An L_2-theory for a class of SPDEs driven by Lévy processes

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Abstract

In this paper we present an L_2-theory for a class of stochastic partial differential equations driven by Lévy processes. The coefficients of the equations are random functions depending on time and space variables, and no smoothness assumption of the coefficients is assumed.

Keywords: Stochastic parabolic partial differential equations, Lévy processes, L_2-theory

AMS 2000 subject classifications: 60H15, 35R60.

1 Introduction

In this article we present an L_2-theory of stochastic partial differential equations (SPDEs in abbreviation) of the type

$$du = \left(\frac{\partial}{\partial x_i} \left(a^{ij} u_{x^j} + \bar{b}^i u \right) + b^i u_{x^i} + cu + f \right) dt + \left(\sigma^{ik} u_{x^i} + \mu^k u + g^k \right) dZ_t^k \tag{1.1}$$

given for $\omega \in \Omega, t \geq 0$ and $x \in \mathbb{R}^d$. Here $\{Z_t^k,\, k = 1, 2 \ldots\}$ are independent one-dimensional Lévy processes defined on a probability space Ω, i and j go from 1 to d, k runs through $\{1, 2, \ldots\}$ with the summation convention on i, j, k being enforced. The coefficients a^{ij}, $\bar{b}^i$, b^i, c, σ^{ik}, μ^k and the free terms f, g^k depend on (ω, t, x).

Stochastic partial differential equations (SPDEs) of type (1.1) arise naturally in many applications, for instance in nonlinear filtering theory of partially observable diffusion processes, in relativistic quantum mechanics and population models with geographical structures (See, for instance, [13]).

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When Z_t^k are independent one-dimensional Wiener processes, the corresponding L_2-theory is well developed and an account of it can be found, for instance, in [12, 13]. Also an L_p-theory of equation (1.1) defined on $\mathbb{R}^d$ was first introduced by Krylov in [7]. Later Krylov's results were extended for SPDEs defined on domains of $\mathbb{R}^d$ (see, for instance, [8, 9, 5]). We referee the reader to [15] and the references therein for other work on SPDEs driven by continuous Banach space-valued processes.

SPDEs (1.1) can be viewed as stochastic differential equations (SDEs) in Banach spaces (see Remark 2.13 below). In [3], Gyöngy studied existence and uniqueness of Hilbert space-valued solution to the following type of SDEs in Banach spaces:

$$du = A(\omega, t, u(t))dN_t + B(\omega, t, u(t))dM_t, \tag{1.2}$$

where N is a non-decreasing real-valued process, M is a Hilbert space-valued quasi-left continuous local martingale, A and B are random operators defined on a reflexive Banach space V continuously and densely embedded into a Hilbert space H. See, for instance, [1, 2, 6, 11, 10] for other related works on Banach space-valued SDEs of type (1.2) driven by Poisson random measures or stable noises.

The purpose of this paper is two-fold. First we show in Section 2 that if each Z_t^k has finite second moment, that is, if

$$\int_{\mathbb{R}} z^2 \nu_k(dz) < \infty \quad \text{for every } k \geq 1, \tag{1.3}$$

where ν_k is the Lévy measure of Z_t^k, then for every $T > 0$, the equation (1.1) admits a unique solution in $\mathbb{H}_2^1(T)$, and the map $\mathcal{R} : (u_0, f, g^k) \to \mathcal{R}u$, where $\mathcal{R}u$ is the solution of (1.1) with initial date u_0, is continuous linear operator from $U_2^1 \times \mathbb{H}_2^{-1}(T) \times \mathbb{L}_2(T, \ell_2)$ to $\mathbb{H}_2^1(T)$ (see Section 2 for the definitions of these spaces). We point out that, while it is possible to deduce this result (Theorem 2.12) from the main results in Gyöngy [3] (see Remark 2.13 below), our approach in this paper is different from his. We use a direct martingale approach and a method of continuity, which may be of independent interest. We then give two extensions in Section 3. First we develop an L_2-theory for the semi-linear equation

$$du = \left(\frac{\partial}{\partial x_i}\left(a^{ij}u_{x^j} + \bar{b}^i u\right) + b^i u_{x^i} + cu + f(u)\right)dt + \left(\sigma^{ik}u_{x^i} + \mu^k u + g^k(u)\right)dZ_t^k,$$

under the condition that $f(u) = f(\omega, t, x, u), g^k(u) = g^k(\omega, t, x, u)$ satisfy **Assumption 3.1** below. Next we weaken the second moment condition (1.3) by assuming that it holds only for sufficiently large k (thus it can be dropped if only finitely many processes Z_t^k appear in the equation) and prove that the equation has a unique pathwise H_2^1-valued solution.

As usual, throughout this paper, $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, ..., x^d)$ and $B_r(x) := \{y \in \mathbb{R}^d : |x-y| < r\}$. For $i = 1, ..., d$, multi-indices $\alpha = (\alpha_1, ..., \alpha_d)$, $\alpha_i \in \{0, 1, 2, ...\}$, and functions $u(x)$, we set

$$u_{x^i} = \partial u/\partial x^i = D_i u, \quad D^\alpha u = D_1^{\alpha_1} \cdot ... \cdot D_d^{\alpha_d} u, \quad |\alpha| = \alpha_1 + ... + \alpha_d.$$

We also use the notation D^m for a partial derivative of order m with respect to x. If we write $c = c(\dots)$, it means that the constant c depends only on what are in parenthesis. For scalar functions f, g defined on $\mathbb{R}^d$, $(f,g) := \int_{\mathbb{R}^d} f(x)g(x)dx$.

2 L_2-theory for linear equations under condition (1.3)

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $(\mathcal{F}_t, t \geq 0)$ satisfying the usual condition. We assume that on Ω we are given independent one-dimensional Lévy processes $Z_t^1, Z_t^2, \dots$ relative to $\{\mathcal{F}_t, t \geq 0\}$. Let $\mathcal{P}$ be the predictable σ-field generated by $\{\mathcal{F}_t, t \geq 0\}$.

For $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, define

$$N_k(t, A) = \# \left\{ 0 \leq s \leq t;\, Z_s^k - Z_{s-}^k \in A \right\}, \quad \widetilde{N}_k(t, A) = N_k(t, A) - t\nu_k(A)$$

where $\nu_k(A) := \mathbb{E}[N_k(1, A)]$ is the Lévy measure of Z_t^k. By Lévy-Itô decomposition, there exist constants α_k, β_k and Brownian motion B_t^k so that

$$Z_t^k = \alpha_k t + \beta_k B_t^k + \int_{|z|<1} z\widetilde{N}_k(t, dz) + \int_{|z|\geq 1} zN_k(t, dz). \tag{2.1}$$

Assumption 2.1 (i) *For each $k \geq 1$,*

$$\widehat{c}_k := \left[\int_{\mathbb{R}} z^2 \nu_k(dz) \right]^{1/2} < \infty. \tag{2.2}$$

(ii) *There exist constants $\delta \in (0,1), K \in [1, \infty)$ so that for any $\omega \in \Omega$, $t > 0$ and $x \in \mathbb{R}^d$*

$$\delta|\xi|^2 \leq (a^{ij} - \alpha^{ij})\xi^i\xi^j \leq a^{ij}\xi^i\xi^j \leq K|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \tag{2.3}$$

where $\alpha^{ij} := \frac{1}{2}\sum_{k=1}^{\infty}(\widehat{c}_k^2 + \beta_k^2)\sigma^{ik}\sigma^{jk}$.

Due to (2.2), $\int_{|z|>1} |z|\nu_k(dz) < \infty$, and thus by absorbing $\int_{|z|>1} z\nu_k(dz)$ into α_k we can rewrite (2.1) as

$$Z_t^k = \tilde{\alpha}_k t + \beta_k B_t^k + \int_{\mathbb{R}} z\widetilde{N}_k(t, dz).$$

For $d \geq 1$, consider the equation for random function $u(t,x)$ on $\Omega \times [0,T] \times \mathbb{R}^d$:

$$du = \left(\frac{\partial}{\partial x_i}\left(a^{ij}u_{x^j} + \bar{b}^i u\right) + b^i u_{x^i} + cu + f \right) dt + \left(\sigma^{ik}u_{x^i} + \mu^k u + g^k \right) dZ_t^k \tag{2.4}$$

in the sense of distributions. See Definition 2.4 below. The coefficients $a^{ij}, \bar{b}^i$, b^i, c, σ^{ik}, μ^k and the free terms f, g^k are random functions depending on $t > 0$ and $x \in \mathbb{R}^d$. Without loss of generality we assume $\tilde{\alpha}_k = 0$, since otherwise we can rewrite (2.4) as follows :

$$du = \left(\frac{\partial}{\partial x_i}\left(a^{ij}u_{x^j} + \bar{b}^i u\right) + (b^i + \tilde{\alpha}_k\sigma^{ik})u_{x^i} + (c + \tilde{\alpha}_k\mu^k)u + f \right) dt + \left(\sigma^{ik}u_{x^i} + \mu^k u + g^k \right) d\tilde{Z}_t^k,$$

where $\tilde{Z}_t^k := Z_t^k - \tilde{\alpha}_k t$.

