

Uniform integrability of exponential martingales and spectral bounds of non-local Feynman-Kac semigroups

Zhen-Qing Chen*

Dedicated to Professor Jiaan Yan on the occasion of his 70th birthday

Abstract

In the first part of this paper, we give a useful criterion for uniform integrability of exponential martingales in the context of Markov processes. The condition of this criterion is easy to verify and is, in general, much weaker than the commonly used Novikov's condition. In the second part of this paper, we present a new approach to the study of spectral bounds of Feynman-Kac semigroups for a large class of symmetric Markov processes. We first establish criteria for the L^p -independence of spectral bounds for Feynman-Kac semigroups generated by continuous additive functionals, using gaugeability results obtained by the author in [3]. We then extend these analytic criteria for the L^p -independence of spectral bounds to non-local Feynman-Kac semigroups via pure jump Girsanov transforms. For this, the uniform integrability of the exponential martingales established in the first part of this paper plays an important role. We use it to show that Kato classes introduced in [3] can only become larger under pure jump Girsanov transforms with symmetric jumping functions.

Keywords: Feynman-Kac transform; Girsanov transform; quadratic form; smooth measure; additive functionals, Kato class; spectral bound; Lévy system; gaugeability; spectral bound

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1 Introduction

Feynman-Kac transform is one of the most important transforms for Markov processes. Suppose that E is a Lusin space (i.e., a space that is homeomorphic to a Borel subset of a compact metric space) and $\mathcal{B}(E)$ denotes the Borel σ -algebra on E . Let m be a Borel σ -finite measure on E with $\text{supp}[m] = E$ and $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}_x, x \in E)$ be an m -symmetric irreducible Borel standard process on E with lifetime ζ (cf. Sharpe [15] for the terminology). For a continuous additive functional A of X having finite variations, one can do Feynman-Kac transform:

$$T_t f(x) = \mathbb{E}_x [e^{A_t} f(X_t)], \quad t \geq 0.$$

It is easy to check (see [1]) that, under suitable Kato class condition on A , $\{T_t; t \geq 0\}$ forms a strongly continuous symmetric semigroup on $L^p(E; m)$ for every $1 \leq p \leq \infty$ and that its L^2 -infinitesimal generator is $\mathcal{L}^\mu := \mathcal{L} + \mu$, where \mathcal{L} is the L^2 -infinitesimal generator of the process X

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and μ is the (signed) Revuz measure for the continuous additive functional A . To emphasize the correspondence between continuous additive functionals and Revuz measures, let's denote A by A^μ . When the process X is discontinuous, it has many discontinuous additive functionals. Let F be a symmetric function on $E \times E$ that vanishes along the diagonal d of $E \times E$. We always extend it to be zero off $E \times E$. Then $\sum_{0 < s \leq t} F(X_{s-}, X_s)$, whenever it is summable, is an additive functional of X . Hence one can also perform non-local Feynman-Kac transform

$$T_t^{\mu, F} f(x) := \mathbb{E}_x \left[\exp \left(A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s) \right) f(X_t) \right], \quad t \geq 0. \quad (1.1)$$

Non-local Feynman-Kac transforms have been investigated in [3, 4, 7, 8]. Let $(N(x, dy), H_t)$ be a Lévy system of X (see Section 2 for its definition). The infinitesimal generator for $\{T_t^{\mu, F}; t \geq 0\}$ of (1.1) is (see Corollary 4.9 and Remark 1 of [7])

$$\mathcal{L}^{\mu, F} := \mathcal{L} + \mu_H \mathbf{F} + \mu,$$

where μ_H is the Revuz measure of the positive continuous additive functional H and

$$\mu_H \mathbf{F} f(dx) := \left(\int_E \left(e^{F(x, y)} - 1 \right) f(y) N(x, dy) \right) \mu_H(dx).$$

Since in this paper we are only concerned with behavior of the Schrödinger semigroup $\{T_t^{\mu, F}; t \geq 0\}$, by considering the 1-subprocess of X if necessary, without loss of generality, we may assume that X is transient; see Remark 4.2 and (5.9) below. Under some suitable Kato class conditions on the Revuz measure μ and the function F , $\{T_t^{\mu, F}; t \geq 0\}$ is a strongly continuous symmetric semigroup on $L^p(E; m)$ for every $1 \leq p \leq \infty$. Hence the limit

$$\lambda_p(X; \mu + F) := - \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T_t^{\mu, F}\|_{p, p}$$

exists, which will be called the L^p -spectral bound of the non-local Feynman-Kac semigroup $\{T_t^{\mu, F}; t \geq 0\}$. We will show in this paper that under suitable conditions, $\lambda_p(X; \mu + F) = \lambda_2(X; \mu + F)$ for all $1 \leq p \leq \infty$ if $\lambda_2(X; \mu + F) \leq 0$. If in addition X is conservative, then $\lambda_2(X; \mu + F) \leq 0$ becomes a necessary and sufficient condition for the independence of $\lambda_p(X; \mu + F)$ in $p \in [1, \infty]$. The L^2 -spectral bound $\lambda_2(X; \mu + F)$ has a variational formula in terms of the Dirichlet form of X , μ and F , see (5.8) below. The spectral bound results obtained in this paper not only extend earlier results in [18, 19, 20, 21, 22] to a larger class of symmetric Markov processes but also give several new criteria (for example, Theorem 4.7(i), Theorem 4.8 and Theorems 5.3-5.5). See Remarks 4.10 and 5.6 below for details.

When $F = 0$, the L^p -independence of spectral bounds for continuous Feynman-Kac transforms was investigated by Takeda in [18, 19] for conservative Feller processes and for symmetric Markov processes with strong Feller property and a tightness assumption, respectively, using a large deviation approach. The results in [18] were extended to purely discontinuous Feynman-Kac transforms (i.e. with $\mu = 0$) first in [20] for rotationally symmetric α -stable processes and then in [22] for conservative doubly Feller processes, both papers again using a large deviation approach. A stochastic process is said to be doubly Feller if it is a Feller process having strong Feller property. See also

[10] for further extensions of above results for doubly Feller processes, which are established along a similar line using large deviation approach. The approach of this paper is completely different. We use the gaugeability results obtained in [3] to establish the L^p -independence of spectral bounds for local Feynman-Kac semigroups for a large class of symmetric Markov processes. These results extend the main results in [18, 19]. We then show that using a pure jump Girsanov transform, we can reduce a non-local Feynman-Kac transform for X into a continuous Feynman-Kac transform for the Girsanov transformed process Y and then apply the L^p -independence result for local Feynman-Kac semigroups. For this, uniform integrability of the exponential martingale used in the Girsanov transform plays a crucial role. Thus in the first part of this paper, we present a useful criterion for the uniform integrability of exponential martingales in the context of Markov processes, which is of independent interest. The condition of this criterion is easy to verify and is, in general, much weaker than the commonly used Novikov's condition. The special cases of this criterion have been used earlier in [9] and [4]. Using a super gauge theorem established in [3], we show that the Kato classes of X introduced in [3] are contained in the corresponding Kato classes of the Girsanov transformed process Y .

The rest of the paper is organized as follows. In Section 2, we give precise setup of this paper, including the definitions of Kato classes and Lévy systems. The criterion of the uniform integrability of exponential martingales in the context of Markov processes is presented and proved in Section 3. Spectral bounds for local Feynman-Kac semigroups and its L^p -independence are studied in Section 4, using gaugeability results for Feynman-Kac transforms obtained by the author in [3]. In Section 5, we first show that the Kato classes of X are contained in the corresponding Kato classes of the Girsanov transformed process Y , and then use it to derive the criteria for the L^p -independence of spectral bounds for non-local Feynman-Kac semigroups. To keep the exposition of this paper as transparent as possible, we have not attempted to present the most general conditions on μ and F .

2 Kato classes and non-local Feynman-Kac transform

Let E be a Lusin space and $\mathcal{B}(E)$ be the Borel σ -algebra on E . Let m be a Borel σ -finite measure on E with $\text{supp}[m] = E$ and $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}_x, x \in E)$ be an m -symmetric irreducible transient Borel standard process on E with lifetime ζ . As mentioned in the Introduction, the transience assumption on X here is just for convenience and is unimportant—we can always consider the 1-subprocess of X instead of X if necessary; see Remark 4.2 and (5.9). Let $(\mathcal{E}, \mathcal{F})$ denote the Dirichlet form of X ; that is, if we use \mathcal{L} to denote the infinitesimal generator of X , then \mathcal{F} is the domain of the operator $\sqrt{-\mathcal{L}}$ and for $u, v \in \mathcal{F}$,

$$\mathcal{E}(u, v) = (\sqrt{-\mathcal{L}}u, \sqrt{-\mathcal{L}}v)_{L^2(E, m)}.$$

We refer readers to [6] or [12] for terminology and various properties of Dirichlet forms such as continuous additive functional, martingale additive functional.

