|^{d+\alpha}} \right) ds \right) \delta_D(y)^{-\beta} dy \\
& + c_1 \sup_{x \in D} \int_{D \setminus K} \left( \int_{\delta_D(y)^\alpha \wedge t}^t s^{-\gamma/\alpha} \left( s^{-d/\alpha} \wedge \frac{s}{|x-y|^{d+\alpha}} \right) ds \right) \delta_D(y)^{\gamma - \beta} dy =: I + II. \quad (4.1)
\end{aligned}$$

Here

$$\begin{aligned}
I &\leq c_1 \sup_{x\in D}\left(\int_D \int_0^{\delta_D(y)^\alpha\wedge|x-y|^\alpha\wedge t} \frac{s}{|x-y|^{d+\alpha}} ds\, \delta_D(y)^{-\beta} dy \right.\\
&\quad \left. + \int_D \int_{\delta_D(y)^\alpha\wedge|x-y|^\alpha\wedge t}^{\delta_D(y)^\alpha\wedge t} s^{-d/\alpha} ds\, \delta_D(y)^{-\beta} dy\right)\\
&\leq c_2 \sup_{x\in D}\int_D \left(\frac{(\delta_D(y)^\alpha\wedge|x-y|^\alpha\wedge t)^2\delta_D(y)^{-\beta}}{|x-y|^{d+\alpha}} + \mathbf{1}_{\{|x-y|<\delta_D(y)\wedge t^{1/\alpha}\}}\frac{\delta_D(y)^{-\beta}}{|x-y|^{d-\alpha}}\right) dy\\
&\leq c_2 \sup_{x\in D}\int_D \left(\frac{(|x-y|\wedge t^{1/\alpha})^{2\alpha-\beta}}{|x-y|^{d}} + \mathbf{1}_{\{|x-y|<\delta_D(y)\wedge t^{1/\alpha}\}}\frac{1}{|x-y|^{d-\alpha+\beta}}\right) dy\\
&\leq 2c_2 t^{(\alpha-\beta)/(2\alpha)} \sup_{x\in D}\int_D \frac{1}{|x-y|^{d-(\alpha-\beta)/2}} dy = c_3\, t^{(\alpha-\beta)/(2\alpha)}, \qquad (4.2)
\end{aligned}$$

while

$$\begin{aligned}
II &\leq c_1 \sup_{x\in D}\int_D \left(\int_{\delta_D(y)^\alpha\wedge t}^{t} \mathbf{1}_{\{s<|x-y|^\alpha\}}\frac{s^{1-\gamma/\alpha}}{|x-y|^{d+\alpha}} ds\right) \delta_D(y)^{\gamma-\beta} dy\\
&\quad + c_1 \sup_{x\in D}\int_D \left(\int_{\delta_D(y)^\alpha\wedge t}^{t} \mathbf{1}_{\{s\geq|x-y|^\alpha\}}\, s^{-(d+\gamma)/\alpha} ds\right) \delta_D(y)^{\gamma-\beta} dy\\
&\leq c_4 \sup_{x\in D}\int_D \frac{\left(|x-y|\wedge t^{1/\alpha}\right)^{2\alpha-\gamma}}{|x-y|^{d+\alpha}} \mathbf{1}_{\{\delta_D(y)<|x-y|\wedge t^{1/\alpha}\}}\, \delta_D(y)^{\gamma-\beta} dy\\
&\quad + c_4 \sup_{x\in D}\int_D |x-y|^{\alpha-d-\gamma}\mathbf{1}_{\{|x-y|\leq t^{1/\alpha}\}}\delta_D(y)^{\gamma-\beta} dy\\
&\leq c_4 t^{\delta/\alpha} \sup_{x\in D}\int_D \frac{1}{|x-y|^{d-\alpha+\varepsilon}\delta_D(y)^{\beta-\gamma}}\, dy, \qquad (4.3)
\end{aligned}$$

