On subhamonicity for symmetric Markov processes

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(October 15, 2010)

Abstract

We establish the equivalence of the analytic and probabilistic notions of subharmonicity in the framework of general symmetric Hunt processes on locally compact separable metric spaces, extending an earlier work of the first named author on the equivalence of the analytic and probabilistic notions of harmonicity. As a corollary, we prove a strong maximum principle for locally bounded finely continuous subharmonic functions in the space of functions locally in the domain of the Dirichlet form under some natural conditions.

AMS 2000 Mathematics Subject Classification: Primary 60J45, 31C05; Secondary 31C25, 60J25.

Keywords and phrases: subharmonic function, uniformly integrable submartingale, symmetric Hunt process, Dirichlet form, Lévy system, strong maximum principle

*The research of this author is supported in part by NSF Grant DMS-0906743.

†The research of this author is partially supported by a Grant-in-Aid for Scientific Research (C) No. 19540220 from Japan Society for the Promotion of Science

1 Introduction

It is known that a function being subharmonic in a domain $D \subset \mathbb{R}^d$ can be defined by $\Delta u \le 0$ on D in the distributional sense; that is, $u \in W^{1,2}_{\rm loc}(D) := \{u \in L^2_{\rm loc}(D) \mid \nabla u \in L^2_{\rm loc}(D)\}$ so that

$$\int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla v(x)\, dx \le 0 \qquad \text{for any non-negative } v \in C_c^\infty(D).$$

When u is continuous, the above is equivalent to the following sub-averaging property by running a Brownian motion $X = (\Omega, X_t, \mathbf{P}_x)_{x\in\mathbb{R}^d}$: for every relatively compact open subset U of D:

$$u(X_{\tau_U}) \in L^1(\mathbf{P}_x) \quad \text{ and } \quad u(x) \le \mathbf{E}_x[u(X_{\tau_U})] \quad \text{ for every } x \in U.$$

Here $\tau_U := \inf\{t > 0 \mid X_t \notin U\}$ is the first exit time from U. A function u is said to be harmonic in D if both u and $-u$ are subharmonic in D. Recently, there have been interest from several areas of mathematics in determining whether the above two notions harmonicity and subharmonicity remain equivalent in a more general context, such as symmetric Hunt processes on locally compact separable metric spaces. For instance, due to their importance in theory and applications, there has been intense interest recently in studying discontinuous processes and non-local (or integro-differential) operators by both analytical and probabilistic approaches. See, e.g. [6, 7] and the references therein. So it is important to identify the connection between the analytic and probabilistic notions of subharmonic functions. Very recently, in [3] the first named author established the equivalence between the analytic and probabilistic notions of harmonic functions for symmetric Markov processes. Subsequently, the above equivalence is extended in [20] to non-symmetric Markov processes associated with sectorial Dirichlet forms.

In this paper, we extend the previous work [3], that is, we address the question of the equivalence of the analytic and probabilistic notions of subharmonicity in the context of symmetric Hunt processes on locally compact separable metric space (Theorem 2.9). As a byproduct of our result, we prove that strong maximum principle holds for locally bounded finely continuous $\mathcal{E}$-subharmonic functions under some conditions (Theorem 2.11). Strong maximum principles for subharmonic functions of second order elliptic operators have been powerful tools for various fields in analysis and geometry. In [16], the second named author established, by using analytic method, a strong maximum principle for finely continuous $\mathcal{E}$-subharmonic functions in the framework of irreducible local semi-Dirichlet forms whose Hunt processes satisfy the absolute continuity condition with respect to the underlying measure, which generalize the classical strong maximum principle for second order elliptic operators (for an extension of strong maximum principle for subharmonicity in the barrier sense, see also [17]). The strong maximum principle developed in [15, 16] can be applied to analysis or geometry for geometric singular spaces; Alexandrov spaces or spaces appeared in the Gromov-Hausdorff limit of Riemannian manifolds with uniform lower Ricci curvature bounds and so on. More concretely in [19], we establish splitting theorems for weighted Alexandrov spaces having measure contraction property, which are striking applications of the strong maximum principle treated in [15, 16] in terms of symmetric diffusion processes. The strong maximum principle established in this paper holds for symmetric Markov processes, which may possibly have discontinuous sample paths, on locally compact separable metric spaces, should have useful implications in the study of non-local operator or jump type symmetric Markov processes.