The transition operators $\{P_t, t \geq 0\}$ of X are defined by

$$P_t f(x) := \mathbb{E}_x[f(X_t)] = \mathbb{E}_x[f(X_t); t < \zeta].$$

(Here and in the sequel, unless mentioned otherwise, we use the convention that a function defined on E takes the value 0 at the cemetery point ∂ .) Throughout this paper, we assume that there is

a Borel symmetric function $G(x, y)$ on $E \times E$ such that

$$\mathbb{E}_x \left[\int_0^\infty f(X_s) ds \right] = \int_E G(x, y) f(y) m(dy)$$

for all measurable function $f \geq 0$. $G(x, y)$ is called the Green function of X . The Green function G will always be chosen so that for each fixed $y \in E$, $x \mapsto G(x, y)$ is an excessive function of X . Note that we do not assume X is a Feller process, nor do we assume X has strong Feller property.

For every $\alpha > 0$, one deduces from the existence of the Green function $G(x, y)$ that there exists a kernel $G_\alpha(x, y)$ so that

$$\mathbb{E}_x \left[\int_0^\infty e^{-\alpha s} f(X_s) ds \right] = \int_E G_\alpha(x, y) f(y) m(dy)$$

for all measurable $f \geq 0$. Clearly, $G_\alpha(x, y) \leq G(x, y)$. Note that by [12, Theorem 4.2.4], for every $x \in E$ and $t > 0$, X_t under \mathbb{P}_x has a density function $p(t, x, y)$ with respect to the measure m .

A set B is said to be m -polar if $\mathbb{P}_m(\sigma_B < \infty) = 0$, where $\sigma_B := \inf\{t > 0 : X_t \in B\}$. We call a positive measure μ on E a smooth measure of X if there is a positive continuous additive functional (PCAF in abbreviation) A of X such that

$$\int_E f(x) \mu(dx) = \uparrow \lim_{t \downarrow 0} \mathbb{E}_m \left[\frac{1}{t} \int_0^t f(X_s) dA_s \right]. \quad (2.1)$$

for any Borel $f \geq 0$. Here $\uparrow \lim_{t \downarrow 0}$ means the quantity is increasing as $t \downarrow 0$. The measure μ is called the Revuz measure of A . We refer to [6, 12] for the characterization of smooth measures in terms of nests and capacity.

For any given positive smooth measure μ , define $G\mu(x) = \int_E G(x, y) \mu(dy)$. It is known (see Stollmann and Voigt [16]) that for any positive smooth measure μ of X ,

$$\int_E u(x)^2 \mu(dx) \leq \|G\mu\|_\infty \mathcal{E}(u, u) \quad \text{for } u \in \mathcal{F}. \quad (2.2)$$

Recall that as X is assumed to have a Green function, any m -polar set is polar. Hence by (2.2) a PCAF A in the sense of [12] with an exceptional set that has a bounded potential (that is, $x \mapsto \mathbb{E}_x[A_\zeta] = G\mu$ is bounded almost everywhere on E , where μ is the Revuz measure of A) can be uniquely refined into a PCAF in the strict sense (as defined on p.195 of [12]). This can be proved by using the same argument as that in the proof of Theorem 5.1.6 of [12].

The following definitions are taken from Chen [3].

Definition 2.1 *Suppose that μ is a signed smooth measure. Let A^μ and $A^{|\mu|}$ be the continuous additive functional and positive continuous additive functional of X with Revuz measures μ and $|\mu|$, respectively.*

(i) *We say μ is in the Kato class of X , $\mathbf{K}(X)$ in abbreviation, if*

$$\limsup_{t \rightarrow 0} \sup_{x \in E} \mathbb{E}_x \left[A_t^{|\mu|} \right] = 0.$$

- (ii) μ is said to be in the class $\mathbf{K}_\infty(X)$ if for any $\varepsilon > 0$, there is a Borel set $K = K(\varepsilon)$ of finite $|\mu|$ -measure and a constant $\delta = \delta(\varepsilon) > 0$ such that for all measurable set $B \subset K$ with $|\mu|(B) < \delta$,

$$\|G(\mathbf{1}_{K^c \cup B}|\mu|)\|_\infty < \varepsilon. \quad (2.3)$$

- (iii) μ is said to be in the class $\mathbf{K}_1(X)$ if there is a Borel set K of finite $|\mu|$ -measure and a constant $\delta > 0$ such that

$$\beta_1(\mu) := \sup_{B \subset K: |\mu|(B) < \delta} \|G(\mathbf{1}_{K^c \cup B}|\mu|)\|_\infty < 1. \quad (2.4)$$

- (iv) A function q is said to be in class $\mathbf{K}(X)$, $\mathbf{K}_\infty(X)$ or $\mathbf{K}_1(X)$ if $\mu(dx) := q(x)m(dx)$ is in the corresponding spaces.

According to [3, Proposition 2.3(i)], $\mathbf{K}_\infty(X) \subset \mathbf{K}(X) \cap \mathbf{K}_1(X)$. Suppose that μ is a positive measure in $\mathbf{K}_1(X)$. By Propositions 2.2 in [3], $G\mu(x) = \mathbb{E}_x[A_\infty^\mu]$ is bounded and so (2.2) is satisfied. Therefore the PCAF corresponding to μ can and is always taken to be in the strict sense.

Let (N, H) be a Lévy system for X (cf. Benveniste and Jacod [2] and Theorem 47.10 of Sharpe [15]); that is, $N(x, dy)$ is a kernel from $(E, \mathcal{B}(E))$ to $(E, \mathcal{B}(E))$ satisfying $N(x, \{x\}) = 0$, and H_t is a PCAF of X with bounded 1-potential such that for any nonnegative Borel function f on $E \times E$ vanishing on the diagonal and any $x \in E$,

$$\mathbb{E}_x \left[\sum_{s \leq t} f(X_{s-}, X_s) \mathbf{1}_{\{s < \zeta\}} \right] = \mathbb{E}_x \left[\int_0^t \int_E f(X_s, y) N(X_s, dy) dH_s \right]. \quad (2.5)$$

The Revuz measure for H will be denoted as μ_H .

Definition 2.2 Suppose F is a bounded function on $E \times E$ vanishing on the diagonal d . It is always extended to be zero off $E \times E$. Define $\mu_F(dx) := \left(\int_E F(x, y) N(x, dy) \right) \mu_H(dx)$. We say F belongs to the class $\mathbf{J}(X)$ (respectively, $\mathbf{J}_\infty(X)$) if the measure

$$\mu_{|F|}(dx) := \left(\int_E |F(x, y)| N(x, dy) \right) \mu_H(dx)$$

belongs to $\mathbf{K}(X)$ (respectively, $\mathbf{K}_\infty(X)$).

See [4, Section 2] for concrete examples of $\mu \in \mathbf{K}_\infty(X)$ and $F \in \mathbf{J}_\infty(X)$. The following result is established in [3, Theorems 2.13 and 2.17].

Theorem 2.3 Assume that a signed measure $\mu \in \mathbf{K}_\infty(X)$ and $F \in \mathbf{J}_\infty(X)$. Let A^μ be the continuous additive functional of X with signed Revuz measure μ , and define the non-local Feynman-Kac functional

$$e_{A^\mu + F}(t) := \exp \left(A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s) \right), \quad t \geq 0.$$

- (i) (Gauge Theorem) The gauge function $g(x) := \mathbb{E}_x [e_{A^\mu + F}(\zeta)]$ is either bounded on E or identically ∞ on E . When g is bounded, we say $(X, A^\mu + F)$, or $(X, \mu + F)$, is gaugeable.
- (ii) (Super Gauge Theorem) Suppose that $(X, A^\mu + F)$ is gaugeable. Then there is an $\varepsilon_0 > 0$ such that $(X, A^{\mu + \varepsilon_0 |\mu|} + F + \varepsilon_0 |F|)$ is gaugeable. In particular, $(X, A^{(1+\varepsilon)\mu} + (1+\varepsilon)F)$ is gaugeable for all $\varepsilon \in [0, \varepsilon_0]$.

3 Uniform integrability of exponential martingales

Our uniform integrability criterion for Doléans-Dade exponential martingales is based on the following simple observation.

Lemma 3.1 *Suppose that $Z = \{Z_t, \mathcal{F}_t\}_{t \geq 0}$ is a non-negative supermartingale and define $Z_\infty = \lim_{t \rightarrow \infty} Z_t$. If there is a constant $c > 0$ so that $Z_t \leq c \mathbb{E}[Z_\infty | \mathcal{F}_t]$, then Z is uniformly integrable.*

Proof. Since Z is a non-negative supermartingale, $Z_\infty = \lim_{t \rightarrow \infty} Z_t$ exists a.s. and so, by Fatou's lemma, $\mathbb{E}[Z_\infty] \leq \mathbb{E}[Z_0] < \infty$. The conclusion of the lemma follows from the fact that $\{\mathbb{E}[Z_\infty | \mathcal{F}_t], t \geq 0\}$ is uniformly integrable. \square

Suppose that $M = \{M_t, \mathcal{F}_t\}_{t \geq 0}$ is a local martingale with $M_0 = 0$. It is well-known (see, e.g., [13, Theorem 9.39]) that

$$Z_t = 1 + \int_0^t Z_{s-} dM_s$$

has a unique solution, which is given by

$$Z_t = \exp\left(M_t - \frac{1}{2}\langle M^c \rangle_t\right) \prod_{0 < s \leq t} (1 + \Delta M_s) e^{-\Delta M_s} \quad \text{for } t \geq 0. \quad (3.1)$$

Here M^c is the continuous local martingale part of M and $\langle M^c \rangle$ is the quadratic variation process of M^c . The quadratic variation process $[M]$ of M is defined as

$$[M]_t = \langle M^c \rangle_t + \sum_{0 < s \leq t} (\Delta M_s)^2, \quad t \geq 0.$$

The local martingale Z is called the Doléans-Dade exponential martingale of M and will be denoted as $\text{Exp}(M)$.