where $\delta := (\alpha-\gamma-d(\beta-\gamma))/2 > 0$ and $\varepsilon := (\alpha+\gamma-d(\beta-\gamma))/2 > 0$. Note that $\varepsilon+\delta = \alpha-d(\beta-\gamma)$ and $\varepsilon-\delta=\gamma$. Let $p = d/(d-\alpha+\varepsilon+\delta/2)$ and $q = d/(\alpha-(\varepsilon+\delta/2))$ so that $1/p+1/q = 1$. Since $D$ is a bounded Lipschitz open set, $p(d-\alpha+\varepsilon) < d$ and $q(\beta-\gamma) < 1$, we have by Young's inequality,

$$\sup_{x\in D}\int_D \frac{1}{|x-y|^{d-\alpha+\varepsilon}\delta_D(y)^{\beta-\gamma}}\, dy \leq \sup_{x\in D}\int_D \left(\frac{1}{p}\frac{1}{|x-y|^{p(d-\alpha+\varepsilon)}} + \frac{1}{q}\frac{1}{\delta_D(y)^{q(\beta-\gamma)}}\right) dy < \infty.$$

This together with (4.1) –(4.3) implies that $\lim_{t\to 0} N^{\alpha,\gamma}_{\mathbf{1}_{D\setminus K}g(x)}(t) = 0$; that is, $\mathbf{1}_{D\setminus K}g \in \mathbf{K}_{\alpha,\gamma}$. This completes the proof of the proposition. □

**Proposition 4.2** *Suppose $\gamma\in[0,\alpha\wedge d)$ and $|F|(z,w) \leq A(|z-w|^\beta\wedge 1)$ for some $A>0$ and $\beta>\alpha$. Then there exists $C_8 = C_8(\beta,d,\alpha,\gamma) > 0$ such that for every arbitrary Borel subset $D$ of $\mathbb{R}^d$,*

$$N_F^{\alpha,\gamma}(t) \;\leq\; C_8\, At. \qquad (4.4)$$

*This in particular implies that $F\in\mathbf{J}_{\alpha,\gamma}$.*

**Proof.** By (2.4), we have that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} q(s,y,w) \left(1+\frac{|z-w|\wedge t^{1/\alpha}}{|y-w|}\right)^\gamma \frac{|F|(z,w)+|F|(w,z)}{|z-w|^{d+\alpha}} dzdwds \\
\le& 2A\left(\int_{\mathbb{R}^d} (|z|^\beta \wedge 1)|z|^{-d-\alpha} dz\right) \int_0^t \int_{\mathbb{R}^d} q(s,y,w)\left(1+\frac{|z-w|\wedge t^{1/\alpha}}{|y-w|}\right)^\gamma dwds \\
\le& c_1 A\left(\int_{B(0,1)} \frac{dz}{|z|^{d+\alpha-\beta}} + \int_{B(0,1)^c} \frac{dz}{|z|^{d+\alpha}}\right) \int_0^t \left(1+\int_D 2^{d+\alpha} \frac{st^{\gamma/\alpha}}{(s^{1/\alpha}+|y-w|)^{d+\alpha}|y-w|^\gamma}\right) dwds \\
\le& c_2 A \int_0^t \left(1+t^{\gamma/\alpha} s\int_0^\infty \frac{r^{d-1}}{r^\gamma (s^{1/\alpha}+r)^{d+\alpha}} dr\right) ds \\
\le& c_2 At + c_3 A\left(\int_0^\infty \frac{u^{d-1-\gamma}}{(1+u)^{d+\alpha}} du\right) t^{\gamma/\alpha} \int_0^t s^{-\gamma/\alpha} ds \le c_4 At
\end{aligned}$$

where the assumption $\gamma \in [0, \alpha\wedge d)$ is used the last inequality. This establishes (4.4). □