Let X be be an m-symmetric Hunt process on a locally compact separable metric space E whose associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular on $L^2(E;m)$. Let D be an open subset of E and τ_D is the first exit time from D by X. Motivated by the example at the beginning of this section, loosely speaking (see next section for precise statements), there are two ways to define a function u being subharmonic in D with respect to X: (a) (probabilistically) $t \mapsto u(X_{t\wedge\tau_D})$ is a $\mathbf{P}_x$-uniformly integrable submartingale for quasi-every $x \in D$; (b) (analytically) $\mathcal{E}(u, g) \leq 0$ for $g \in \mathcal{F} \cap C_c^+(D)$. We will show in Theorem 2.9 below that these two definitions are equivalent under some integrability conditions as imposed in the previous work [3] by the first author. Note that even in the Brownian motion case, a function u that is subharmonic in D is typically not in the domain $\mathcal{F}$ of the Dirichlet form. Denote by $\mathcal{F}_{D,\mathrm{loc}}$ the family of functions u on E such that for every relatively compact open subset D_1 of D, there is a function $f \in \mathcal{F}$ so that $u = f$ m-a.e. on D_1. To show these two definitions are equivalent, the crux of the difficulty is to

(i) appropriately extend the definition of $\mathcal{E}(u, v)$ to functions u in $\mathcal{F}_{D,\mathrm{loc}}$ that satisfy some minimal integrability condition when X is discontinuous so that $\mathcal{E}(u, v)$ is well defined for every $v \in \mathcal{F} \cap C_c(D)$;

(ii) show that if u is subharmonic in D in the probabilistic sense, then $u \in \mathcal{F}_{D,\mathrm{loc}}$ and $\mathcal{E}(u, v) \leq 0$ for every non-negative $v \in \mathcal{F} \cap C_c(D)$.

The question (i) is solved in the previous work [3]. The main focus of this paper is to address the second question (ii). For (ii), we establish a Riesz type decomposition theorem (Lemma 3.6) for $\mathcal{E}$-subharmonic functions, which is a crucial step in proving our main result.

If one assumes a priori that $u \in \mathcal{F}$, then the equivalence of (a) and (b) is easy to establish. In next section, we give precise definitions, statements of the main results and their proofs. Four examples are given to illustrate the main results of this paper. We use ":="as a way of definition. For two real numbers a and b, $a \wedge b := \min\{a, b\}$.

The results of this paper can be extended to non-symmetric Hunt processes associated with sectorial Dirichlet forms. We will not pursuit this generalization here in this paper.

2 Main result

Let $X = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \zeta, \mathbf{P}_x, x \in E)$ be an m-symmetric right Markov process on a space E, where m is a positive σ-finite measure with full topological support on E. A cemetery state ∂ is added to E to form $E_\partial := E \cup \{\partial\}$, and Ω is the totality of right-continuous, left-limited sample paths from $[0, \infty)$ to E_∂ that hold the value ∂ once attaining it. Throughout this paper, every function f on E is automatically extended to be a function on E_∂ by setting $f(\partial) = 0$. For any $\omega \in \Omega$, we set $X_t(\omega) := \omega(t)$. Let $\zeta(\omega) := \inf\{t \geq 0 \mid X_t(\omega) = \partial\}$ be the life time of X. Throughout this paper, we use the convention that $X_\infty(\omega) := \partial$. As usual, $\mathcal{F}_\infty$ and $\mathcal{F}_t$ are the minimal augmented σ-algebras obtained from $\mathcal{F}^0_\infty := \sigma\{X_s \mid 0 \leq s < \infty\}$ and $\mathcal{F}^0_t := \sigma\{X_s \mid 0 \leq s \leq t\}$ under

$\{\mathbf{P}_x : x \in E\}$. For a Borel subset B of E, $\tau_B := \inf\{t \geq 0 \mid X_t \notin B\}$ (the *exit time* of B) is an $(\mathcal{F}_t)$-stopping time.