In the remainder of this section, X is a general strong Markov process (not necessarily symmetric).

Theorem 3.2 *Suppose that M is a martingale additive functional of a strong Markov process X with $M_0 = 0$ and $\sup_{x \in E} \mathbb{E}_x[M]_\infty < \infty$ and that there is a constant $\delta \in (0, 1)$ so that $\Delta M_s \geq \delta - 1$ for every $s \geq 0$ a.s. Then $\text{Exp}(M)$ is uniformly integrable under \mathbb{P}_x for every $x \in E$.*

Proof. Under the assumption that $\sup_{x \in E} \mathbb{E}_x[M]_\infty < \infty$, M is a square-integrable martingale under \mathbb{P}_x for every $x \in E$. Thus $M_\infty = \lim_{t \rightarrow \infty} M_t$ exists \mathbb{P}_x -a.s. and $\mathbb{E}_x[M_\infty] = 0$ for every $x \in E$. Observe that $Z := \text{Exp}(M)$ is a non-negative local martingale and hence a non-negative supermartingale under \mathbb{P}_x for every $x \in E$. It is also a multiplicative functional of X . By (3.1), the Markov property of X and Jensen's inequality, for every $x \in E$ and $t \geq 0$,

$$\begin{aligned} \mathbb{E}_x[Z_\infty/Z_t | \mathcal{F}_t] &= \mathbb{E}_x[Z_\infty \cdot \theta_t | \mathcal{F}_t] = \mathbb{E}_{X_t}[Z_\infty] \\ &\geq \exp\left(\mathbb{E}_{X_t}\left[M_\infty - \frac{1}{2}\langle M^c \rangle_\infty + \sum_{0 < s < \infty} (\log(1 + \Delta M_s) - \Delta M_s)\right]\right) \\ &\geq \exp\left(\mathbb{E}_{X_t}\left[-\frac{1}{2}\langle M^c \rangle_\infty - \frac{1}{2\delta^2}(\Delta M_s)^2\right]\right) \end{aligned}$$

$$\geq \exp\left(-\frac{1}{2\delta^2} \sup_{x \in \infty} \mathbb{E}_x[M]_\infty\right),$$

where the second to last inequality is due to the fact that

$$\log(1+x) - x \geq x^2/(2\delta^2) \quad \text{for } x \geq -1 + \delta.$$

The conclusion of the theorem now follows from Lemma 3.1. \square

Corollary 3.3 *Suppose that X is a transient Borel standard process on E with Lévy system (N, H) . Let b be a function on $E \times E$ vanishing on the diagonal d such that $b(x, y) \geq \delta - 1$ for some constant $\delta > 0$ and that $G\mu_{b^2}$ is bounded for*

$$\mu_{b^2}(dx) := \left(\int_E b(x, y)^2 N(x, dy) \right) \mu_H(dx).$$

Then there is a unique purely discontinuous square integrable martingale additive functional of X with $M_0 = 0$ and $\Delta M_t = b(X_{t-}, X_t) \mathbf{1}_{\{t < \zeta\}}$ for $t > 0$. Moreover, $\text{Exp}(M)$ is uniformly integrable under \mathbb{P}_x for every $x \in E$.

Proof. Using the Lévy system, the assumption that μ_{b^2} has a bounded potential is equivalent to the assumption that

$$\sup_{x \in E} \mathbb{E}_x \left[\sum_{s > 0} b(X_{s-}, X_s)^2 \mathbf{1}_{\{s < \zeta\}} \right] < \infty.$$

Thus there is a unique purely discontinuous square integrable martingale additive functional of X with $M_0 = 0$ and $\Delta M_t = b(X_{t-}, X_t) \mathbf{1}_{\{t < \zeta\}}$ for $t > 0$ (see, e.g., [13]). Since

$$[M]_\infty = \sum_{s > 0} b(X_{s-}, X_s)^2 \mathbf{1}_{\{s < \zeta\}},$$

it follows immediately from Theorem 3.2 that $\text{Exp}(M)$ is uniformly integrable. \square

Remark 3.4 (i) Suppose that $E = D$ is a connected open subset of \mathbb{R}^n , X is Brownian motion killed upon leaving domain D , and $M_t = \int_0^t b(X_s) dX_s$. Note that $M_t = \int_0^{t \wedge \tau_D} b(X_s) dX_s$ is a martingale additive functional of X , where τ_D is the first exit time (or lifetime) of the killed Brownian motion X from D . We have by Theorem 3.2 that the exponential martingale $\text{Exp}(M)$ is uniformly integrable if

$$\sup_{x \in D} \mathbb{E}_x \left[\int_0^\infty |b(X_s)|^2 ds \right] = \sup_{x \in D} \mathbb{E}_x \left[\int_0^{\tau_D} |b(X_s)|^2 ds \right] < \infty.$$

The result in this particular case was first derived in passing on page 746 of [9]. The above condition is in general much weaker than the Novikov's condition for the uniform integrability of $\text{Exp}(M)$. When M is a continuous local martingale M , for $\text{Exp}(M)$ to be uniformly integrable, Novikov's condition requires $\mathbb{E}_x \exp(\langle M \rangle_\infty / 2) < \infty$ (see, e.g., [14, Proposition VIII.1.15]). In this concrete example, the latter condition amounts to the assumption that

$$\mathbb{E}_x \exp\left(\frac{1}{2} \int_0^\infty |b(X_s)|^2 ds\right) = \mathbb{E}_x \exp\left(\frac{1}{2} \int_0^{\tau_D} |b(X_s)|^2 ds\right) < \infty.$$

(ii) Suppose that X is an m -symmetric irreducible transient Borel standard process on E , and $F \in \mathbf{J}(X)$. Then

$$M_t := \sum_{0 < s \leq t} F(X_{s-}, X_s) - \int_0^t \left(\int_{E_\partial} F(X_s, y) N(X_s, dy) \right) dH_s, \quad t \geq 0, \quad (3.2)$$

is the purely discontinuous martingale additive functional of X with $M_0 = 0$ and $\Delta M_t = b(X_{t-}, X_t)$ for $t > 0$. Define

$$A_t^F := \int_0^t \left(\int_E \left(e^{F(x,y)} - 1 \right) N(X_s, dy) \right) dH_s, \quad (3.3)$$

which is a continuous additive functional of X . Then by (3.1), we have

$$\text{Exp}(M)_t = \exp \left(\sum_{s \leq t} F(X_{s-}, X_s) - A_t^F \right), \quad t \geq 0. \quad (3.4)$$

Suppose now that $F \in \mathbf{J}_\infty(X)$. Then the condition of Corollary 3.3 is satisfied for $b(x, y) := e^{F(x,y)} - 1$ and so $\text{Exp}(M)$ is uniformly integrable under \mathbb{P}_x for every $x \in E$. This fact was first established in [4, page 241].

(iii) If the condition $\sup_{x \in E} \mathbb{E}_x[M]_\infty < \infty$ in Theorem 2.3 is replaced by a weaker condition $\sup_{x \in E} \mathbb{E}_x[M]_T < \infty$ for some fixed constant $T > 0$, the same proof of Theorem 2.3 yields that $\{\text{Exp}(M)_t, t \in [0, T]\}$ is a \mathbb{P}_x -martingale for every $x \in E$. \square

The above uniform integrability results for exponential martingales will be used in Section 5.

4 Spectral bounds for local Feynman-Kac semigroup

For a signed measure μ , we use μ^+ and μ^- to denote the positive part and negative part of μ appearing in the Hahn-Jordan decomposition of μ . Observe that if μ_1 and μ_2 are two non-negative measures so that $\mu_1 - \mu_2 = \mu$, then $\mu_1 \geq \mu^+$ and $\mu_2 \geq \mu^-$.

In the rest of this paper, we work under the setting of Section 2. Let μ be a signed smooth measure so that $\mu^+ \in \mathbf{K}_1(X)$ and $G\mu^-$ is bounded. Define the Feynman-Kac semigroup $\{T_t^\mu, t \geq 0\}$ by

$$T_t^\mu f(x) = \mathbb{E}_x \left[e^{A_t^\mu} f(X_t) \right].$$

As is explained in the paragraph proceeding [3, Theorem 2.12], $\{T_t^\mu, t \geq 0\}$ is a strongly continuous symmetric semigroup in $L^p(E, m)$ for every $1 \leq p \leq \infty$ and its associated symmetric quadratic form is $(\mathcal{E}^\mu, \mathcal{F})$, where

$$\mathcal{E}^\mu(u, v) = \mathcal{E}(u, v) - \int_E u(x)v(x)\mu(dx) \quad \text{for } u, v \in \mathcal{F}.$$

If we use \mathcal{L} to denote the infinitesimal generator for the semigroup of X , then the infinitesimal generator for the semigroup P_t^μ is $\mathcal{L} + \mu$. For $1 \leq p \leq \infty$, we use $\|T_t^\mu\|_{p,p}$ to denote the operator norm of $T_t^\mu : L^p(E; m) \rightarrow L^p(E; m)$.