Remark 2.2 Conditions (2.2) and (2.3) will be weakened in section 3. In particular, one can completely drop the condition (2.2) if there are only finitely many processes Z_t^k in equation (2.4).

For $n = 0, 1, 2, ...$, define the Banach spaces

$$H_2^n := H_2^n(\mathbb{R}^d) = \left\{ u : D^\alpha u \in L_2(\mathbb{R}^d), |\alpha| \le n \right\}.$$

In general, for $\gamma \in \mathbb{R}$ define the space $H_2^\gamma = H_2^\gamma(\mathbb{R}^d) = (1-\Delta)^{-\gamma/2} L_2$ as the set of all distributions u on $\mathbb{R}^d$ such that $(1-\Delta)^{\gamma/2} u \in L_2$. For $u \in H_2^\gamma$, we define

$$\|u\|_{H_2^\gamma} := \|(1-\Delta)^{\gamma/2} u\|_{L_2} := \|\mathcal{F}^{-1}\left[(1+|\xi|^2)^{\gamma/2}\mathcal{F}(u)(\xi)\right]\|_{L_2(\mathbb{R}^d)}, \tag{2.5}$$

where $\mathcal{F}$ is the Fourier transform. If n is a nonnegative integer then the norm $\|u\|_{H_2^n}$ is equivalent to $\sum_{|\alpha| \le n} \|D^\alpha u\|_{L_2}$. Let $\mathcal{P}^{dP \times dt}$ be the completion of $\mathcal{P}$ with respect to $dP \times dt$. For $n \in \mathbb{Z}$ and $T > 0$, we write $u \in \mathbb{H}_2^n(T)$ if u is an H_2^n-valued $\mathcal{P}^{dP \times dt}$-measurable process defined on $\Omega \times [0, T]$ so that

$$\|u\|_{\mathbb{H}_2^n(T)} := \left(\mathbb{E}\left[\int_0^T \|u(t,\cdot)\|_{H_2^n}^2 \, dt \right] \right)^{1/2} < \infty.$$

Denote $\mathbb{L}_2(T) := \mathbb{H}_2^0(T)$. For an ℓ_2-valued process $g = (g^1, g^2, ...)$, we write $g \in \mathbb{L}_2(T, \ell_2)$ if $g^k \in \mathbb{L}_2(T)$ for every $k \ge 1$ and

$$\|g\|_{\mathbb{L}_2(T,\ell_2)} := \left(\sum_{k=1}^\infty (\beta_k^2 + \widehat{c}_k^2) \mathbb{E}\left[\int_0^T \|g^k\|_{L_2}^2 \, dt \right] \right)^{1/2} < \infty.$$

Finally we use U_2^1 to denote the family of $L_2(\mathbb{R}^d)$-valued $\mathcal{F}_0$-measurable random variables u_0 having

$$\|u_0\|_{U_2^1} := \left(\mathbb{E}\|u_0\|_{L_2}^2 \right)^{1/2} < \infty.$$

Remark 2.3 (i) Since we are assuming $\tilde{\alpha}_k = 0$, Z_t^k is a square integrable martingale, whose quadratic variation will be denoted by $[Z^k]$. For every process H in $L_2(\Omega \times [0,T])$, which has a predictable $dP \times dt$-version $\widetilde{H}$, $M_t := \int_0^t H_s dZ_s^k$ $(= \int_0^t \widetilde{H}_s dZ_s^k$ in $L_2(\Omega))$ is well defined and M_t is a martingale with

$$\mathbb{E}\left[M_t^2\right] = \mathbb{E}\left[\int_0^t H_s^2 \, d[Z^k]_s \right] = (\beta_k^2 + \widehat{c}_k^2)\, \mathbb{E}\left[\int_0^t H_s^2 ds \right], \quad t \le T.$$

(ii) For any $g = (g^1, g^2, \dots) \in \mathbb{L}_2(T, \ell_2)$ and $\phi \in C_0^\infty(\mathbb{R}^d)$, the finite sum $\sum_{k=1}^n \int_0^t (g^k, \phi) dZ_t^k$ is a square integrable martingale with quadratic variation $\sum_{k=1}^n (\beta_k^2 + \widehat{c}_k^2) \int_0^t (g^k, \phi)^2 ds$. Since,

$$\sum_{k=1}^\infty \mathbb{E} \int_0^T (\beta_k^2 + \widehat{c}_k^2)(g^k, \phi)^2 ds \le \|\phi\|_{L_2}^2 \|g\|_{\mathbb{L}_2(T,\ell_2)}^2 < \infty, \tag{2.6}$$

it follows that the series of stochastic integral $\sum_{k=1}^\infty \int_0^t (g^k, \phi)\, dZ_t^k$ converges uniformly on $t \in [0, T]$ in probability.

(iii) In many articles, the equations of the type

$$du = (Au + f)dt + g(u(t-))dZ_t$$

has been considered to study path-wise solutions. The expression $u(t-)$ is used because in those type of equations one requires the solution to be adapted and càdlàg. In this article we do not use such notation since we only require $g(u)$ has a predictable version.

Definition 2.4 *We say $u \in \mathcal{H}^1_2(T)$ if $u \in \mathbb{H}^1_2(T)$, $u(0) \in U^1_2$, u is right continuous with left limits in L_2 a.s., and for some $f \in \mathbb{H}^{-1}_2(T)$ and $g = (g^1, g^2, \dots) \in \mathbb{L}_2(T, \ell_2)$*

$$du(t) = f(t)dt + g^k(t)dZ^k_t \qquad \forall \quad 0 \le t \le T$$

in the sense of distributions, that is, for any $\phi \in C^\infty_0(\mathbb{R}^d)$, the equality

$$(u(t), \phi) = (u(0), \phi) + \int_0^t (f(s), \phi)ds + \int_0^t (g^k(s), \phi)dZ^k_s \tag{2.7}$$

holds for all $t \le T$ a.s.. In this case, we write

$$\mathbb{D}u := f, \quad \mathbb{S}^k u = g^k, \quad \mathbb{S}u := (\mathbb{S}^1 u, \mathbb{S}^2 u, \dots) = (g^1, g^2, \dots), \tag{2.8}$$

and define

$$\|u\|_{\mathcal{H}^1_2(T)} := \|u\|_{\mathbb{H}^1_2(T)} + \|\mathbb{D}u\|_{\mathbb{H}^{-1}_2(T)} + \|\mathbb{S}u\|_{\mathbb{L}_2(T,\ell_2)} + \|u(0)\|_{U^1_2}.$$

Lemma 2.5 *Let $u \in \mathcal{H}^1_2(T)$, then*

(i) *for any $\phi \in C^\infty_0(\mathbb{R}^d)$, $(u(t), \phi)$ is a progressively measurable right continuous process having left limits ;*

(ii) *for each fixed $t > 0$, $u(t) = u(t-)$ in L_2 a.s.*

Proof. (i). For any positive integer n, the process $\sum_{k=1}^n \int_t^t (g^k, \phi)dZ^k_s$ is progressively measurable, right continuous and has left limits. Thus the claim follows immediately from Remark 2.3(ii).