## 4.1 Stable-like processes on closed $d$-sets

A Borel subset $D$ in $\mathbb{R}^d$ with $d \ge 1$ is said to be a $d$-set if there exist constants $r_0 > 0$, $\mathcal{C}_2 > \mathcal{C}_1 > 0$ so that

$$\mathcal{C}_1\, r^d \le |B(x,r)\cap D| \le \mathcal{C}_2\, r^d \qquad \text{for all } \ x\in D \text{ and } 0<r\le r_0, \tag{4.5}$$

where for a Borel set $A \subset \mathbb{R}^d$, we use $|A|$ to denote its Lebesgue measure. The notion of a $d$-set arises both in the theory of function spaces and in fractal geometry. It is known that if $D$ is a $d$-set, then so is its Euclidean closure $\overline{D}$. Every uniformly Lipschitz open set in $\mathbb{R}^d$ is a $d$-set, so is its Euclidean closure. It is easy to check that the classical von Koch snowflake domain in $\mathbb{R}^2$ is an open 2-set. A $d$-set can have very rough boundary since every $d$-set with a subset of zero Lebesgue measure removed is still a $d$-set.

Suppose that $D$ is a closed $d$-set $D \subset \mathbb{R}^d$ and $c(x,y)$ is a symmetric function on $D\times D$ that is bounded between two strictly positive constants $\mathcal{C}_4 > \mathcal{C}_3 > 0$, that is,

$$\mathcal{C}_3 \le c(x,y) \le \mathcal{C}_4 \qquad \text{for a.e. } \ x,y\in D. \tag{4.6}$$

For $\alpha \in (0,2)$, we define

$$\mathcal{F} = \left\{u\in L^2(D;dx) : \int_{D\times D} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\, dx\, dy < \infty\right\} \tag{4.7}$$

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{D\times D} (u(x)-u(y))(v(x)-v(y)) \frac{c(x,y)}{|x-y|^{d+\alpha}}\, dx\, dy, \quad u,v\in\mathcal{F}. \tag{4.8}$$

It is easy to check that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(D,dx)$ and therefore there is an associated symmetric Hunt process $X$ on $D$ starting from every point in $D$ except for an exceptional set that has zero capacity. The process $X$ is called a symmetric $\alpha$-stable-like process on $D$ in [11]. When $c(x,y)$ is a constant function, $X$ is the reflected $\alpha$-stable process appeared in [2]. Note that when $D=\mathbb{R}^d$ and $c(x,y)$ is a constant function, then $X$ is nothing but a symmetric $\alpha$-stable process on $\mathbb{R}^d$.

It follows as a special case from [11, Theorem 1.1] that the symmetric stable-like process $X$ on a closed $d$-set in $\mathbb{R}^d$ has a Hölder continuous transition density function $p(t,x,y)$ with respect to the Lebesgue measure on $D$ that satisfies the estimate (1.1) with $\gamma = 0$ and the comparison constant $C_0$ depending only on $d$, $\alpha$, $r_0$ and the constants $\mathcal{C}_k$, $k = 1, \cdots, 4$ in (4.5) and (4.6). In particular, this implies that the process $X$ can be refined so it can start from every point in $D$. Thus as a special case of Theorem 1.3, we have the following.

**Theorem 4.3** *Suppose that $X$ is a symmetric $\alpha$-stable-like process on a closed $d$-set $D$ in $\mathbb{R}^d$. Assume $\mu \in \mathbf{K}_{\alpha,0}$ and $F \in \mathbf{J}_{\alpha,0}$. Let $q$ be the density of the Feynman-Kac semigroup of $X$ corresponding to $A^{\mu,F}$. For any $T > 0$, there exists a constant $C_9 > 1$ such that for all $(t,x,y) \in (0,T] \times D \times D$,*

$$C_9^{-1} q(t,x,y) \le q(t,x,y) \le C_9 q(t,x,y).$$

**Remark 4.4** Let $n \ge 1$ be an integer and $d \in (0,n]$. In general, a Borel subset $D$ in $\mathbb{R}^n$ is said to be a $d$-set if there exist a measure $\mu$ and constants $r_0 > 0$, $\mathcal{C}_2 > \mathcal{C}_1 > 0$ so that

$$\mathcal{C}_1\, r^d \le \mu(B(x,r) \cap D) \le \mathcal{C}_2\, r^d \qquad \text{for all } \; x \in D \text{ and } 0 < r \le r_0, \tag{4.9}$$