For $1 \leq p \leq \infty$, define the L^p -spectral bound of semigroup $\{T_t^\mu, t \geq 0\}$ by

$$\lambda_p(X, \mu) := - \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T_t^\mu\|_{p,p} = - \inf_{t > 0} \frac{1}{t} \log \|T_t^\mu\|_{p,p}.$$

Clearly

$$\|T_t^\mu\|_{\infty, \infty} = \|T_t^\mu 1\|_\infty = \sup_{x \in E} \mathbb{E}_x [e_A(t); t < \zeta].$$

It is well-known that the L^2 -spectral bound $\lambda_2(X, \mu)$ of $\{T_t^\mu, t \geq 0\}$ can be represented in terms of its quadratic form $(\mathcal{E}^\mu, \mathcal{F})$:

$$\begin{aligned} \lambda_2(X, \mu) &= \inf \left\{ \mathcal{E}^\mu(u, u) : u \in \mathcal{F} \text{ with } \int_E u(x)^2 m(dx) = 1 \right\} \\ &= \inf \left\{ \mathcal{E}(u, u) - \int_E u(x)^2 \mu(dx) : u \in \mathcal{F} \text{ with } \int_E u(x)^2 m(dx) = 1 \right\}. \end{aligned} \quad (4.1)$$

By duality, we have $\|T_t^\mu\|_{1,1} = \|T_t^\mu\|_{\infty, \infty}$. Consequently, it follows from the Cauchy-Schwartz inequality that

$$\|T_t^\mu f\|_2^2 \leq \|T_t^\mu 1\|_\infty \|T_t^\mu(f^2)\|_1 \leq \|T_t^\mu\|_{\infty, \infty}^2 \|f\|_2^2 \quad \text{for } f \in L^2(E, m).$$

Thus we have $\|T_t^\mu\|_{2,2} \leq \|T_t^\mu\|_{\infty, \infty}$. We now deduce by interpolation that

$$\|T_t^\mu\|_{2,2} \leq \|T_t^\mu\|_{p,p} \leq \|T_t^\mu\|_{\infty, \infty} \quad \text{for } 1 < p < \infty.$$

Hence

$$\lambda_\infty(X, \mu) \leq \lambda_p(X, \mu) \leq \lambda_2(X, \mu) \quad \text{for } 1 < p < \infty. \quad (4.2)$$

The following theorem is proved as Theorem 2.12 in [3], however condition (4.3) is missing from its statement. For reader's convenience, we reproduce the proof here.

Theorem 4.1 *Assume that $m(E) < \infty$, $\|G1\|_\infty < \infty$. Let μ be a signed smooth measure such that $\mu^+ \in \mathbf{K}_1(X)$ and $G\mu^-$ is bounded. Then (X, μ) is gaugeable if and only if $\lambda_2(X, \mu) > 0$. Assume in addition that $\mu^+ \in \mathbf{K}(X)$ and that*

$$\text{there is some } t_0 > 0 \text{ so that } P_{t_0} \text{ is a bounded operator from } L^2(E; m) \text{ into } L^\infty(E; m). \quad (4.3)$$

Then $\lambda_p(X, \mu)$ is independent of $p \in [1, \infty]$ if $\lambda_2(X, \mu) > 0$.

Proof. By [3, Theorem 2.11], if (X, μ) is gaugeable, then $\lambda_\infty(X, \mu) > 0$ and therefore $\lambda_2(X, \mu) > 0$. Conversely suppose $\lambda_2(X, \mu) > 0$. Then for any $\varepsilon \in (0, \lambda_2(X, \mu))$, there is $\delta(\varepsilon) > 0$ such that

$$\|T_t^\mu\|_{2,2} \leq e^{-t(\lambda_2(X, \mu) - \varepsilon)} \quad \text{for } t \geq \delta(\varepsilon). \quad (4.4)$$

Since $1 \in L^2(E, m)$, $\int_0^\infty T_t^\mu 1 dt$ is $L^2(E; m)$ -integrable. Hence by [3, Theorem 2.11], (X, μ) is gaugeable.

Assume now that (4.3) holds and that $\mu^+ \in \mathbf{K}(X)$. By duality, P_{t_0} is a bounded operator from $L^1(E; m)$ to $L^2(E; m)$. It follows that $P_{2t_0} = P_{t_0} \circ P_{t_0}$ is a bounded operator from $L^1(E; m)$ into $L^\infty(E; m)$, whose operator norm will be denoted as $\|P_{2t_0}\|_{1, \infty}$. On the other hand, since $\mu^+ \in \mathbf{K}(X)$, there is some $\delta > 0$ so that $\sup_{x \in E} \mathbb{E}_x [A_\delta^{\mu^+}] < 1/2$. By Khasminskii's inequality,

$$c_1 := \sup_{x \in E} \mathbb{E}_x \left[\exp(2A_\delta^{\mu^+}) \right] \leq \frac{1}{1 - \sup_{x \in E} \mathbb{E}_x [2A_\delta^{\mu^+}]} < \infty.$$

Thus by the Markov property of X , we have $\sup_{x \in E} \mathbb{E}_x \left[\exp(A_t^{\mu^+}) \right] < \infty$ for every $t > 0$. Thus for every $f \in L^2(E; m)$ and $x \in E$, by Cauchy-Schwartz inequality,

$$|T_{2t_0}^\mu f(x)|^2 = \left(\mathbb{E}_x \left[\exp(A_{2t_0}^\mu) f(X_{2t_0}) \right] \right)^2 \leq \mathbb{E}_x \left[f(X_{2t_0})^2 \right] \mathbb{E}_x \left[\exp(2A_{2t_0}^{\mu^+}) \right] \leq c_1 \|P_{2t_0}\|_{1,\infty} \|f\|_2^2. \quad (4.5)$$

Suppose $\lambda_2(X, \mu) > 0$. For any $\varepsilon \in (0, \lambda_2(X, \mu))$, there is $\delta(\varepsilon) > 0$ so that (4.4) holds. Then for $t > \delta(\varepsilon) + 2t_0$, by (4.5) and then (4.4),

$$\|T_t^\mu\|_{\infty,\infty} = \|T_t^\mu 1\|_\infty = \|T_{2t_0}^\mu (T_{t-2t_0}^\mu 1)\|_\infty \leq c_2 \|T_{t-2t_0}^\mu 1\|_2 \leq c_2 \sqrt{m(E)} e^{-(t-1)(\lambda_2(X,\mu)-\varepsilon)}.$$

This implies that $\lambda_\infty(X, \mu) \geq \lambda_2(X, \mu) - \varepsilon$ and so $\lambda_\infty(X, \mu) \geq \lambda_2(X, \mu)$. Hence by (4.2), $\lambda_\infty(X, \mu) = \lambda_2(X, \mu) = \lambda_p(X, \mu)$ for all $p \in [1, \infty]$. \square

For $\alpha > 0$, let $X^{(\alpha)}$ denote the α -subprocess of X ; that is, $X^{(\alpha)}$ is the subprocess of X killed at exponential rate α . Let $G^{(\alpha)}$ be the 0-resolvent (or Green operator) of $X^{(\alpha)}$. Then $G^{(\alpha)} = G_\alpha$, the α -resolvent of X . Thus for $\beta > \alpha > 0$, $\mathbf{K}_1(X) \subset \mathbf{K}_1(X^{(\alpha)}) \subset \mathbf{K}_1(X^{(\beta)})$ and $\mathbf{K}_\infty(X) \subset \mathbf{K}_\infty(X^{(\alpha)}) \subset \mathbf{K}_\infty(X^{(\beta)})$. In fact, it follows from the resolvent equation $G_\alpha = G_\beta + (\beta - \alpha)G_\alpha G_\beta$ that $\mathbf{K}_\infty(X^{(\alpha)}) = \mathbf{K}_\infty(X^{(\beta)})$ for every $\beta > \alpha$. Consequently, $\mathbf{J}_\infty(X^{(\alpha)}) = \mathbf{J}_\infty(X^{(\beta)})$ for every $\beta > \alpha$.

Remark 4.2 For any signed measure μ on E with $\mu \in \mathbf{K}_1(X^{(\alpha)})$ and $G_\alpha \mu^-$ bounded for some $\alpha > 0$, $\{T_t^\mu, t \geq 0\}$ is still well defined as a strongly continuous symmetric semigroup in $L^p(E; m)$ for every $p \in [1, \infty]$, and relation (4.2) continues to hold. To see this, let $\{T_t^{\mu,\alpha}, t \geq 0\}$ denote the Feynman-Kac semigroup of $X^{(\alpha)}$ associated with smooth measure μ . Recall the following simple facts. Let R be an exponential random variable with mean $1/\alpha$ that is independent of X . The α -subprocess $X^{(\alpha)}$ of X can be realized as follows: $X_t^{(\alpha)}(\omega) = X_t(\omega)$ if $t < R(\omega)$ and $X_t^{(\alpha)} = \partial$ if $t \geq R(\omega)$. Let A^μ be the continuous additive functional of X with (signed) Revuz measure μ . Then $t \mapsto A_{t \wedge R}$ is a continuous additive functional of $X^{(\alpha)}$ with Revuz measure μ . It follows immediate that $\mathbf{K}(X) = \mathbf{K}(X^{(\alpha)})$ and

$$T_t^{\mu,\alpha} = e^{-\alpha t} T_t^\mu \quad \text{for every } t \geq 0.$$

Since $\{T_t^{\mu,\alpha}, t \geq 0\}$ is a strongly continuous symmetric semigroup in $L^p(E; m)$ for every $1 \leq p \leq \infty$, so is $\{T_t^\mu, t \geq 0\}$. Moreover,

$$\lambda_p(X^{(\alpha)}, \mu) = \alpha + \lambda_p(X, \mu) \quad \text{for every } p \in [1, \infty] \text{ and } \beta > 0. \quad (4.6)$$

Since relation (4.2) holds for $\lambda_p(X^{(\alpha)}, \mu)$, the same holds for $\lambda_p(X, \mu)$.