(ii). By assumption $u(t-)$ exists. Let $\{\phi_n, : \phi_n \in H^1_2, n = 1, 2, ...\}$ be an orthonormal basis in $L_2(\mathbb{R}^d)$. Then the process $t \mapsto (u(t-), \phi_n)$ is predictable by (i). Since $\int_0^t (g^k, \phi_n)dZ^k_t$ is stochastically continuous, we have for each fixed t and $n \ge 1$, $(u(t), \phi_n) = (u(t-), \phi_n)$ a.s. Therefore

$$u(t-) = \sum_n (u(t-), \phi_n)\phi_n = u(t) \quad \text{a.s.}$$

The lemma is now proved. □

Lemma 2.6 *For any integer n and $f \in \mathbb{H}^n_2(T)$ there exist $f_0, f_1, \dots, f_d \in \mathbb{H}^{n+1}_2(T)$ so that*

$$f = f_0 + D_i f_i, \quad \sum_{i=0}^d \|f_i\|_{\mathbb{H}^{n+1}_2(T)} \le N\|f\|_{\mathbb{H}^n_2(T)}.$$

Proof. This is a classical result and we give a proof only for the completeness. By definition (2.5) the map $(1-\Delta): H_2^{n+2} \to H_2^n$ is an isometry. Denote

$$f_0 = (1-\Delta)^{-1} f \qquad \text{and} \qquad f_i = -\frac{\partial f_0}{\partial x^i} \quad \text{for} \quad i = 1, 2, ..., d.$$

Then $f = (1-\Delta)(1-\Delta)^{-1} f = f_0 + D_i f_i$, and

$$\sum_{i=0}^{d} \|f_i\|_{\mathbb{H}_2^{n+1}(T)} \le N \|f_0\|_{\mathbb{H}_2^{n+2}(T)} \le N \|f\|_{\mathbb{H}_2^n(T)}.$$

□

Theorem 2.7 *The space $\mathcal{H}_2^1(T)$ is a Banach space, and for any $u \in \mathcal{H}_2^1(T)$ we have*

$$\mathbb{E}\left[\sup_{t \le T} \|u(t)\|_{L_2}^2\right] \le c\left(\|Du\|_{\mathbb{L}_2(T)}^2 + \|\mathbb{D}u\|_{\mathbb{H}_2^{-1}(T)}^2 + \|\mathbb{S}u\|_{\mathbb{L}_2(T,\ell_2)}^2 + \|u(0)\|_{U_2^1}^2\right), \tag{2.9}$$

where c is independent of u.

Proof. First we prove (2.9). Let $u(0) = u_0$ and $du = f dt + g^k dZ_t^k$. Then for any $\phi \in C_0^\infty(\mathbb{R}^d)$,

$$(u(t), \phi) = (u_0, \phi) + \int_0^t (f(s), \phi) ds + \int_0^t (g^k(s), \phi) dZ_t^k \tag{2.10}$$

for all $t \le T$ (a.s.). For $f \in \mathbb{H}_2^{-1}(T)$, by Lemma 2.6, we can write it as

$$f = f_0 + \frac{\partial}{\partial x_i} f_i$$

with $f_i \in \mathbb{L}_2(T)$ for $0 \le i \le d$ and

$$\sum_{i=0}^{d} \|f_i\|_{\mathbb{L}_2(T)} \le c \|f\|_{\mathbb{H}_2^{-1}(T)}. \tag{2.11}$$

For a moment, additionally assume that u, f, g, u_0 are infinitely differentiable in x, and therefore

$$u(t) = u_0 + \int_0^t f dt + \int_0^t g^k dZ_t^k, \quad \forall t \le T \, (a.s.). \tag{2.12}$$

The stochastic integral in (2.12) doesn't change if we replace g by its predictable version, thus we may assume that g is predictable.

Applying Ito's formula to $|u(t)|^2$ in (2.12) (see, for instance, Theorem 4.4.7 of [4]) and integrating over $\mathbb{R}^d$, we get

$$\begin{aligned}
\|u(t)\|_{L_2}^2 &= \|u_0\|_{L_2}^2 + 2\int_0^t (u(s), f(s))ds + \sum_k \beta_k^2 \int_0^t \|g^k(s)\|_{L_2}^2 ds \\
&\quad +2\sum_k \int_0^t (u(s-), g^k(s))dZ_s^k + \sum_k \sum_{0<s\le t} \|g^k(s)\Delta Z_s^k\|_{L_2}^2 \\
&= \|u_0\|_{L_2}^2 + 2\int_0^t \left((u(s),\, f_0(s)) - \sum_{i=1}^d (u_{x^i}(s), f_i(s)) \right) ds + \sum_k \beta_k^2 \int_0^t \|g^k(s)\|_{L_2}^2 ds \\
&\quad +2\sum_k \int_0^t (u(s-), g^k(s))dZ_s^k + \sum_k \sum_{0<s\le t} \|g^k(s)\Delta Z_s^k\|_{L_2}^2, \qquad (2.13)
\end{aligned}$$

where we have used the fact that Z^k's are independent and so with probability one at most one of the $Z_s^1, Z_s^2 \dots$ can jump at any given time. By virtue of the Lévy system of the Lévy process Z_s^k, it follows that

$$\sum_{0<s\le t} \|g^k(s)\Delta Z_s^k\|_{L_2}^2 = M_t^k + \hat{c}_k^2 \int_0^t \|g^k\|_{L_2}^2 ds, \qquad (2.14)$$

where M^k is a purely discontinuous square integrable martingale with

$$M_t^k - M_{t-}^k = \|g^k(t)\Delta Z_t^k\|_{L_2}^2 \qquad \text{for } t > 0.$$

For any $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le T} \left| \int_0^t \left((u(s), f_0(s)) - \sum_{i=1}^d (u_{x^i}(s), f_i(x)) \right) ds \right| \right] \\
&\le \varepsilon \|Du\|_{\mathbb{L}_2(T)}^2 + \varepsilon \mathbb{E} \sup_{t\le T} \|u(t)\|_{L_2}^2 + c(\varepsilon, T) \sum_{i=0}^d \|f^i\|_{\mathbb{L}_2(T)}^2.
\end{aligned}$$

By Davis's inequality,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T} |M_t^k| \right] &\le c\,\mathbb{E}\left[[M^k, M^k]_T^{1/2} \right] \le c\,\mathbb{E}\left[\sum_{0<t\le T} \|g^k(t)\Delta Z_t^k\|_{L_2}^2 \right] \\
&\le c\,\hat{c}_k^2\,\mathbb{E}\left[\int_0^T \|g^k(t)\|_{L_2}^2 dt \right],
\end{aligned}$$

and

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\sum_{k=1}^{\infty}\Big|\int_0^t (u(s-),g^k(s))dZ^k_s\Big|\right]\\
\le\ & c\sum_{k=1}^{\infty}\mathbb{E}\left[\left(\sum_{0<t\le T}(u(t-),g^k(t))^2(\Delta Z^k_t)^2\right)^{1/2}\right]\\
\le\ & c\sum_{k=1}^{\infty}\mathbb{E}\left[\sup_{t\le T}\|u(t)\|_{L_2}\left(\sum_{0<t\le T}\|g^k(t)\|^2_{L_2}(\Delta Z^k_t)^2\right)^{1/2}\right]\\
\le\ & \varepsilon\mathbb{E}\left[\sup_{t\le t}\|u(t)\|^2_{L_2}\right]+c(\varepsilon)\sum_{k=1}^{\infty}\mathbb{E}\left[\sum_{0<t\le T}\|g^k(t)\|^2_{L_2}(\Delta Z^k_t)^2\right]\\
\le\ & \varepsilon\mathbb{E}\left[\sup_{t\le T}\|u(t)\|^2_{L_2}\right]+c(\varepsilon)\sum_{k=1}^{\infty}\hat{c}_k^2\,\mathbb{E}\int_0^T\|g^k(t)\|^2_{L_2}\,dt.
\end{aligned}
$$

By choosing $\varepsilon>0$ sufficiently small, one gets

$$
\mathbb{E}\left[\sup_{t\le T}\|u(t)\|^2_{L_2}\right]\le c\left(\|u_0\|^2_{U^1_2}+\|Du\|^2_{\mathbb{L}_2(T)}+\|f_i\|^2_{\mathbb{L}_2(T)}+\|g\|^2_{\mathbb{L}_2(T,\ell_2)}\right). \tag{2.15}
$$

To drop the assumption that u,f,g,u_0 are sufficiently smooth in x, we take a nonnegative function $\psi\in C_0^\infty(B_1(0))$ with unit integral, and for $\varepsilon>0$ define $\psi_\varepsilon(x)=\varepsilon^{-d}\psi(x/\varepsilon)$. For any generalized function v, define $v^{(\varepsilon)}(x)=v*\psi_\varepsilon(x):=(v(\cdot),\psi_\varepsilon(x-\cdot))$, then $v^{(\varepsilon)}(x)$ is an infinitely differentiable function of x. By plugging $\psi_\varepsilon(x-\cdot)$ instead of ϕ in (2.10),

$$
u^{(\varepsilon)}(t,x)=u_0^{(\varepsilon)}(x)+\int_0^t(f_0^{(\varepsilon)}+D_if_i^{(\varepsilon)})dt+\int_0^t g^{k(\varepsilon)}dZ^k_t. \tag{2.16}
$$