It is established in [11] that for every $\alpha \in (0,2)$, a symmetric $\alpha$-stable-like process $X$ can always be constructed on any closed $d$-set $D$ in $\mathbb{R}^n$ via the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D;\mu)$ defined by (4.7)–(4.8) but with the $d$-measure $\mu(dx)$ in place of the Lebesgue measure $dx$ there. Moreover by [11, Theorem 1.1], the process $X$ has a jointly Hölder continuous transition density function $p(t,x,y)$ with respect to the $d$-measure $\mu$ on $D$ that satisfies the estimate (1.1) with $\gamma = 0$. The proof of Theorem 1.3 also works for such process $X$; in other words, Theorem 4.3 continues to hold for such kind of symmetric stable-like processes. □

## 4.2 Killed symmetric $\alpha$-stable processes

A symmetric $\alpha$-stable process $X$ in $\mathbb{R}^d$ is a Lévy process whose characteristic function is given by $\mathbb{E}_0\left[\exp(i\xi \cdot X_t)\right] = e^{-t|\xi|^\alpha}$. It is well-known that the process $X$ has a Lévy intensity function $J(x,y) = \mathcal{A}(d,-\alpha)|x-y|^{-(d+\alpha)}$, where

$$\mathcal{A}(d,-\alpha) = \alpha 2^{-1+\alpha}\Gamma(\frac{d+\alpha}{2})\pi^{-d/2}(\Gamma(1-\frac{\alpha}{2}))^{-1}. \tag{4.10}$$

Here $\Gamma$ is the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1}e^{-t}dt$ for every $\lambda > 0$. Let $X^D$ be the killed symmetric $\alpha$-stable process $X^D$ in a $C^{1,1}$ open set $D$. It follows from [7] that $X^D$ satisfies the assumption of Section 1 with $\gamma = \alpha/2$. Thus as a special case of Theorem 1.3, we have the following.

**Theorem 4.5** *Suppose that $X$ is a killed symmetric $\alpha$-stable process in a $C^{1,1}$ open set $D$. Assume $\mu \in \mathbf{K}_{\alpha,\alpha/2}$ and $F \in \mathbf{J}_{\alpha,\alpha/2}$. Let $q_D$ be the density of the Feynman-Kac semigroup of $X$ corresponding to $A^{\mu,F}$. For any $T > 0$, there exists a constant $C_{10} > 1$ such that for all $(t,x,y) \in (0,T] \times D \times D$,*

$$C_{10}^{-1}\psi_{\alpha/2}(t,x,y)q(t,x,y) \le q_D(t,x,y) \le C_{10}\psi_{\alpha/2}(t,x,y)q(t,x,y).$$

Let $X^m$ be a relativistic $\alpha$-stable process in $\mathbb{R}^d$ with mass $m > 0$, i.e., $X^m$ is a Lévy process in $\mathbb{R}^d$ with

$$\mathbb{E}_0\left[\exp(i\xi \cdot X_t^m)\right] = \exp\left(t\left(m^\alpha - (|\xi|^2 + m^2)^{\alpha/2}\right)\right).$$

$X^m$ has a Lévy intensity function $J^m(x,y) = \mathcal{A}(d,-\alpha)\varphi(m^{1/\alpha}|x-y|)|x-y|^{-d-\alpha}$ where

$$\varphi(r) := 2^{-(d+\alpha)}\,\Gamma\left(\frac{d+\alpha}{2}\right)^{-1}\int_0^\infty s^{\frac{d+\alpha}{2}-1}e^{-\frac{s}{4}-\frac{r^2}{s}}\,ds, \tag{4.11}$$

which is decreasing and is a smooth function of $r^2$ satisfying $\varphi(0) = 1$ and

$$\varphi(r) \asymp e^{-r}(1 + r^{(d+\alpha-1)/2}) \qquad \text{on } [0,\infty) \tag{4.12}$$

(see [12, pp. 276-277] for details).