Corollary 4.3 *Assume that (4.3) holds and that $m(E) < \infty$. Let μ be a signed smooth measure with $\mu^+ \in \cup_{\alpha>0} \mathbf{K}_1(X^{(\alpha)}) \cap \mathbf{K}(X)$ and $G_{\alpha_0} \mu^-$ bounded for some $\alpha_0 > 0$. Then $\lambda_p(X, \mu)$ is independent of $p \in [1, \infty]$.*

Proof. There is an $\alpha > 0$ sufficiently large so that $\mu^+ \in \mathbf{K}_1(X^{(\alpha)}) \cap \mathbf{K}(X)$ with $G_\alpha \mu^-$ bounded. Note that since $\mu \in \mathbf{K}(X)$, $\lambda_2(X, \mu)$ is finite. By increasing the value of α if necessary, we may and do assume $\lambda_2(X^{(\alpha)}, \mu) > 0$. On the other hand, $G^{(\alpha)} 1 \leq 1/\alpha$. Therefore by Theorem 4.1, we

have $\lambda_2(X^{(\alpha)}, \mu) = \lambda_\infty(X^{(\alpha)}, \mu)$. This together with (4.6) yields $\lambda_2(X, \mu) = \lambda_\infty(X, \mu)$ and so the conclusion of the theorem follows. \square

Next, we investigate the independence of $\lambda_p(X, \mu)$ without assuming $m(E) < \infty$. The next result is a slight extension of [17, Lemma 3.5].

Lemma 4.4 *Suppose that $\mu = \mu_1 - \mu_2$, where μ_1 and μ_2 are non-negative smooth measures so that $\|G_\alpha \mu_1\|_\infty < 1$ for some $\alpha > 0$. If*

$$\inf \left\{ \mathcal{E}(u, u) - \int_E u(x)^2 \mu(dx); u \in \mathcal{F} \text{ with } \int_E u(x)^2 m(dx) = 1 \right\} > 0,$$

then

$$\inf \left\{ \mathcal{E}(u, u) + \int_E u(x)^2 \mu_2(dx); u \in \mathcal{F} \text{ with } \int_E u(x)^2 \mu_1(dx) = 1 \right\} > 1.$$

Proof. The proof is the same as that for [17, Lemma 3.5]. For reader's convenience, we spell out its proof here. Let $\delta := \|G_\alpha \mu_1\|_\infty < 1$. By (2.2) applied to the α -subprocess $X^{(\alpha)}$ of X , there is some $C > 0$ such that

$$\int_E u(x)^2 \mu_1(dx) \leq \delta \mathcal{E}(u, u) + C \int_E u(x)^2 m(dx) \quad \text{for } u \in \mathcal{F}. \quad (4.7)$$

Suppose that $\lambda := \inf \left\{ \mathcal{E}(u, u) - \int_E u(x)^2 \mu(dx); u \in \mathcal{F} \text{ with } \int_E u(x)^2 m(dx) = 1 \right\} > 0$; that is,

$$\mathcal{E}(u, u) - \int_E u(x)^2 \mu(dx) \geq \lambda \int_E u(x)^2 m(dx) \quad \text{for every } u \in \mathcal{F}.$$

Thus for $u \in \mathcal{F}$, we have by (4.7) that

$$\int_E u(x)^2 \mu_1(dx) \leq \delta \mathcal{E}(u, u) + \frac{C}{\lambda} \left(\mathcal{E}(u, u) - \int_E u(x)^2 (\mu_1 - \mu_2)(dx) \right)$$

and so

$$\frac{1 + (C/\lambda)}{\delta + (C/\lambda)} \int_E u(x)^2 \mu_1(dx) \leq \mathcal{E}(u, u) + \frac{C/\lambda}{\delta + (C/\lambda)} \int_E u(x)^2 \mu_2(dx) \leq \mathcal{E}(u, u) + \int_E u(x)^2 \mu_2(dx).$$

The conclusion of the lemma now follows. \square

Remark 4.5 (i) Note that $\lim_{\alpha \rightarrow \infty} \|G_\alpha \mu_1\|_\infty = 0$ for $\mu_1 \in \mathbf{K}_\infty(X)$. Moreover, by [3, Proposition 2.3(ii)],

$$\lim_{\alpha \rightarrow \infty} \|G_\alpha \mu_1\|_\infty < 1 \quad \text{for } \mu_1 \in \mathbf{K}_1(X). \quad (4.8)$$

(ii) The assumption that $\|G_\alpha \mu_1\| < 1$ in Lemma 4.4 is only used to deduce inequality (4.7). So the result holds under the assumption that μ_1 satisfies the Hardy class condition (4.7) for some $\delta < 1$. \square

Let μ be a non-negative smooth measure and $\alpha \geq 0$. We say $G_\alpha\mu$ is an α -potential if $\sup_{x \in E} G_\alpha\mu(x) = 0$.

Lemma 4.6 *Suppose that the process X admits no killings inside, that is $\mathbb{P}_x(X_{\zeta^-} \in E, \zeta < \infty) = 0$ for q.e. $x \in E$. If μ is a non-negative measure in $\mathbf{K}_\infty(X^{(1)})$, then $G_\alpha\mu$ is an α -potential for every $\alpha > 0$.*

Proof. Since $\mu \in \mathbf{K}_\infty(X^{(1)}) = \mathbf{K}_\infty(X^{(\alpha)})$, for every $\varepsilon > 0$, there is a Borel subset K and a constant $\delta > 0$ so that for every Borel subset $B \subset K$ with $\mu(B) < \varepsilon$, $\|G_\alpha(\mathbf{1}_{K^c \cup B}\mu)\|_\infty < \varepsilon$. On the other hand, by [6, Theorem 2.3.15], there is an increasing sequence $\{F_k, k \geq 1\}$ of closed sets so that $\mu(\cap_{k \geq 1} F_k^c) = 0$, $\mu(F_k) < \infty$, and $\mathbf{1}_{F_k}\mu$ is a measure of finite energy for every $k \geq 1$. Let $j \geq 1$ be large enough so that $\mu(K \setminus F_j) < \delta$. Let $u := G_\alpha(\mathbf{1}_{F_j \cap K}\mu)$. Then $u \in \mathcal{F}$ and by [6, Corollary 3.5.3], $\mathbb{P}_x(\lim_{t \rightarrow \zeta} u(X_t) = 0) = 1$ for q.e. $x \in E$. It follows that

$$\inf_{x \in E} G_\alpha\mu(x) \leq \inf_{x \in E} G_\alpha(\mathbf{1}_{F_j \cap K}\mu)(x) + \sup_{x \in E} G_\alpha(\mathbf{1}_{E \setminus F_j}\mu)(x) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\inf_{x \in E} G_\alpha\mu(x) = 0$. \square

Theorem 4.7 *Suppose that μ is a signed smooth measure with $\mu^+ \in \mathbf{K}_\infty(X^{(1)})$ and $G_1\mu^-$ bounded.*

- (i) $\lambda_\infty(X, \mu) \geq \min\{\lambda_2(X, \mu), 0\}$. Consequently, $\lambda_p(X, \mu)$ is independent of $p \in [1, \infty]$ if $\lambda_2(X, \mu) \leq 0$.
- (ii) Assume in addition that X is conservative and that either $G\mu^-$ is bounded or $G_\alpha\mu^-$ is an α -potential for some $\alpha > 0$. Then $\lambda_\infty(X, \mu) = 0$ if $\lambda_2(X, \mu) > 0$. Hence $\lambda_p(X, \mu)$ is independent of $p \in [1, \infty]$ if and only if $\lambda_2(X, \mu) \leq 0$.

Proof. First note that by the resolvent equation, $G_1\mu^-$ is bounded if and only if $G_\alpha\mu^-$ is bounded.