By (2.15),

$$
\mathbb{E}\left[\sup_{t\le T}\|u^{(\varepsilon)}(t)\|^2_{L_2}\right]\le c\left(\|u_0^{(\varepsilon)}\|^2_{U^1_2}+\|Du^{(\varepsilon)}\|^2_{\mathbb{L}_2(T)}+\|f_i^{(\varepsilon)}\|^2_{\mathbb{L}_2(T)}+\|g^{(\varepsilon)}\|^2_{\mathbb{L}_2(T,\ell_2)}\right), \tag{2.17}
$$

and similarly by considering $u^{(\varepsilon)}-u^{(\varepsilon')}$ instead of $u^{(\varepsilon)}$,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\le T}\|u^{(\varepsilon)}(t)-u^{(\varepsilon')}(t)\|^2_{L_2}\right] &\le c\|u_0^{(\varepsilon)}-u_0^{(\varepsilon')}\|^2_{U^1_2}+c\|Du^{(\varepsilon)}-Du^{(\varepsilon')}\|^2_{\mathbb{L}_2(T)}\\
&+ c\|f_i^{(\varepsilon)}-f_i^{(\varepsilon')}\|^2_{\mathbb{L}_2(T)}+c\|g^{(\varepsilon)}-g^{(\varepsilon')}\|^2_{\mathbb{L}_2(T,\ell_2)}.
\end{aligned} \tag{2.18}
$$

By using the fact that for any $h\in L_2$, $\|h^{(\varepsilon)}\|_{L_2}\le\|h\|_{L_2}$ and $\|h^{(\varepsilon)}-h\|_{L_2}\to 0$ as $\varepsilon\to 0$, we get

$$
\|u^{(\varepsilon)}-u\|_{\mathbb{L}_2(T)}+\|Du^{(\varepsilon)}-Du\|_{\mathbb{L}_2(T)}+\|f_i^{(\varepsilon)}-f_i\|_{\mathbb{L}_2(T)}+\|g^{(\varepsilon)}-g\|_{\mathbb{L}_2(T,\ell_2)}+\|u_0^{(\varepsilon)}-u_0\|_{U^1_2}\to 0
$$

as $\varepsilon \to 0$. This and (2.18) show that there exists an L_2-valued càdlàg process v so that

$$\mathbb{E}\left[\sup_{t\le T}\|u^{(\varepsilon)}(t)-v(t)\|^2_{L_2}\right]\to 0 \quad \text{as} \quad \varepsilon\to 0.$$

Since $u^{(\varepsilon)}\to u$ in $\mathbb{H}^1_2(T)$ we conclude $v=u$, and we get (2.9) by letting $\varepsilon\to 0$ in (2.17).

Now we prove the completeness of the space $\mathcal{H}^1_2(T)$. Let $\{u_n : n=1,2,...\}$ be a Cauchy sequence in $\mathcal{H}^1_2(T)$. Let $f_n := \mathbb{D}u_n$, $g_n := \mathbb{S}u_n$ and $u_{n0} := u_n(0)$. Then there exist $u\in\mathbb{H}^1_2(T)$, $f\in\mathbb{H}^{-1}_2(T)$, $g\in\mathbb{L}_2(T,\ell_2)$ and $u_0\in U^1_2$ so that u_n, f_n, g_n and u_{n0} converge to u,f,g and u_0, respectively. Furthermore, by (2.9),

$$\mathbb{E}\left[\sup_{t\le T}\|u_n-u_m\|^2_{L_2}\right]\le c\|u_n-u_m\|^2_{\mathcal{H}^1_2(T)}\to 0$$

as $m,n\to\infty$. Also since $u^n\to u$ in $\mathbb{H}^1_2(T)$ as $n\to\infty$, it follows that $\mathbb{E}\left[\sup_{t\le T}\|u_n-u\|^2_{L_2}\right]\to 0$ as $n\to\infty$ and u is an L_2-valued càdlàd process.

Now let $\phi\in C^\infty_0(\mathbb{R}^d)$. Since (a.s.)

$$(u_n(t),\phi)=(u_{n0},\phi)+\int_0^t(f_n(s),\phi)ds+\int_0^t(g^k_n(s),\phi)dZ^k_s,\quad \forall t\le T,$$

taking $n\to\infty$, we have for each $t>0$,

$$(u(t),\phi)=(u_0,\phi)+\int_0^t(f(s),\phi)ds+\int_0^t(g^k(s),\phi)dZ^k_s \quad \text{a.s.} \tag{2.19}$$

Thus equality (2.19) holds almost everywhere in $\Omega\times[0,T]$. Using the fact that both sides of (2.19) are càdlàd processes, we conclude that equality (2.19) holds for all $t\le T$ (a.s.). Consequently $u\in\mathcal{H}^1_2(T)$ and $u^n\to u$ in $\mathcal{H}^1_2(T)$. □

Assumption 2.8 (i) *The coefficients $a^{ij},\bar{b}^i,b^i,c,\sigma^{ik}$ and μ^k are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable functions.*

(ii) *For each ω,t,x,i,j,*

$$|a^{ij}|+|\bar{b}^i|+|b^i|+|c|+\Big(\sum_{k=1}^\infty(\beta_k^2+\hat{c}_k^2)(|\sigma^{ik}|^2+|\mu^k|^2)\Big)^{1/2}\le K. \tag{2.20}$$

Theorem 2.9 *(A priori estimate) Let* **Assumptions 2.1 and 2.8** *hold. Then for any solution $u\in\mathcal{H}^1_2(T)$ of equation (2.4), we have*

$$\|u\|_{\mathcal{H}^1_2(T)}\le c\left(\|f\|_{\mathbb{H}^{-1}_2(T)}+\|g\|_{\mathbb{L}_2(T,\ell_2)}+\|u_0\|_{U^1_2}\right), \tag{2.21}$$

where $c=c(\delta,K,T)$.

Proof. We proceed as in the proof of Theorem 2.7. As before, rewrite $f \in \mathbb{H}_2^{-1}(T)$ as

$$f = f_0 + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} f_i \quad \text{with} \quad f_i \in \mathbb{L}_2(T)$$

and

$$\sum_{i=0}^{d} \|f_i\|_{\mathbb{L}_2(T)} \le c\|f\|_{\mathbb{H}_2^{-1}(T)}.$$

As in the proof of Theorem 2.7, without loss of generality, we may and do assume that u, f, g, u_0 are sufficiently smooth in x. By h^k we denote the predictable version of $\sigma^{ik}u_{x^i} + \mu^k u + g^k$. Applying Ito's formula to $|u(t)|^2$ in (2.4),

$$\begin{aligned}
\mathbb{E}\|u(t)\|_{L_2}^2 &= \mathbb{E}\|u_0\|_{L_2}^2 + 2\mathbb{E}\left[\int_0^t \left(-(a^{ij}u_{x^j} + \bar{b}^i u + f_i, u_{x^i}) + (b^i u_{x^i} + cu + f_0, u)\right) ds\right] \\
&\quad + \sum_k \beta_k^2 \int_0^t \|h^k\|_{L_2}^2\, ds + 2\mathbb{E}\left[\sum_k \int_0^t (h^k, u(s-))dZ_s^k\right] \\
&\quad + \sum_k \mathbb{E}\left[\sum_{0<s\le t} \|h^k \Delta Z_s^k\|_{L_2}^2\right].
\end{aligned} \tag{2.22}$$