Let $X^{m,D}$ be a killed relativistic $\alpha$-stable process in a bounded $C^{1,1}$ open set. Define

$$K_t^m := \exp\left(\sum_{0<s\le t}\ln(\varphi(m^{1/\alpha}|X_s^D - X_{s-}^D|)) + m\,(t\wedge\tau_D)\right).$$

Since $\int_{\mathbb{R}^d} J(x,y) - J^m(x,y)dy = m$ for all $x \in \mathbb{R}^d$ (see [21]), it follows from [12, p.279] that $X^{m,D}$ can be obtained from the killed symmetric $\alpha$-stable process $X^D$ in $D$ through the non-local Feynman-Kac transform $K_t^m$. That is, $\mathbb{E}_x\left[f(X_t^{m,D})\right] := \mathbb{E}_x\left[K_t^m f(X_t^D)\right]$. By (4.11), for any $M > 0$, there exists a constant $c = c(d,\alpha,M,\mathrm{diam}(D)) > 0$ such that for all $m \in (0,M]$, $|\ln(\varphi(m^{1/\alpha}|x-y|))| \le c(|x-y|^2 \wedge 1)$ and so, by Proposition 4.2, $F_m(x,y) := \ln(\varphi(m^{1/\alpha}|x-y|)) \in \mathbf{J}_{\alpha,\alpha/2}$. The constant function $m$ is in $\mathbf{K}_{\alpha,\alpha/2}$ and so $N^{\alpha,\alpha/2}_{m,F_m}(t)$ goes to zero as $t$ goes to zero uniformly on $m \in (0,M]$. Thus, as an application of Theorem 1.3, we arrive at the following result, which is the bounded open set case of a more general result recently obtained in [9] by a different method.

**Theorem 4.6** *Suppose that $D$ is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. For any $m > 0$, let $p_D^m$ be the transition density of the killed relativistic $\alpha$-stable process with weight $m$ in $D$. For any $M > 0$ and $T > 0$, there exists a constant $C_{11} > 1$ such that for all $m \in (0,M]$ and $(t,x,y) \in (0,T] \times D \times D$,*

$$C_{11}^{-1}\psi_{\alpha/2}(t,x,y)q(t,x,y) \le p_D^m(t,x,y) \le C_{11}\psi_{\alpha/2}(t,x,y)q(t,x,y).$$

## 4.3 Censored stable processes

Fix an open set $D$ in $\mathbb{R}^d$ with $d \ge 1$. Recall that $\mathcal{A}(d,-\alpha)$ is the constant defined in (4.10). Define a bilinear form $\mathcal{E}$ on $C_c^\infty(D)$ by

$$\mathcal{E}(u,v) := \frac{1}{2}\int_D\int_D (u(x)-u(y))(v(x)-v(y))\frac{\mathcal{A}(d,-\alpha)}{|x-y|^{d+\alpha}}dxdy, \quad u,v \in C_c^\infty(D). \tag{4.13}$$

Using Fatou's lemma, it is easy to check that the bilinear form $(\mathcal{E}, C_c^\infty(D))$ is closable in $L^2(D,dx)$. Let $\mathcal{F}$ be the closure of $C_c^\infty(D)$ under the Hilbert inner product $\mathcal{E}_1 := \mathcal{E} + (\,\cdot\,,\,\cdot\,)_{L^2(D,dx)}$. As noted in [2], $(\mathcal{E},\mathcal{F})$ is Markovian and hence a regular symmetric Dirichlet form on $L^2(D,dx)$, and therefore there is an associated symmetric Hunt process $Y = \{Y_t, t \ge 0, \mathbb{P}_x, x \in D\}$ taking values in $D$ (cf.