(i) For any $\lambda < \min\{\lambda_2(X, \mu), 0\}$, there is $\alpha > 0$ so that $\lambda + \alpha < \min\{\lambda_2(X, \mu), 0\}$. Clearly,

$$\inf \left\{ \mathcal{E}_\alpha(u, u) - \int_E u(x)^2 (\mu + (\lambda + \alpha)m)(dx); u \in \mathcal{F} \text{ with } \int_E u(x)^2 m(dx) = 1 \right\} = \lambda_2(X; \mu) - \lambda > 0.$$

Note that $(\mathcal{E}_\alpha, \mathcal{F})$ is the Dirichlet form for the α -subprocess $X^{(\alpha)}$ of X and $\|G^{(\alpha)}\mathbf{1}\|_\infty \leq 1/\alpha$. Since $\mu^+ - (\mu^- + (-\lambda - \alpha)m) = \mu + (\lambda + \alpha)m$, one has

$$(\mu + (\lambda + \alpha)m)^+ \leq \mu^+ \quad \text{and} \quad (\mu + (\lambda + \alpha)m)^- \leq \mu^- + (-\lambda - \alpha)m. \quad (4.9)$$

We thus have by Lemma 4.4 that

$$\inf \left\{ \mathcal{E}_\alpha(u, u) + \int_E u(x)^2 ((-\lambda - \alpha)m + \mu^-)(dx); u \in \mathcal{F} \text{ with } \int_E u(x)^2 \mu^+(dx) = 1 \right\} > 1.$$

It follows from the elementary inequality $\frac{a}{b} \geq \frac{a+c}{b+c}$ for $a \geq b \geq 0$ and $c > 0$ that

$$\inf_{u \in \mathcal{F}} \frac{\mathcal{E}_\alpha(u, u) + \int_E u(x)^2 (\mu + (\lambda + \alpha)m)^-(dx)}{\int_E u(x)^2 (\mu + (\lambda + \alpha)m)^+(dx)} \geq \inf_{u \in \mathcal{F}} \frac{\mathcal{E}_\alpha(u, u) + \int_E u(x)^2 (\mu^- + (-\lambda - \alpha)m)(dx)}{\int_E u(x)^2 \mu^+(dx)} > 1.$$

Since $\mu^+ \in \mathbf{K}_\infty(X^{(1)}) = \mathbf{K}_\infty(X^{(\alpha)})$ and $G_\alpha((-\lambda - \alpha)m + \mu^-)$ is bounded, it follows from (4.9) and [3, Theorem 5.4] that $(X^{(\alpha)}, \mu + (\lambda + \alpha)m)$ is gaugeable. Let $\zeta^{(\alpha)}$ denote the lifetime of $X^{(\alpha)}$, which can be realized as $\zeta \wedge R$ for an exponential random variable R with mean $1/\alpha$ that is independent of X , and let $\mathbb{E}_x^{(\alpha)}$ denote the expectation under the probability law of $X^{(\alpha)}$ starting from x . Then

$$\begin{aligned} \sup_{t \geq 0} e^{\lambda t} \|T^\mu 1\|_\infty &= \sup_{t \geq 0} \left(e^{\lambda t} \sup_{x \in E} \mathbb{E}_x \left[e^{A_t^\mu}; t < \zeta \right] \right) = \sup_{t \geq 0} \left(e^{(\lambda + \alpha)t} \sup_{x \in E} e^{-\alpha t} \mathbb{E}_x \left[e^{A_t^\mu}; t < \zeta \right] \right) \\ &= \sup_{t \geq 0} \left(e^{(\lambda + \alpha)t} \sup_{x \in E} \mathbb{E}_x \left[e^{A_t^\mu}; t < \zeta^{(\alpha)} \right] \right) \leq \sup_{x \in E} \mathbb{E}_x^{(\alpha)} \left[\sup_{t < \zeta} e^{(\lambda + \alpha)t + A_t^\mu} \right] < \infty, \end{aligned}$$

where the last inequality is due to [3, Corollary 2.9(5)]. This implies that

$$\lambda_\infty(X; \mu) = - \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T^\mu 1\|_\infty \geq \lambda.$$

Since the above holds for every $\lambda < \min\{\lambda_2(X; \mu), 0\}$, we conclude that $\lambda_\infty(X; \mu) \geq \min\{\lambda_2(X; \mu), 0\}$. In particular, when $\lambda_2(X, \mu) \leq 0$, we have $\lambda_\infty(X, \mu) \geq \lambda_2(X, \mu)$. This together with (4.2) yields that $\lambda_p(X, \mu)$ is independent of $p \in [1, \infty]$ when $\lambda_2(X, \mu) \leq 0$.

(ii) Since $\lambda_2(X, \mu) > 0$, we have from (i) that $\lambda_\infty(X, \mu) \geq 0$. Assume now that X is conservative. If $G\mu^-$ is bounded, then

$$\|T_t^\mu 1\|_\infty = \sup_{x \in E} \mathbb{E}_x \left[\exp(A_t^\mu) \right] \geq \sup_{x \in E} \mathbb{E}_x \left[\exp(-A_t^{\mu^-}) \right] \geq \sup_{x \in E} \exp(-\mathbb{E}_x[A_\infty^{\mu^-}]) \geq \exp(-\|G\mu^-\|_\infty).$$

If $G_\alpha \mu^-$ is a potential for some $\alpha > 0$, then

$$\|T_t^\mu 1\|_\infty \geq \sup_{x \in E} \mathbb{E}_x \left[\exp(-A_t^{\mu^-}) \right] \geq \sup_{x \in E} \exp(-\mathbb{E}_x[A_t^{\mu^-}]) \geq \exp\left(-\inf_{x \in E} e^{\alpha t} G_\alpha \mu^-(x)\right) = 1.$$

In either cases, we have

$$\lambda_\infty(X, \mu) = - \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T_t^\mu 1\|_\infty \leq 0.$$

Therefore $\lambda_\infty(X, \mu) = 0 < \lambda_2(X, \mu)$. □

Theorem 4.8 *Suppose that $1 \in \mathbf{K}_\infty(X^{(1)})$ and $\mu \in \mathbf{K}_\infty(X^{(1)})$. Then $\lambda_p(X, \mu)$ is independent of $p \in [1, \infty]$.*

Proof. Since $\mu \in \mathbf{K}_\infty(X^{(1)})$, $\|G_1 \mu\|_\infty < \infty$. In view of (2.2), there exists $\beta > 0$ so that $\lambda_2(X^{(1)}, \mu) < \beta$; or, equivalently, $\lambda_2(X^{(1)}, \mu + \beta m) < 0$. Since $1 \in \mathbf{K}_\infty(X^{(1)})$, we have by Theorem 4.7 that $\lambda_p((X^{(1)}, \mu + \beta m)) = \lambda_2((X^{(1)}, \mu + \beta m))$ for all $p \in [1, \infty]$. On the other hand,

$$\lambda_p(X^{(1)}, \mu + \beta m) = -\beta + \lambda_p(X^{(1)}, \mu) = -\beta + 1 + \lambda_p(X, \mu) \quad \text{for } p \in [1, \infty],$$

which yields the L^p -independence of $\lambda_p(X, \mu)$. □

The reason that we need to assume $1 \in \mathbf{K}_\infty(X^{(1)})$ in Theorem 4.8 is because the gaugeability results for $(X^{(1)}, \nu)$, Theorems 2.12 and 5.2 of [3], require $\nu^+ \in \mathbf{K}_\infty(X^{(1)})$ and $G_1 \nu^-$ bounded, and they are applied to the measure $\nu = \mu + \beta m$. In Theorem 4.7, these gaugeability results are applied to $(X^{(\alpha)}, \nu)$ for measure $\nu = \mu + (\lambda + \alpha)m$ with $\lambda + \alpha < 0$ so we do not need to assume $1 \in \mathbf{K}_\infty(X)$.

The following sufficient condition for $1 \in \mathbf{K}_\infty(X^{(1)})$ is established in [3, Theorem 4.2] (together with its proof).

Lemma 4.9 *Suppose that $\alpha > 0$ and that G_α maps bounded functions into continuous functions. If for every $\varepsilon > 0$, there is a compact set $K \subset E$ such that $\sup_{x \in E} G_\alpha \mathbf{1}_{K^c}(x) < \varepsilon$, then $1 \in \mathbf{K}_\infty(X^{(\alpha)}) = \mathbf{K}_\infty(X^{(1)})$.*

Remark 4.10 (i) Theorem 4.7(ii) extends the main result (Theorem 3.1) of [18], where it is shown by a large deviation argument that, for any m -symmetric irreducible conservative Feller process X that has jointly continuous transition density function and signed measure $\mu \in \mathbf{K}_\infty(X)$, $\lambda_p(X, \mu)$ is independent of $p \in [1, \infty]$ if and only if $\lambda_2(X, \mu) \leq 0$. Theorem 4.8 extends a corresponding result in [19]. Our approach also reveals where the role of conservativeness of Y is played in part (ii) of Theorem 4.7 in connection with part (i). See [3, 17] for related results on the L^p -independence of the spectral radius of the transition semigroup of X (that is, for $\lambda_p(X, 0)$ corresponding to $\mu = 0$).

(ii) Assume that X is an m -symmetric irreducible process X satisfying strong Feller property (that is, its transition semigroup maps bounded Borel measurable functions into bounded continuous functions) and the following tightness assumption: for every $\varepsilon > 0$, there is a compact subset K so that $\sup_{x \in E} G_1 \mathbf{1}_K(x) \leq \varepsilon$. For such a process, as an application of a large deviation result established in [19, Theorem 1.1], it is shown in [19] that $\lambda_p(X, \mu)$ is independent of $p \in [1, \infty]$ for every $\mu \in \mathbf{K}_\infty(X)$. This result is a special case of our Theorem 4.8 in view of Lemma 4.9.