By using $2ab \le \varepsilon a^2 + \varepsilon^{-1}b^2$, we get for any $\varepsilon > 0$

$$\begin{aligned}
\sum_k \mathbb{E}\beta_k^2 \int_0^t \|h^k\|_{L_2}^2\, ds &= \mathbb{E}\sum_k \int_0^t \beta_k^2 \|\sigma^{ik}u_{x^i} + \mu^k u + g^k\|_{L_2}^2 dt \\
&= \mathbb{E}\sum_k \int_0^t \int_{\mathbb{R}^d} \beta_k^2 \left(\sigma^{ik}\sigma^{jk}u_{x^i}u_{x^j} + 2\sigma^{ik}u_{x^i}(\mu^k u + g^k) + |\mu|_{\ell_2}^2|u|^2 + |g|_{\ell_2}^2\right) dxds \\
&\le 2\mathbb{E}\left[\int_0^t (\alpha_1^{ij}u_{x^i}, u_{x^j})_{L_2} ds\right] + \varepsilon\|Du\|_{\mathbb{L}_2(t)}^2 + c(\varepsilon)\|u\|_{\mathbb{L}_2(t)}^2 + c(\varepsilon)\|g\|_{\mathbb{L}_2(t,\ell_2)}^2,
\end{aligned}$$

where $\alpha_1^{ij} = \frac{1}{2}\sum_k \beta_k^2 \sigma^{ik}\sigma^{jk}$. Similarly,

$$\begin{aligned}
&\sum_k \mathbb{E}\left[\sum_{0<s\le t} \|h^k \Delta Z_s^k\|_{L_2}^2\right] \\
&= \sum_k \widehat{c}_k^2 \mathbb{E}\left[\int_0^t \|\sigma^{ik}u_{x^i} + \mu^k u + g^k\|_{L_2}^2 ds\right] \\
&\le 2\mathbb{E}\left[\int_0^t (\alpha_2^{ij}u_{x^i},\, u_{x^j}) ds\right] + \varepsilon\|Du\|_{\mathbb{L}_2(t)}^2 + c(\varepsilon)\|u\|_{\mathbb{L}_2(t)}^2 + c(\varepsilon)\|g\|_{\mathbb{L}_2(t,\ell_2)}^2
\end{aligned}$$

where $\alpha_2^{ij} = \frac{1}{2}\sum_k \widehat{c}_k^2 \sigma^{ik}\sigma^{jk}$. Also,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \left(-(\bar{b}^i u, u_{x^i}) - \sum_{i=1}^{d}(f_i, u_{x^i}) + (b^i u_{x^i} + cu + f_0, u)\right) ds\right] \\
&\le \varepsilon\|Du\|_{\mathbb{L}_2(t)}^2 + c(\varepsilon)\|u\|_{\mathbb{L}_2(t)}^2 + c(\varepsilon)\sum_{i=0}^{d} \|f_i\|_{\mathbb{L}_2(t)}^2.
\end{aligned}$$

Thus we have from (2.22) that for each $t \le T$,

$$\begin{aligned}
&\mathbb{E}\|u(t)\|_{L_2}^2 + 2\mathbb{E}\sum_{i,j=1}^{d}\int_0^t \big((a^{ij}-\alpha^{ij})u_{x^i},\, u_{x^j}\big)\,ds \\
\le\;\; & \|u_0\|^2_{U_2^1} + \varepsilon\|Du\|^2_{\mathbb{L}_2(t)} + c\left(\int_0^t \mathbb{E}\|u(s)\|^2_{L_2}ds + \sum_{i=0}^{d}\|f_i\|^2_{\mathbb{L}_2(t)} + \|g\|^2_{\mathbb{L}_2(t,\ell_2)}\right), \qquad (2.23)
\end{aligned}$$

where $c = c(\varepsilon, K)$ is independent of t. On the other hand, we know from condition (2.3) that

$$((a^{ij}-\alpha^{ij})u_{x^i},\, u_{x^j}) \ge \delta\|Du\|^2_{L_2}.$$

The above two displays together with Gronwall's inequality yield

$$\|u\|_{\mathbb{H}^1_2(T)} \le c\left(\|u_0\|_{U_2^1} + \|f\|_{\mathbb{H}_2^{-1}(T)} + \|g\|_{\mathbb{L}_2(T,\ell_2)}\right),$$

where $c = c(\delta, K, T)$. The theorem is proved. □

Remark 2.10 *The proof of Theorem 2.9 shows that if* $\bar{b}^i = b^i = c = \mu^k = 0$*, one can drop the term* $\int_0^t \mathbb{E}\|u\|^2_{L_2}$ *in (2.23), and therefore by taking* $\varepsilon < \delta/2$*,*

$$\|Du\|_{\mathbb{L}_2(T)} \le c\left(\|f\|_{\mathbb{H}_2^{-1}(T)} + \|g\|_{\mathbb{L}_2(T,\ell_2)} + \|u_0\|_{U_2^1}\right)$$

where c *is* **independent** *of* T*.*

To use the method of continuity we introduce linear operators L_λ and Δ_λ as follows: For $\lambda \in [0,1]$, denote

$$a^{ij}_\lambda = \lambda a^{ij} + (1-\lambda)\delta^{ij}, \quad \sigma^{ik}_\lambda = \lambda\sigma^{ik},$$

$$\bar{b}^i_\lambda = \lambda\bar{b}^i, \quad b^i_\lambda = \lambda b^i, \quad c_\lambda = \lambda c, \quad \mu^k_\lambda = \lambda\mu^k.$$

$$Lu = \frac{\partial}{\partial x_i}(a^{ij}u_{x^j} + \bar{b}^i) + b^i u_{x^i} + cu, \quad \Lambda^k u = \sigma^{ik}u_{x^i} + \mu^k u,$$

$$L_\lambda u := \lambda Lu + (1-\lambda)\Delta u = \frac{\partial}{\partial x_i}(a^{ij}_\lambda u_{x^j} + \bar{b}^i_\lambda) + b^i_\lambda u_{x^i} + c_\lambda u,$$

$$\Lambda^k_\lambda u := \lambda\Lambda^k u = \sigma^{ik}_\lambda u_{x^i} + \mu^k_\lambda u \qquad \text{for } k \ge 1.$$

Note that

$$L_{\lambda_1}u - L_{\lambda_2}u = (\lambda_1-\lambda_2)(L-\Delta)u, \qquad \Lambda_{\lambda_1}u - \Lambda_{\lambda_2}u = (\lambda_1-\lambda_2)\Lambda u,$$

where $\Lambda_\lambda u := (\Lambda^1_\lambda u, \Lambda^2_\lambda u, \dots)$, $\Lambda u := (\Lambda^1 u, \Lambda^2 u, \dots)$, and

$$\|L_{\lambda_1}u - L_{\lambda_2}u\|_{H_2^{-1}} + \|\Lambda_{\lambda_1}u - \Lambda_{\lambda_2}u\|_{L_2(\ell_2)} \le c(K)|\lambda_1-\lambda_2|\|u\|_{H_2^1}. \qquad (2.24)$$

Corollary 2.11 *There exists a constant $c = c(\delta, K, T)$ so that for any $\lambda \in [0,1]$ and any solution $u \in \mathcal{H}_2^1(T)$ of the equation*

$$du = (L_\lambda u + f)dt + (\Lambda_\lambda^k u + g^k)dZ_t^k, \quad u(0) = u_0$$

we have

$$\|u\|_{\mathcal{H}_2^1(T)} \le c \left(\|f\|_{\mathbb{H}_2^{-1}(T)} + \|g\|_{\mathbb{L}_2(T,\ell_2)} + \|u_0\|_{U_2^1} \right).$$

Proof. For $\lambda \in [0,1]$, denote $\alpha_\lambda^{ij} := \frac{1}{2}\sum_{k=1}^\infty (\widehat{c}_k^2 + \beta_k^2)\sigma_\lambda^{ik}\sigma_\lambda^{jk} = \lambda^2 \alpha^{ij}$. Then for any $\xi \in \mathbb{R}^d$,

$$(a_\lambda^{ij} - \alpha_\lambda^{ij})\xi^i\xi^j \le a_\lambda^{ij}\xi^i\xi^j = \lambda a^{ij}\xi^i\xi^j + (1-\lambda)|\xi|^2 \le K|\xi|^2,$$

$$\delta|\xi|^2 \le \lambda(a^{ij} - \alpha^{ij})\xi^i\xi^j + (1-\lambda)|\xi|^2 \le \left(\lambda a^{ij} + (1-\lambda)\delta^{ij} - \lambda^2\alpha^{ij}\right)\xi^i\xi^j = (a_\lambda^{ij} - \alpha_\lambda^{ij})\xi^i\xi^j.$$

Also, the new coefficients $a_\lambda^{ij}, \bar{b}_\lambda^i, b_\lambda^i, c_\lambda, \sigma_\lambda^{ik}, \nu_\lambda^k$ satisfy (2.3) and (2.20). Now the assertion of the corollary follows from Theorem 2.9.

□

Here is the main result of this section.