Theorem 3.1.1 of [17]). The process $Y$ is the censored $\alpha$-stable process in $D$ that is studied in [2]. By (4.13), the jumping kernel $J(x,y)$ of the censored $\alpha$-stable process $Y$ is given by

$$J(x,y) = \frac{\mathcal{A}(d,-\alpha)}{|x-y|^{d+\alpha}} \qquad \text{for } x,y \in D.$$

As a particular case of a more general result established in [8, Theorem 1.1], when $\alpha \in (1,2)$ and $D$ is a $C^{1,1}$ open subset of $\mathbb{R}^d$, the censored $\alpha$-stable process on $D$ satisfies the assumption of Section 1 with $\gamma = \alpha - 1$. Thus as a special case of Theorem 1.3, we have the following:

**Theorem 4.7** *Suppose that $\alpha \in (1,2)$ and that $Y$ is a censored stable process in a $C^{1,1}$ open set $D$. Assume $\mu \in \mathbf{K}_{\alpha,\alpha-1}$ and $F \in \mathbf{J}_{\alpha,\alpha-1}$. Let $q_D$ be the density of the Feynman-Kac semigroup of $Y$ corresponding to $A^{\mu,F}$. For any $T > 0$, there exists a constant $C_{12} > 1$ such that for all $(t,x,y) \in (0,T] \times D \times D$,*

$$C_{12}^{-1}\psi_{\alpha-1}(t,x,y)q(t,x,y) \le q_D(t,x,y) \le C_{12}\psi_{\alpha-1}(t,x,y)q(t,x,y).$$

Similar to [2], we can define a censored relativistic $\alpha$-stable process in $D$. Alternatively, with

$$K_t := \exp\left(\sum_{0<s\le t} \ln(\varphi(m^{1/\alpha}(|Y_{s-} - Y_s|)) + \mathcal{A}(d,-\alpha)\int_0^t \int_D \frac{1-\varphi(m^{1/\alpha}|Y_s - y|)}{|Y_s - y|^{\alpha+d}} dyds\right),$$

if $D$ is a bounded $C^{1,1}$ open set, a censored relativistic stable process $Y^m$ can also be obtained from the censored stable process $Y$ through the Feynman-Kac transform $K_t$. That is, $\mathbb{E}_x[f(Y_t^m)] = \mathbb{E}_x[K_t f(Y_t)]$ (see [6, 12]). By an argument similar to that of Subsection 4.2, one can see that $F_m := \ln(\varphi(m^{1/\alpha}|x-y|)) \in \mathbf{J}_{\alpha,\alpha/2}$. Moreover, since

$$g_m(x) := \int_D (1-\varphi(m^{1/\alpha}|x-y|))|x-y|^{-\alpha-d}dy \le \int_{\mathbb{R}^d} (1-\varphi(m^{1/\alpha}|x-y|))|x-y|^{-\alpha-d}dy = m,$$

$g_m \in \mathbf{K}_{\alpha,\alpha/2}$ and $N^{\alpha,\alpha/2}_{g_m,F_m}(t)$ goes to zero as $t$ goes to zero uniformly on $m \in (0,M]$. Thus as a particular case of Theorem 4.7, we have the following.

**Theorem 4.8** *Suppose that $\alpha \in (1,2)$ and that $D$ is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. For any $m > 0$, let $q_D^m$ be the transition density of the censored relativistic $\alpha$-stable process with weight $m$ in $D$. For any $M > 0$ and $T > 0$, there exists a constant $C_{13} > 1$ such that for all $m \in (0,M]$ and $(t,x,y) \in (0,T] \times D \times D$,*

$$C_{13}^{-1}\psi_{\alpha-1}(t,x,y)q(t,x,y) \le q_D^m(t,x,y) \le C_{13}\psi_{\alpha-1}(t,x,y)q(t,x,y).$$

In fact, Theorems 4.7 and 4.8 are applicable to certain class of censored stable-like processes whose Dirichlet heat kernel estimates are given in [8].