(iii) By the same argument as that for [19, Proposition 4.1] that for smooth measure μ with $\mu^+ \in \mathbf{K}_1(X)$ and $G\mu^+$ bounded,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}_x \left[e^{A_t^\mu}; t < \zeta \right] \geq -\lambda_2(X, \mu), \quad x \in E.$$

So whenever $\lambda_2(X, \mu) = \lambda_\infty(X, \mu)$, one has for every $x \in E$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}_x \left[e^{A_t^\mu}; t < \zeta \right] = \lim_{t \rightarrow \infty} \frac{1}{t} \log \sup_{x \in E} \mathbb{E}_x \left[e^{A_t^\mu}; t < \zeta \right] = -\lambda_2(X, \mu).$$

□

5 Spectral bounds for non-local Feynman-Kac semigroups

Throughout this section, F is a bounded symmetric function in the Kato class $\mathbf{J}(X)$. Let M be the purely discontinuous square-integrable martingale additive functional of X with

$$\Delta M_t = e^{F(X_{t-}, X_t)} - 1,$$

which has the expression (3.2). Let $\text{Exp}(M)$ be the Doléans-Dade exponential martingale of M . It defines a family of probability measures $\{\mathbb{Q}_x, x \in E\}$ by $d\mathbb{Q}_x/d\mathbb{P}_x = \text{Exp}(M)_t$ on \mathcal{F}_t . For emphasis, the Girsanov transformed process $\{X_t, \mathbb{Q}_x\}$ is denoted by Y . The process Y is still m -symmetric and its associated Dirichlet form on $L^2(E; m)$ is $(\mathcal{E}^Y, \mathcal{F})$, where

$$\mathcal{E}^Y(u, u) = \mathcal{E}(u, u) + \frac{1}{2} \int_{E \times E} (u(x) - u(y))^2 \left(e^{F(x, y)} - 1 \right) N(x, dy) \mu_H(dx) \quad (5.1)$$

(see [8]). So the symmetric process Y has Lévy system $(e^{F(x, y)} N(x, dy), H)$. If $F \in \mathbf{J}_\infty(X)$, then by Remark 3.4(ii), $\text{Exp}(M)$ is a uniformly integrable martingale under \mathbb{P}_x for every $x \in E$.

Lemma 5.1 *If A is a PCAF of X with Revuz measure ν , then the Revuz measure of A as a PCAF of Y is still ν .*

Proof. The proof is exactly the same as that for Lemma 4.4 in [11] so it is omitted here. \square

The next result says that the corresponding Kato classes become larger after Girsanov transform.

Theorem 5.2 *Let $F \in \mathbf{J}_\infty(X)$ be symmetric and Y be the above Girsanov transformed process in terms of F . Then*

$$\mathbf{K}_\infty(X) \subset \mathbf{K}_\infty(Y) \quad \text{and} \quad \mathbf{J}_\infty(X) \subset \mathbf{J}_\infty(Y).$$

Moreover, if ν is a non-negative measure so that $G\nu$ is bounded, then so is $G^Y\nu$.

Proof. For notational convenience, let $Z_t = \text{Exp}(M)_t$. Since $\mathbb{E}_x[Z_\zeta] = \mathbb{E}_x[Z_0] = 1$ for every $x \in E$, $(X, F - A^F)$ is gaugeable. By the Super Gauge theorem (Theorem 2.3(ii)), there is an $\varepsilon > 0$ so that

$$c_0 := \sup_{x \in E} \left(\mathbb{E}_x \left[Z_\zeta^{1+\varepsilon} \right] \right)^{1/(1+\varepsilon)} < \infty. \quad (5.2)$$

Clearly, Y has a Green function $G^Y(x, y)$ defined by

$$\int_E G^Y(x, y) f(y) = \mathbb{E}_x \left[\int_0^\infty f(Y_s) ds \right] = \mathbb{E}_x \left[Z_\zeta \int_0^\zeta f(X_s) ds \right].$$

In fact, $G^Y(x, y) = \mathbb{E}_x^y[Z_\zeta]G(x, y)$, where \mathbb{E}_x^y is the expectation under the law of \mathbb{P}_x^y which is obtained from \mathbb{P}_x through Doob's h -transform with $h(z) = G(z, y)$; see [3]. Hence for each fixed y , $x \mapsto G^Y(x, y)$ is an excessive function of Y .

Let $k \geq 2$ be an integer so that $p := k/(k-1) < 1 + \varepsilon$. Then for any positive smooth measure ν with $\|G\nu\|_\infty = \|\mathbb{E}.A_\zeta^\nu\|_\infty < \infty$, by Hölder's inequality and (5.2),

$$G^Y\nu(x) = \mathbb{E}_x [Z_\zeta A_\zeta^\nu] \leq \left(\mathbb{E}_x [Z_\zeta^p] \right)^{1/p} \left(\mathbb{E}_x [(A_\zeta^\nu)^k] \right)^{1/k} \leq c_0 (k!)^{1/k} \|G\nu\|_\infty. \quad (5.3)$$

This implies that $\mathbf{K}_\infty(X) \subset \mathbf{K}_\infty(Y)$ and so $\mathbf{J}_\infty(X) \subset \mathbf{J}_\infty(Y)$. In particular, (5.3) implies that $G^Y\nu$ is bounded if $G\nu$ is bounded. \square

Clearly, the 1-subprocess $Y^{(1)}$ of Y can be obtained from $X^{(1)}$ through the Girsanov transform $\text{Exp}(M)$. When $F \in \mathbf{J}_\infty(X^{(1)})$, we have by applying Theorem 5.2 to $X^{(1)}$ that

$$\mathbf{K}_\infty(X^{(1)}) \subset \mathbf{K}_\infty(Y^{(1)}) \quad \text{and} \quad \mathbf{J}_\infty(X^{(1)}) \subset \mathbf{J}_\infty(Y^{(1)}). \quad (5.4)$$

Assume that μ is a signed smooth measure with $\mu^+ \in \mathbf{K}(X)$ and $G\mu^-$ bounded, and $F \in \mathbf{J}(X)$ symmetric. Define the non-local Feynman-Kac semigroup

$$T_t^{\mu, F} f(x) := \mathbb{E}_x \left[\exp \left(A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s) \right) f(X_t) \right], \quad t \geq 0.$$

It follows from the proof of [7, Proposition 2.3] and Hölder inequality that $\{T_t^{\mu, F}; t \geq 0\}$ is a strongly continuous semigroup in $L^p(E; m)$ for every $1 \leq p \leq \infty$. Moreover, it is easy to verify that $T_t^{\mu, F}$ is a symmetric operator in $L^2(E; m)$. The L^p -spectral bound of $\{T_t^{\mu, F}; t \geq 0\}$ is defined to be

$$\lambda_p(X, \mu + F) := - \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T_t^{\mu, F}\|_{p,p}.$$

The purpose of this section is to give necessary and sufficient conditions for $\lambda_p(X, \mu + F)$ to be independent of $1 \leq p \leq \infty$.

Assume from now that F is a bounded symmetric function in $\mathbf{J}_\infty(X^{(1)})$. Let A^F be the continuous additive functional defined by (3.3), whose Revuz measure is

$$\nu_F(dx) := \left(\int_E (e^{F(x,y)} - 1) N(x, dy) \right) \mu_H(dx).$$

By (3.4), we have

$$T_t^{\mu, F} f(x) = \mathbb{E}_x \left[\text{Exp}(M)_t e^{A_t^\mu + A_t^F} f(X_t) \right] = \mathbb{E}_x^\mathbb{Q} \left[e^{A_t^\mu + A_t^F} f(Y_t) \right] =: Q_t f(x). \quad (5.5)$$

Thus for any $1 \leq p \leq \infty$ and $t \geq 0$,

$$\|P_t^{\mu, F}\|_{p,p} = \|Q_t\|_{p,p} \quad \text{and so} \quad \lambda_p(X; \mu + F) = \lambda_p(Y; \mu + \nu_F). \quad (5.6)$$

Since F is bounded, $c_1|F| \leq |e^F - 1| \leq c_2|F|$. Thus in view of Lemma 5.1 and (5.4),

$$\text{the signed Revuz measure } \nu_F \text{ of } A^F \text{ belongs to } \mathbf{K}_\infty(X^{(1)}) \subset \mathbf{K}_\infty(Y^{(1)}). \quad (5.7)$$

In particular, it follows from (4.1), (5.1) and (5.6)-(5.7) that

$$\begin{aligned} & \lambda_2(X; \mu + F) = \lambda_2(Y; \mu + \nu_F) \\ &= \inf \left\{ \mathcal{E}^Y(u, u) - \int_E u(x)^2 \left(\int_E (e^{F(x,y)} - 1) N(x, dy) \right) \mu_H(dx) - \int_E u(x)^2 \mu(dx); \right. \\ & \quad \left. u \in \mathcal{F} \text{ with } \int_E u(x)^2 m(dx) = 1 \right\} \\ &= \inf \left\{ \mathcal{E}(u, u) - \int_{E \times E} u(x)u(y) (e^{F(x,y)} - 1) N(x, dy) \mu_H(dx) - \int_E u(x)^2 \mu(dx); \right. \\ & \quad \left. u \in \mathcal{F} \text{ with } \int_E u(x)^2 m(dx) = 1 \right\}. \end{aligned} \quad (5.8)$$

We start with an analogy of Corollary 4.3.