Theorem 2.12 *Suppose* **Assumptions 2.1** *and* **2.8** *hold. Then for any $f \in \mathbb{H}_2^{-1}(T)$, $g \in \mathbb{L}_2(T,\ell_2)$ and $u_0 \in U_2^1$, equation (2.4) with initial data u_0 has a unique solution $u \in \mathcal{H}_2^1(T)$, and*

$$\|u\|_{\mathcal{H}_2^1(T)} \le c(\|f\|_{\mathbb{H}_2^{-1}(T)} + \|g\|_{\mathbb{L}_2(T,\ell_2)} + \|u_0\|_{U_2^1}), \tag{2.25}$$

where $c = c(\delta, K, T)$.

Proof. In view of the a priori estimate (2.21), it suffices to show that there is a solution to (2.4).

Step 1. We show that for any given $f \in \mathbb{H}_2^{-1}(T), g \in \mathbb{L}_2(T,\ell_2)$ and $u_0 \in U_2^1$, the equation

$$du = (\Delta u + f)dt + g^k dZ_t^k, \quad u(0) = u_0 \tag{2.26}$$

has a solution $u \in \mathcal{H}_2^1(T)$. For a moment, assume that $g^k = 0$ for all $k \ge N$ for some $N \ge 1$, and each g^k is of the type

$$g^k(t) = \sum_{i=1}^{m(k)} I_{(\tau_{i-1},\tau_i]}(t)\varphi_i(x), \tag{2.27}$$

where τ_i are bounded stopping times and $\varphi_i \in C_0^\infty(\mathbb{R}^d)$. Define

$$v(t) = \sum_{k=1}^N \int_0^t g^k(s)dZ_s^k.$$

Then it is easy to see that $v \in \mathcal{H}_2^1(T)$. Note that u satisfies (2.26) if and only if $\bar{u} := u - v$ satisfies

$$d\bar{u} = (\Delta\bar{u} + \Delta v + f)dt \qquad \text{with} \quad \bar{u}(0) = u_0.$$

Since this equation has a solution in $\mathcal{H}_2^1(T)$ (see Theorem 5.1 in [7]), we conclude that equation (2.26) has a solution u in $\mathcal{H}_2^1(T)$.

In general, by Theorem 3.10 of [7], we can take a sequence $g_n \in \mathbb{L}_2(T, \ell_2)$ so that $\|g_n - g\|_{\mathbb{L}_2(T,\ell_2)} \to 0$ as $n \to \infty$, $g_n^k = 0$ for $k \geq N(n)$ and g_n^k are of type (2.27). By the above result we can define $u_n \in \mathcal{H}_2^1(T)$ as the solution of

$$du_n = (\Delta u_n + f)dt + g_n^k dZ_t^k, \quad u(0) = u_0,$$

and by Theorem 2.9 (or Corollary 2.11)

$$\|u_n - u_m\|_{\mathcal{H}_2^1(T)} \leq c\|g_n - g_m\|_{\mathbb{L}_2(T,\ell_2)} \to 0$$

as $n, m \to \infty$. Now it is clear the limit of this Cauchy sequence is a solution of (2.26).

Step 2. Let $J \subset [0, 1]$ denote the set of λ, so that for any f, g, u_0, the equation

$$du = (L_\lambda u + f)dt + (\Lambda_\lambda^k u + g^k)dZ_t^k, \quad u(0) = u_0 \tag{2.28}$$

has a solution $u \in \mathcal{H}_2^1(T)$. Then as proved above, $0 \in J$. Now assume $\lambda_0 \in J$, and note that u is a solution of equation (2.28) if and only if

$$du = (L_{\lambda_0} u + (L_\lambda u - L_{\lambda_0} u + f))\, dt + \Big(\Lambda_{\lambda_0}^k u + (\Lambda_\lambda^k u - \Lambda_{\lambda_0}^k u + g^k)\Big)\, dZ_t^k. \tag{2.29}$$

Remember that $D : H_2^n \to H_2^{n-1}$ is a bounded operator. Thus for any $u \in \mathcal{H}_2^1(T)$, $k \geq 1$ and $\lambda \in [0, 1]$, we have

$$L_\lambda u \in \mathbb{H}_2^{-1}(T) \qquad \text{and} \qquad \Lambda_\lambda u \in \mathbb{L}_2(T, \ell_2).$$

Recall $\lambda_0 \in J$. Denote $u^0 = u_0$ and for $n = 1, 2, \dots$ we define $u^{n+1} \in \mathcal{H}_2^1(T)$ as the solution of the equation

$$du^{n+1} = (L_{\lambda_0} u^{n+1} + f_n)dt + (\Lambda_{\lambda_0}^k u^{n+1} + g_n^k)dZ_t^k, \quad u^{n+1}(0) = u_0$$

where

$$f_n := L_\lambda u^n - L_{\lambda_0} u^n + f \quad \text{and} \quad g_n^k := \Lambda_\lambda^k u^n - \Lambda_{\lambda_0}^k u^n + g^k.$$

By Corollary 2.11 and the inequality (2.24), we have

$$\begin{aligned}
\|u^{n+1} - u^n\|_{\mathcal{H}_2^1(T)} &\leq c\|(L_\lambda - L_{\lambda_0})(u^n - u^{n-1})\|_{\mathbb{H}_2^{-1}(T)} + c\|(\Lambda_\lambda - \Lambda_{\lambda_0})(u^n - u^{n-1})\|_{\mathbb{L}_2(T,\ell_2)} \\
&\leq c\|\lambda - \lambda_0\|\|u^n - u^{n-1}\|_{\mathbb{H}_2^1(T)}.
\end{aligned}$$

Let $\varepsilon_0 = 1/(2c)$. Then for $\lambda \in (\lambda_0 - \varepsilon_0, \lambda + \varepsilon_0)$, $\|u^{n+1} - u^n\|_{\mathcal{H}_2^1(T)} \leq \frac{1}{2}\|u^n - u^{n-1}\|_{\mathcal{H}_2^1(T)}$ for every $n \geq 1$ and so u^n converges to some u in $\mathcal{H}_2^1(T)$. It follows that u solves equation (2.29). This proves that $(\lambda_0 - \varepsilon_0, \lambda_0 + \varepsilon_0) \cap [0, 1] \subset J$. Consequently we conclude $J = [0, 1]$. □

Remark 2.13 Theorem 2.12 can be deduced from the main results in Gyöngy [3] by regarding the SPDE (1.1) as a Hilbert space-valued SDE of type (1.2). To see this, by using the same notation as that in [3], let H denote the Hilbert space $L_2(\mathbb{R}^d)$, $V = H_2^1(\mathbb{R}^d)$ and E the Hilbert space ℓ^2 of all square summable real-valued sequences with the usual orthonormal basis $\{e_k, k = 1, 2, \dots\}$. Here V is viewed as a separable reflexive Banach space which is embedded continuously and densely into the Hilbert space H. Let V^* be the dual space of V. By identifying Hilbert space H with its dual H^*, we have

$$V \subset H \equiv H^* \subset V^*.$$

Assume that **Assumptions 2.1 and 2.8** hold. Define the E-valued martingale

$$Z_t := \sum_{k=1}^{\infty} \frac{1}{(\beta_k^2 + \widehat{c}_k^2)^{1/2}(1+k^2)^{1/2}} Z_t^k e_k.$$

Let Q be the linear operator from E to itself defined by $Qe_k = \frac{1}{1+k^2} e_k$. We now define three linear operators from $V \subset H$ into $L(E, H)$, the space of bounded linear operators from E ro H.

$$B_0(\omega, t, \cdot): \quad u \mapsto \left((z^k) \mapsto \sum_{k=1}^{\infty} c_k \sqrt{1+k^2} \mu^k(t) u z^k e_k \right)$$

$$B_i(\omega, t, \cdot): \quad u \mapsto \left((z^k) \mapsto \sum_{k=1}^{\infty} c_k \sqrt{1+k^2} \sigma^k(t) u_{x^i} z^k e_k \right), \qquad i = 1, 2, \dots, d,$$

$$B_{d+1}(\omega, t, \cdot): \quad u \mapsto \left((z^k) \mapsto \sum_{k=1}^{\infty} c_k \sqrt{1+k^2} g^k(t) z^k e_k \right).$$

It is easy to verify that each $B_j Q^{1/2}$ is a Hilbert-Schmidt operator from E to H with $j = 0, 1, \dots, d+1$. For $v \in V$, let

$$A(\omega, t, v) = \frac{\partial}{\partial x_i} \left(a^{ij} v_{x^j} + \bar{b}^i v \right) + b^i v_{x^i} + cv + f,$$

which is viewed as an element in V^*. Then SPDE (1.1) can be rewritten as an SDE in Hilbert space:

$$du(t) = A(\omega, t, u(t))dt + \left(\sum_{j=0}^{d+1} B_j(\omega, t, u(t)) \right) dZ_t.$$