## 4.4 Stable processes with drifts

Let $\alpha \in (1,2)$ and $d \ge 2$. In this subsection, we apply our main result to a non-symmetric process.

For $b = (b_1,\dots,b_d)$ with $b_i \in \mathbb{K}_{d,\alpha-1}$, a Feller process $Z$ on $\mathbb{R}^d$ with infinitesimal generator $\mathcal{L}^b := \Delta^{\alpha/2} + b(x)\cdot\nabla$ is constructed in [3] through the fundamental solution of $\mathcal{L}^b$. Let $Z^D$ be the subprocess of $Z$ killed upon leaving $D$. The following result is established in [10].

**Theorem 4.9** *If $\alpha \in (1,2)$, $d \geq 2$ and $D$ is a bounded $C^{1,1}$ open set, then $Z^D$ has a jointly continuous transition density function $p_D(t,x,y)$ that satisfies* (1.1) *with $\gamma = \alpha/2$.*

Thus as a special case of Theorem 1.3, we also have the following:

**Theorem 4.10** *Suppose that $\alpha \in (1,2)$, $d \geq 2$, that $D$ is a bounded $C^{1,1}$ open set and that $Z^D$ is the subprocess of $Z$ killed upon leaving $D$. Assume $\mu \in \mathbf{K}_{\alpha,\alpha/2}$ and $F \in \mathbf{J}_{\alpha,\alpha/2}$. Let $q_D$ be the density of the Feynman-Kac semigroup of $Z^D$ corresponding to $A^{\mu,F}$. For any $T > 0$, there exists a constant $C_{14} > 1$ such that for all $(t,x,y) \in (0,T] \times D \times D$,*

$$C_{14}^{-1}\psi_{\alpha/2}(t,x,y)q(t,x,y) \leq q_D(t,x,y) \leq C_{14}\psi_{\alpha/2}(t,x,y)q(t,x,y).$$

# References


[1] Ph. Blanchard and Z. M. Ma. Semigroup of Schrödinger operators with potentials given by Radon measures. In *Stochastic processes, physics and geometry*, 160–195, World Sci. Publishing, Teaneck, NJ, 1990.

[2] K. Bogdan, K. Burdzy and Z.-Q. Chen. Censored stable processes. *Probab. Theory Related Fields* **127** (2003), 89–152

[3] K. Bogdan and T. Jakubowski. Estimates of heat kernel of fractional Laplacian perturbed by gradient operators. *Comm. Math. Phys.* **271(1)** (2007), 179–198.

[4] Z.-Q. Chen, P. J. Fitzsimmons, K. Kuwae and T.-S. Zhang. Perturbation of symmetric Markov processes. *Probab. Theory Relat. Fields* **140** (2008), 239-275.

[5] Z.-Q. Chen, P. J. Fitzsimmons, K. Kuwae and T.-S. Zhang. On general perturbations of symmetric Markov processes. *J. Math. Pures et Appliquées* **92** (2009), 363-374.

[6] Z.-Q. Chen and P. Kim. Stability of Martin boundary under non-local Feynman-Kac perturbations. *Probab. Theory Related Fields* **128** (2004), 525–564.

[7] Z.-Q. Chen, P. Kim, and R. Song. Heat kernel estimates for Dirichlet fractional Laplacian. *J. European Math. Soc.* **12** (2010), 1307–1329.

[8] Z.-Q. Chen, P. Kim and R. Song. Two-sided heat kernel estimates for censored stable-like processes. *Probab. Theory Relat. Fields* **146** (2010), 361–399.

[9] Z.-Q. Chen, P. Kim and R. Song. Sharp heat kernel estimates for relativistic stable processes in open sets. *Ann. Probab.* **40** (2012), 213–244.