Theorem 5.3 *Assume that (4.3) holds and $m(E) < \infty$. Let μ be a signed smooth measure with $\mu^+ \in \mathbf{K}_\infty(X^{(\alpha)})$ and $G_\alpha \mu^-$ bounded for some $\alpha \geq 0$, and $F \in \mathbf{J}_\infty(X^{(\alpha)})$ symmetric. Then $\lambda_p(X, \mu + F)$ is independent of $p \in [1, \infty]$.*

Proof. By the same reasoning as that in Remark 4.2, we have

$$\lambda_p(X^{(\alpha)}, \mu + F) = \lambda_p(X, \mu + F) + \alpha \quad \text{for every } p \in [1, \infty]. \quad (5.9)$$

So it suffices to show that $\lambda_p(X^{(\alpha)}, \mu + F)$ is independent of $p \in [1, \infty]$. By regarding $X^{(\alpha)}$ as X , we may assume, without loss of generality, that the condition of the theorem holds with $\alpha = 0$.

Since P_{t_0} is a bounded linear operator from $L^2(E; m)$ to $L^\infty(E; m)$, by duality, P_{t_0} is a bounded linear operator from $L^1(E; m)$ to $L^2(E; m)$. Hence $P_{2t_0} : L^1(E; m) \rightarrow L^\infty(E; m)$ is bounded. Let

$$Z_t = \text{Exp}(M)_t = \exp \left(\sum_{0 < s \leq t} F(X_{s-}, X_s) - A_t^F \right), \quad t \geq 0.$$

Since $F \in \mathbf{J}_\infty(X)$, it follows from Khasminskii's inequality and the Markov property that (cf. [4, (3.11)]) that there are constants $c_1, c_2 > 0$ so that $\sup_{x \in E} \mathbb{E} [Z_t^2] \leq c_1 e^{c_2 t}$ for every $t > 0$. Denote by Y the Girsanov transformed process of X via Z . Then for every $f \in L^2(E; m)$,

$$|P_{2t_0}^Y f(x)| := |\mathbb{E}_x[f(Y_{2t_0})]| = \mathbb{E}_x [M_{2t_0} f(X_t)] \leq (\mathbb{E}_x [M_{2t_0}^2] \mathbb{E}_x [f(X_{2t_0})^2])^{1/2} \leq c \|f\|_{L^2(E; m)}.$$

This proves that condition (4.3) holds for Y with $2t_0$ in place of t_0 . Since

$$(\mu + \nu_F)^+ \leq \mu^+ + (\nu_F)^+ \quad \text{and} \quad (\mu + \nu_F)^- \leq \nu^- + (\nu_F)^-,$$

we deduce by Theorem 5.2 and (5.3) that $(\mu + \nu_F)^+ \in \mathbf{K}_\infty(Y)$ and $G^Y(\mu + \nu_F)^-$ is bounded. Hence by (5.6) and Corollary 4.3, $\lambda_p(X, \mu + F) = \lambda_p(Y, \mu + \nu_F)$ is independent in $p \in [1, \infty]$. \square

The next result is a non-local Feynman-Kac semigroup counterpart of Theorem 4.7.

Theorem 5.4 *Suppose that μ is a signed smooth measure with $\mu^+ \in \mathbf{K}_\infty(X^{(1)})$ and $G_1 \mu^-$ bounded, and $F \in \mathbf{J}_\infty(X^{(1)})$ symmetric.*

- (i) $\lambda_\infty(X, \mu + F) \geq \min\{\lambda_2(X, \mu + F), 0\}$. Consequently, $\lambda_p(X, \mu + F)$ is independent of $p \in [1, \infty]$ if $\lambda_2(X, \mu + F) \leq 0$.
- (iii) Assume in addition that X is conservative and that $\mu \in \mathbf{K}_\infty(X^{(1)})$. Then $\lambda_\infty(X, \mu + F) = 0$ if $\lambda_2(X, \mu + F) > 0$. Hence $\lambda_p(X, \mu + F)$ is independent of $p \in [1, \infty]$ if and only if $\lambda_2(X, \mu + F) \leq 0$.

Proof. For notational convenience, let $Z_t := \text{Exp}(M)_t$. By Remark 3.4(ii), $\{Z_t, t \geq 0\}$ is a uniformly integrable martingale under each \mathbb{P}_x . It follows that the Girsanov transformed process Y is transient and has a Green function G^Y . Furthermore, Y is conservative if so is X . It is clear that Y is m -irreducible since $\text{Exp}(M)_t > 0$ a.s.. Note that it follows from Theorem 5.2 applied to the subprocess $X^{(1)}$ that if $G_1 \mu^-$ is bounded, then so is $G_1^Y \mu^-$. Thus in view of (5.4) and Lemma 4.6, $\mu + \nu_F$ satisfies the condition of Theorem 4.7 for the symmetric process Y . The conclusion of the theorem now follows from Theorem 4.7 applied to $(Y, \mu + \nu_F)$ and relation (5.6). \square

The following theorem extends Theorem 4.8 to non-local Feynman-Kac semigroups.

Theorem 5.5 *Suppose that $1 \in \mathbf{K}_\infty(X^{(1)})$, $\mu \in \mathbf{K}_\infty(X^{(1)})$ and $F \in \mathbf{J}_\infty(X^{(1)})$ symmetric. Then $\lambda_p(X, \mu + F)$ is independent of $p \in [1, \infty]$.*

Proof. Let $Y^{(1)}$ be the Girsanov transformed process from $X^{(1)}$ via the function F . By the same reason as that for Theorem 5.4, we can apply Theorem 4.8 to $(Y^{(1)}, \mu + \nu_F)$ to conclude that $\lambda_p(Y^{(1)}, \mu + \nu_F)$ is independent of $p \in [1, \infty]$. Consequently, in view of (5.6) applied to $X^{(1)}$, $\lambda_p(X^{(1)}, \mu + F)$ is independent of $p \in [1, \infty]$. The conclusion of the theorem follows once one notices that $\lambda_p(X^{(1)}, \mu + F) = 1 + \lambda_p(X, \mu + F)$ for every $p \in [1, \infty]$. \square

Remark 5.6 (i) When $\mu = 0$, the conclusion of Theorem 5.4(i) recovers and extends the main result of [22], which was established by using a large deviation approach. The latter extends an earlier result of [20] where X is a rotationally symmetric α -stable process.

(ii) It follows from (5.5) and Remark 4.10(iii) that for any $\mu \in \mathbf{K}_\infty(X)$ and symmetric $F \in \mathbf{J}_\infty(X)$,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp \left(A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s) \right); t < \zeta \right] \geq -\lambda_2(X, \mu + F), \quad x \in E.$$

Thus whenever $\lambda_2(X, \mu + F) = \lambda_\infty(X, \mu + F)$, we have for every $x \in E$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp \left(A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s) \right); t < \zeta \right] = -\lambda_2(X, \mu + F).$$

(iii) The idea of using pure Girsanov transform to reduce a non-local Feynman-Kac transform to a continuous (local) Feynman-Kac transform of a new process is applicable in many other situations. For example, using this idea, one can easily deduce a large deviation result for general non-local Feynman-Kac functionals [21, Theorem 2.1] from the corresponding result of local Feynman-Kac transforms [19, Theorem 1.1]. In fact, such an approach shows that the large deviation result in [21, Theorem 2.1] holds in fact for $\mu \in \mathbf{K}_\infty(X)$ and symmetric $F \in \mathbf{J}_\infty(X)$, while [21, Theorem 2.1] requires $\mu \in \mathbf{K}_\infty(X)$ and symmetric $F \in \mathbf{A}_2(X)$, a subclass of $\mathbf{J}_\infty(X)$ introduced in [3]. It should be mentioned that [21, Theorem 2.1] implies the independence of $\lambda_p(X, \mu + F)$ in $p \in [1, \infty]$ under the same condition on X as in [19] (see Remark 4.10(ii) above) with $\mu \in \mathbf{K}_\infty(X)$ and symmetric $F \in \mathbf{A}_2(X)$. \square

We refer the reader to [5, Section 5] for concrete examples of functions in Kato classes $\mathbf{K}_\infty(X^{(1)})$ and $\mathbf{J}_\infty(X^{(1)})$. For example, it is shown in [5] that when X is a symmetric α -stable-like process on a global d -set E with d -measure m , then $L^p(E; m) \subset \mathbf{K}_\infty(X^{(1)})$ for every $p > d/\alpha$ when $\alpha \leq d$, and $L^p(E; m) \subset \mathbf{K}_\infty(X^{(1)})$ for every $p \geq 1$ when $0 < d < \alpha$. It is further shown there that when X is a symmetric diffusion on \mathbb{R}^d associated with a uniformly elliptic and bounded divergence form operator, then $L^p(\mathbb{R}^n; dx) \subset \mathbf{K}_\infty(X^{(1)})$ for every $p > n/2$ when $n \geq 3$, and $L^p(\mathbb{R}^n; dx) \subset \mathbf{K}_\infty(X^{(1)})$ for every $p \geq 1$ when $d = 1$ or 2 .

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Zhen-Qing Chen

Department of Mathematics, University of Washington, Seattle, WA 98195, USA

E-mail: zqchen@uw.edu