Assumptions 2.1 and 2.8 imply that the conditions **(I)**-**(V)** on [3, page 235] are satisfied. Now our Theorem 2.12 follows from Theorems 2.9, 2.10 and 4.1 of Gyöngy [3]. The approach in [3] is different from ours. □

For a stopping time τ relative to $\{\mathcal{F}_t\}$, denote

$$(\!(0, \tau]\!] := \{(\omega, t) : 0 < t \leq \tau(\omega)\}.$$

Then obviously the process $1_{(\!(0,\tau]\!]}(\omega,t)$ is left-continuous and predictable. Actually by definition the predictable σ-field $\mathcal{P}$ is the σ-field generated by all such processes. For an H^1_2-valued $\mathcal{P}^{dP\times dt}$-measurable process u, write $u\in\mathbb{H}^1_2(\tau)$ if

$$\|u\|^2_{\mathbb{H}^1_2(\tau)} := \mathbb{E}\left[\int_0^\tau \|u\|^2_{H^1_2}ds\right] < \infty.$$

We define the Banach spaces $\mathbb{L}_2(\tau,\ell_2)$ and $\mathcal{H}^1_2(\tau)$ similarly. The following theorem plays the key role when we weaken condition (2.2) later in the next section.

Theorem 2.14 *Suppose that τ is a stopping time bounded by T. Theorem 2.12 holds with the deterministic time T replaced by the stopping time τ.*

Proof. First we prove the existence. As mentioned above $1_{(\!(0,\tau]\!]}$ is predictable and therefore

$$\bar{f} := 1_{(\!(0,\tau]\!]}\, f \in \mathbb{H}^{-1}_2(T), \quad \bar{g} := 1_{(\!(0,\tau]\!]}\, g \in \mathbb{L}_2(T,\ell_2).$$

Let $u\in\mathcal{H}^1_2(T)$ be the solution of (2.4) with $\bar{f}$ and $\bar{g}$ instead of f and g respectively. Then, since $\tau\le T$, we have $u\in\mathcal{H}^1_2(\tau)$ and

$$\begin{aligned}\|u\|_{\mathcal{H}^1_2(\tau)} \le \|u\|_{\mathcal{H}^1_2(T)} &\le c\left(\|\bar{f}\|_{\mathbb{H}^{-1}_2(T)} + \|\bar{g}\|_{\mathbb{L}_2(T,\ell_2)} + \|u_0\|_{U^1_2}\right)\\ &= c\left(\|f\|_{\mathbb{H}^{-1}_2(\tau)} + \|g\|_{\mathbb{L}_2(\tau,\ell_2)} + \|u_0\|_{U^1_2}\right).\end{aligned}$$

Now we prove the uniqueness. Let $u\in\mathcal{H}^1_2(\tau)$ be a solution of equation (2.4). Then since $\tau\le T$,

$$1_{(\!(0,\tau]\!]}\cdot(\mathbb{D}u-\Delta u)\in\mathbb{H}^{-1}_2(T), \quad 1_{(\!(0,\tau]\!]}\cdot\mathbb{S}u\in\mathbb{L}_2(T,\ell_2).$$

See (2.8) for the definitions of $\mathbb{D}u,\mathbb{S}u$ and $\mathbb{S}^k u$. According to Theorem 2.12 we can define $v\in\mathcal{H}^1_2(T)$ as the solution of

$$dv = (\Delta v + 1_{(\!(0,\tau]\!]}\,(\mathbb{D}u-\Delta u))dt + 1_{(\!(0,\tau]\!]}\,\mathbb{S}^k u\; dZ^k_t, \quad v(0)=u(0). \tag{2.30}$$

Then for $t\le\tau$, $d(u-v)=\Delta(u-v)dt$ and therefore using a classical result for the heat equation (see, for instance, Theorem 5.1 of [7]), we conclude that $u(t)=v(t)$ for all $t\le\tau$, a.s.. Thus, equation (2.4) becomes (just replace u by v for $t\le\tau$)

$$\begin{aligned} dv &= \left(\frac{\partial}{\partial x^i}\left(\tilde{a}^{ij}v_{x^j}+\tilde{\bar{b}}^i v\right)+\tilde{b}^i v_{x^i}+\tilde{c}v+f1_{(\!(0,\tau]\!]}\right)dt\\ &\quad+\left(\tilde{\sigma}^{ik}v_{x^i}+\tilde{\mu}^k v+g^k 1_{(\!(0,\tau]\!]}\right)dZ^k_t, \end{aligned}\tag{2.31}$$

where

$$\tilde{a}^{ij}=a^{ij}1_{(\!(0,\tau]\!]}+\delta^{ij}(1-1_{(\!(0,\tau]\!]}), \quad \tilde{\bar{b}}^i=\bar{b}^i 1_{(\!(0,\tau]\!]}, \quad \tilde{b}^i=b^i 1_{(\!(0,\tau]\!]}, \quad \ldots,\tilde{\mu}^k=\mu^k 1_{(\!(0,\tau]\!]}.$$

Note that since finite sum or product of predictable functions is predictable, these new coefficients are predictable, and obviously they satisfy (2.3) and (2.20). Thus it follows from Theorem 2.12 that v is the **unique** solution of equation (2.31) in the class $\mathcal{H}_2^1(T)$. We proved that if $u \in \mathcal{H}_2^1(\tau)$ is a solution of equation (2.4) then $u(t) = v(t)$ for all $t \le \tau$. This proves the uniqueness of solution of equation (2.4) in the class $\mathcal{H}_2^1(\tau)$. The theorem is proved. $\square$

3 Further Results

In this section we give two extensions of Theorem 2.12. First, we consider the nonlinear equation

$$\begin{aligned} du &= \left(\frac{\partial}{\partial x_i} \left(a^{ij} u_{x^j} + \bar{b}^i u \right) + b^i u_{x^i} + cu + f(u) \right) dt \\ &\quad + \left(\sigma^{ik} u_{x^i} + \mu^k u + g^k(u) \right) dZ_t^k, \end{aligned} \tag{3.1}$$

where $f(u) = f(\omega, t, x, u)$ and $g^k(u) = g^k(\omega, t, x, u)$.

Assumption 3.1 (i) For any $u \in H_2^1$,

$$f(u) \in H_2^{-1} \quad \text{and} \quad g(u) := (g^1(u), g^2(u), \dots) \in L_2(\ell_2).$$

(ii) For every $\varepsilon > 0$, there exists a constant $K_1 = K_1(\varepsilon)$ so that for any $u, v \in \mathbb{H}_2^1(T)$ and $t \le T$,

$$\|f(u) - f(v)\|^2_{\mathbb{H}_2^{-1}(t)} + \|g(u) - g(v)\|^2_{\mathbb{L}_2(t,\ell_2)} \le \varepsilon \|u - v\|^2_{\mathbb{H}_2^1(t)} + K_1 \|u - v\|^2_{\mathbb{L}_2(t)}. \tag{3.2}$$

Theorem 3.2 *Suppose* **Assumptions 2.1**, **2.8** *and* **3.1** *hold. Then for any* $u_0 \in U_2^1$, *equation* (3.1) *with initial data* u_0 *has a unique solution* $u \in \mathcal{H}_2^1(T)$, *and*

$$\|u\|_{\mathcal{H}_2^1(T)} \le c \left(\|f(0)\|_{\mathbb{H}_2^{-1}(T)} + \|g(0)\|_{\mathbb{L}_2(T,\ell_2)} + \|u_0\|_{U_2^1} \right) \tag{3.3}$$

where $f(0) = f(\omega, t, x, 0), g(0) = g(\omega, t, x, 0)$ *and* $c = c(\delta, K, T)$.