[10] Z.-Q. Chen, P. Kim and R. Song. Dirichlet heat kernel estimates for fractional Laplacian under gradient perturbation. *Ann. Probab.*, to appear, 2011. arXiv:1011.3273

[11] Z.-Q. Chen and T. Kumagai. Heat kernel estimates for stable-like processes on $d$-sets. *Stoch. Proc. Appl.* **108** (2003), 27–62.

[12] Z.-Q. Chen and R. Song. Drift transforms and Green function estimates for discontinuous processes. *J. Funct. Anal.* **201** (2003), 262–281.

[13] K. L. Chung. Doubly-Feller process with multiplicative functional. *Seminar on Stochastic Processes, 1985*, 63–78. Progr. Probab. Statist., **12**, Birkhäuser Boston, MA, 1986.

[14] K. L. Chung and Z. Zhao. *From Brownian Motion to Schrödinger's Equation.* Springer-Verlag, Berlin, 1995.

[15] C. Doléans-Dade. Quelques applications de la formule de changement de variables pour les semimartingales. *Z. Wahrsch.* **16** (1970), 181-194.

[16] P. J. Fitzsimmons and R. K. Getoor. Smooth measures and continuous additive functionals of right Markov processes. In *Itô's Stochastic Calculus and Probability Theory*, 31-49, Springer, Tokyo, 1996.

[17] M. Fukushima, Y. Oshima and M. Takeda. *Dirichlet Forms and Symmetric Markov Processes.* 2nd Edition. Walter de Gruyter, 2011.

[18] P. Kim and R. Song. Two-sided estimates on the density of Brownian motion with singular drift. *Illinois J. Math.* **50(3)** (2006), 635–688.

[19] L. Riahi. Estimates of Green functions and their applications for parabolic operators with singular potentials. *Colloq. Math.* **95(2)** (2003), 267–283.

[20] L. Riahi. Dirichlet Green functions for parabolic operators with singular lower-order terms. *JIPAM. J. Inequal. Pure Appl. Math.* **8(2)** (2007), Article 36.

[21] M. Ryznar. Estimates of Green function for relativistic $\alpha$-stable process. *Potential Anal.* **17(1)** (2002), 1–23.

[22] M. Sharpe. *General Theory of Markov Processes.* Academic Press, Boston, 1988.

[23] B. Simon. Schrödinger semigroups. *Bull. Amer. Math. Soc. (N.S.)* **7** (1982), 447–526.

[24] R. Song. Probabilistic approach to the Dirichlet problem of perturbed stable processes. *Probab. Theory Relat. Fields* **95** (1993), 371–389.

[25] R. Song. Feynman-Kac semigroups with discontinuous additive functionals. *J. Theor. Probab.* **8** (1995), 727-762.

[26] R. Song. Two-sided estimates on the density of the Feynman-Kac semigroups of stable-like processes. *Electron. J. Probab.* **11 (6)** (2006), 146–161.

[27] C. Wang. On estimates of the density of Feynman-Kac semigroups of $\alpha$-stable-like processes. *J. Math. Anal. Appl.* **348** (2008), 938–970.

[28] Qi S. Zhang. Gaussian bounds for the fundamental solutions of $\nabla(A\nabla u) + B\nabla u - u_t = 0$. *Manuscripta Math.* **93** (1997), 381–390.

[29] Z. Zhao. A probabilistic principle and generalized Schrödinger perturbation. *J. Funct. Anal.* **101** (1991), 162–176.

**Zhen-Qing Chen**
Department of Mathematics, University of Washington, Seattle, WA 98195, USA
E-mail: zqchen@uw.edu

**Panki Kim**
Department of Mathematics, Seoul National University, Building 27, 1 Gwanak-ro, Gwanak-gu, Seoul 151-747, Republic of Korea
E-mail: pkim@snu.ac.kr

**Renming Song**
Department of Mathematics, University of Illinois, Urbana, IL 61801, USA
E-mail: rsong@math.uiuc.edu