The transition semigroup $\{P_t : t \geq 0\}$ of X is defined by

$$P_t f(x) := \mathbf{E}_x[f(X_t)] = \mathbf{E}_x[f(X_t) : t < \zeta], \qquad t \geq 0.$$

Each P_t may be viewed as an operator on $L^2(E;m)$, and taken as a whole these operators form a strongly continuous semigroup of self-adjoint contractions. The Dirichlet form associated with X is the bilinear form

$$\mathcal{E}(u,v) := \lim_{t \downarrow 0} t^{-1}(u - P_t u, v)_m$$

defined on the space

$$\mathcal{F} := \left\{ u \in L^2(E;m) \,\middle|\, \sup_{t>0}\ t^{-1}(u - P_t u, u)_m < \infty \right\}.$$

Here we use the notation $(f,g)_m := \int_E f(x)g(x)\, m(dx)$ and we shall use $\|f\|_2 := \sqrt{(f,f)_m}$ for $f, g \in L^2(E;m)$. P_t is extended to be a strongly continuous semigroup $\{T_t; t \geq 0\}$ on $L^2(E;m)$. Without loss of generality, we may assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E;m)$ and the X is an m-symmetric Hunt process, where E is a locally compact separable metric space having a one point compactification $E_\partial := E \cup \{\partial\}$ and m is a positive Radon measure with full topological support (see [8]).

A set $B \subset E_\partial$ is called *nearly Borel* if for each probability measure μ on E_∂, there exist Borel sets $B_1, B_2 \subset E_\partial$ such that $B_1 \subset B \subset B_2$ and $\mathbf{P}_\mu(X_t \in B_2 \setminus B_1 \text{ for some } t \geq 0) = 0$. Any hitting time $\sigma_B := \inf\{t > 0 \mid X_t \in B\}$ is an $(\mathcal{F}_t)$-stopping time for nearly Borel subset of E_∂ (see Theorem 10.7 and the remark after Definition 10.21 in [1]). A subset B of E_∂ is said to be X-invariant if B is nearly Borel and

$$\mathbf{P}_x(X_t \in B_\partial, X_{t-} \in B_\partial \ \text{ for all } t \geq 0) = 1 \qquad \text{for any } x \in B.$$

A set A is *finely open* if for each $x \in A$ there exists a nearly Borel subset $B = B(x)$ of E such that $B \supset E \setminus A$ and $\mathbf{P}_x(\sigma_B > 0) = 1$. A set N is called *properly exceptional* if $E \setminus N$ is X-invariant and $m(N) = 0$. A nearly Borel set N is called *m-polar* if $\mathbf{P}_m(\sigma_N < \infty) = 0$ and any subset N of E is called *exceptional* if there exists an m-polar set $\tilde{N}$ containing N. Clearly any properly exceptional set N is exceptional. A function defined q.e. on an open subset D of E is said to be *q.e. finely continuous* on D if there exists a properly exceptional Borel set N such that u is Borel measurable and finely continuous on $D \setminus N$. It is known (cf. [12]) a quasi-continuous function on D is q.e. finely continuous on D.

Let $\mathcal{F}_e$ be the family of m-measurable functions u on E such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of $\mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. We call $\{u_n\}$ as above an approximating sequence for $u \in \mathcal{F}_e$. For any $u, v \in \mathcal{F}_e$ and its approximating sequences $\{u_n\}$, $\{v_n\}$ the limit $\mathcal{E}(u,v) = \lim_{n\to\infty} \mathcal{E}(u_n, v_n)$ exists and does not depend on the choices of the approximating sequences for u, v. It is known that $\mathcal{E}^{1/2}$ on $\mathcal{F}_e$ is a semi-norm and $\mathcal{F} = \mathcal{F}_e \cap L^2(E;m)$. We call

$(\mathcal{E},\mathcal{F}_e)$ *the extended Dirichlet space* of $(\mathcal{E},\mathcal{F})$. Any $u\in\mathcal{F}_e$ admits a quasi-continuous m-version $\tilde{u}$. Throughout this paper, we always take quasi-continuous m-version of the element of $\mathcal{F}_e$, that is, we omit *tilde* from $\tilde{u}$ for $u\in\mathcal{F}_e$

Let D be an open subset of E. We define

$$\begin{cases} \mathcal{F}_D := \{u\in\mathcal{F}\mid u=0 \ \ \mathcal{E}\text{-q.e. on } E\setminus D\}, \\ \mathcal{E}^D(u,v) := \mathcal{E}(u,v) \qquad \text{for } u,v\in\mathcal{F}_D. \end{cases}$$