Proof. We will use a fixed point theorem to show the existence and uniqueness of the solution to (3.1). Estimate (3.3) follows from (2.21), (3.2), (2.9) and the Gronwall's inequality. Let $\mathcal{R}(f, g) \in \mathcal{H}_2^1(T)$ denote the solution of (2.4) with initial data u_0. Then by Theorem 2.12,

$$\mathcal{R}u := \mathcal{R}(f(u), g(u)) \qquad \text{for } u \in \mathcal{H}_2^1(T)$$

is well defined and $\mathcal{R}$ is a map from $\mathcal{H}_2^1(T)$ to $\mathcal{H}_2^1(T)$. Define $u^0 = \mathcal{R}(f(0), g(0))$ and $u^{n+1} = \mathcal{R}(f(u^n), g(u^n))$. Then by Theorem 2.12 and (3.2), for any $t \le T$,

$$\begin{aligned} \|\mathcal{R}u - \mathcal{R}v\|^2_{\mathcal{H}_2^1(t)} &\le c\varepsilon \|u - v\|^2_{\mathcal{H}_2^1(t)} + cK_1 \|u - v\|^2_{\mathbb{L}_2(t)} \\ &\le c\varepsilon \|u - v\|^2_{\mathcal{H}_2^1(t)} + cK_1 \int_0^t \|u - v\|^2_{\mathcal{H}_2^1(s)} ds \end{aligned}$$

where the last inequality is from (2.9). The proof of Theorem 6.4 in [7] implies that $\mathcal{R}^m$ is a contraction in $\mathcal{H}_2^1(T)$ with the coefficient 1/2 for all sufficiently large m, that is, $\|\mathcal{R}^m u - \mathcal{R}^m v\|_{\mathcal{H}_2^1(T)} < 1/2\|u-v\|_{\mathcal{H}_2^1(T)}$. This yields all the assertions of the theorem. The theorem is proved. □

Here is an application of Theorem 3.2 to SPDEs with the fractional Laplacian.

Example 3.3 *For simplicity assume $g^k(u) = 0$ for $k \geq 2$. Take $f(u) = (-\Delta)^{\alpha/2}u$ and $g(u) = g^1(u) = (-\Delta)^{\beta/2}u$ where $\alpha < 2$ and $\beta < 1$, then obviously for any $\varepsilon > 0$,*

$$\|f(u)-f(v)\|^2_{\mathbb{H}_2^{-1}(t)} + \|g(u)-g(v)\|^2_{\mathbb{L}_2(t)} \leq c\|u-v\|^2_{\mathbb{H}_2^{-1+\alpha}(t)} + c\|u-v\|^2_{\mathbb{H}_2^{\beta}(t)}$$

$$\leq \varepsilon\|u-v\|^2_{\mathbb{H}_2^1(t)} + K_1\|u-v\|^2_{\mathbb{L}_2(t)},$$

where for the second inequality we use the following classical fact (see section 2.4.7 of [14]): if $\gamma = \kappa\gamma_1 + (1-\kappa)\gamma_0$ and $\kappa \in [0,1]$ then $\|u\|_{H_2^\gamma} \leq N\|u\|^{\kappa}_{H_2^{\gamma_1}}\|u\|^{1-\kappa}_{H_2^{\gamma_0}}$. Thus the existence and uniqueness of equation (3.1) with $f(u)$ and $g(u)$ given as above are guaranteed by Theorem 3.2.

The following is a weakened version of **Assumption 2.1**.

Assumption 3.4 *There exists an integer $N_0 \geq 1$ so that*
(i) *$\widehat{c}_k < \infty$ for all integer $k > N_0$;*
(ii) *for some $\delta > 0$,*

$$\delta|\xi|^2 < (a^{ij} - \alpha^{ij}_{N_0})\xi^i\xi^j, \qquad \forall \xi \in \mathbb{R}^d, \tag{3.4}$$

where $\alpha^{ij}_{N_0} := \frac{1}{2}\sum_{k=N_0+1}^{\infty}(\widehat{c}_k^2 + \beta_k^2)\sigma^{ik}\sigma^{jk}$.

For a stopping time $\tau \leq T$, write $u \in \mathbb{H}^1_{2,\mathrm{loc}}(\tau)$ if there exists a sequence of stopping times $\tau_n \uparrow \infty$ so that $u \in \mathbb{H}_2^1(\tau \wedge \tau_n)$ for each n. Here is our second extension.

Theorem 3.5 *Let* **Assumption 3.4** *hold and $\sigma^i_k = 0$ for $k \leq N_0$. Then for any $u_0 \in U_2^1$, $f \in \mathbb{H}_2^{-1}(T)$ and process $g = (g^1, g^2, \dots)$ having entries in $\mathbb{L}_2(T)$ so that $\sum_{N_0+1}^{\infty}(\widehat{c}_k^2 + \beta_k^2)\|g^k\|^2_{\mathbb{L}_2(T)} < \infty$, there exists a unique $u \in \mathbb{H}^1_{2,\mathrm{loc}}(T)$ such that*
(i) *$u(t)$ is right continuous with left limits in L_2 a.s.,*
(ii) *for any $\phi \in C_0^\infty(\mathbb{R}^d)$, the equality*

$$\begin{aligned}(u(t),\phi) &= (u_0,\phi) + \int_0^t \big((-a^{ij}u_{x^j} - \bar{b}^i u,\, \phi_{x^i}) + (b^i u_{x^i} + cu + f,\, \phi)\big)\, ds \\ &\quad + \int_0^t \Big((\sigma^{ik}u_{x^i},\phi) + (\mu^k,\phi) + (g^k,\phi)\Big)\, dZ^k_s \end{aligned} \tag{3.5}$$

holds for all $t \leq T$ a.s..

We say that $u \in \mathbb{H}^1_{2,\mathrm{loc}}(\tau)$ is a path-wise solution if u satisfies the conditions (i) and (ii) in the theorem for $t < \tau$.

Proof. Step 1. First assume that **Assumption 2.1** holds. Then the existence of pathwise solution under **Assumption 2.1** in $\mathbb{H}^1_2(\tau)$ (hence in $\mathbb{H}^1_{2,\mathrm{loc}}(\tau)$) follows from Theorem 2.14. Now we show that the pathwise solution is unique in $\mathbb{H}^1_{2,\mathrm{loc}}(\tau)$. Let $u \in \mathbb{H}^1_{2,\mathrm{loc}}(\tau)$ be a path-wise solution. Define $\tau_n = \tau \wedge \inf\{t : \int_0^t \|u\|^2_{H^1_2} ds > n\}$. Then $u \in \mathbb{H}^1_2(\tau_n)$ and $\tau_n \uparrow \tau$ since $\int_0^t \|u\|^2_{H^1_2} ds < \infty$ for all $t < \tau$, a.s. By Theorem 2.14,

$$\|u\|_{\mathbb{H}^1_2(\tau_n)} \leq c(T, d, K)(\|f\|_{\mathbb{H}^{-1}_2(\tau_n)} + \|g\|_{\mathbb{L}_2(\tau_n, \ell_2)} + \|u(0)\|_{U^1_2}).$$

By letting $n \to \infty$ we find that $u \in \mathcal{H}^1_2(\tau)$, and the uniqueness of the pathwise solution under **Assumption 2.1** follows from the uniqueness result of Theorem 2.14.

Step 2. For the general case, note that for each $n > 0$ and $k \leq N_0$,

$$\widehat{c}_{k,n} := \left(\int_{\{z \in \mathbb{R} : |z| \leq n\}} |z|^2 \nu_k(dz) \right)^{1/2} < \infty.$$

Consider Lévy processes $(Z^1_n, \dots, Z^{N_0}_n, Z^{N_0+1}, \dots)$ in place of $(Z^1, Z^2 \dots)$, where $Z^k_n (k \leq N_0)$ is obtained from Z^k by removing all the jumps that has absolute size strictly large than n. Note that condition (2.3) is valid with $\widehat{c}_k$ replaced by $\widehat{c}_{k,n}$ since σ^{ik} are assumed to be zero for all $k \leq N_0$.

By Step 1, there is a unique pathwise solution $v_n \in \mathcal{H}^1_2(T)$ with Z^k_n in place of Z^k for $k = 1, 2, \dots, N_0$. Let T_n be the first time that one of the Lévy processes $\{Z^k, 1 \leq k \leq N_0\}$ has a jump of (absolute) size in (n, ∞). Define $u(t) = v_n(t)$ for $t < T_n \wedge T$. Note that for $n < m$, by Step 1, we have $v_n(t) = v_m(t)$ for $t < T_n$. This is because, for $t < T_n$, both v_n and v_m satisfy (3.5) with each term inside the stochastic integral multiplied by $1_{s < T_n}$ (and with Z^k_n, $k \leq N_0$, in place of Z^k). Thus u is well defined. By letting $n \to \infty$, one constructs a unique pathwise solution u in $\mathbb{H}^1_{2,loc}(T)$. The theorem is proved.

□

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