Then $(\mathcal{E}^D,\mathcal{F}_D)$ is again a regular Dirichlet form on $L^2(D;m)$, which is called the *part space* in D. Denote by $\mathcal{F}_{D,\mathrm{loc}}$ (resp. $(\mathcal{F}_D)_{\mathrm{loc}}$) *the space of functions locally in* $\mathcal{F}$ on D (resp. *the space of functions locally in* $\mathcal{F}_D$); that is, $u\in\mathcal{F}_{D,\mathrm{loc}}$ (resp. $u\in(\mathcal{F}_D)_{\mathrm{loc}}$) if and only if for any relatively compact open set U with $\overline{U}\subset D$ there exists $u_U\in\mathcal{F}$ (resp. $u_U\in\mathcal{F}_D$) such that $u=u_U$ m-a.e. on U. Clearly $(\mathcal{F}_D)_{\mathrm{loc}}\subset\mathcal{F}_{D,\mathrm{loc}}$ and $\mathbf{1}_D\in(\mathcal{F}_D)_{\mathrm{loc}}$. Any $u\in\mathcal{F}_{D,\mathrm{loc}}$ admits an m-version $\tilde{u}$ of u which is quasi-continuous on D. As remarked above, we always take such m-version and omit *tilde* from $\tilde{u}$ for $u\in\mathcal{F}_{D,\mathrm{loc}}$. We can see that $\mathcal{F}_{D,\mathrm{loc}}\cap L^\infty_{\mathrm{loc}}(D;m)\subset(\mathcal{F}_D)_{\mathrm{loc}}$. Indeed, for $u\in\mathcal{F}_{D,\mathrm{loc}}\cap L^\infty_{\mathrm{loc}}(D;m)$, we can take $u_U\in\mathcal{F}_b$ such that $u=u_U$ m-a.e. on U, because $u_U=(-\|u\|_{U,\infty})\vee(u_U\wedge\|u\|_{U,\infty})$ m-a.e. on U, where $\|u\|_{U,\infty}:=m\text{-ess-}\sup_U|u|$. Taking $\phi\in\mathcal{F}\cap C_c(E)$ with $\phi=1$ on U and $\phi=0$ on D^c. Then $u_U\phi\in\mathcal{F}_D$ and $u=u_U\phi$ m-a.e. on U.

Definition 2.1 (Sub/Super-harmonicity) Let D be an open set in E. We say that a nearly Borel measurable function u defined on E is *subharmonic (resp. superharmonic) in D* if for any relatively compact open subset U of D with $\overline{U}\subsetneq D$, $t\mapsto u(X_{t\wedge\tau_U})$ is a uniformly integrable right continuous $\mathbf{P}_x$-submartingale (resp. $\mathbf{P}_x$-supermartingale) for q.e. $x\in E$. A nearly Borel function u on E is said to be *harmonic in D* if u is both superharmonic and subharmonic in D. If $t\mapsto u(X_{t\wedge\tau_U})$ is a uniformly integrable right continuous $\mathbf{P}_x$-submartingale for all $x\in E$, u is called *subharmonic in D without exceptional set*. The *subharmonicity/harmonicity in D without exceptional set* is analogously defined.

Definition 2.2 (Sub/Super-harmonicity in the weak sense) Let D be an open set in E. We say that a nearly Borel function u defined on E is *subharmonic (resp. superharmonic) in D in the weak sense* if u is q.e. finely continuous in D and for any relatively compact open subset U with $\overline{U}\subsetneq D$, $\mathbf{E}_x[|u|(X_{\tau_U})]<\infty$ for q.e. $x\in E$ and $u(x)\le\mathbf{E}_x[u(X_{\tau_U})]$ (resp. $u(x)\ge\mathbf{E}_x[u(X_{\tau_U})]$) holds for q.e. $x\in E$. A nearly Borel measurable function u on E is said to be *harmonic in D in the weak sense* if u is both superharmonic and subharmonic in D in the weak sense. If u is finely continuous (nearly) Borel in D and for any relatively compact open subset U with $\overline{U}\subsetneq D$, $\mathbf{E}_x[|u|(X_{\tau_U})]<\infty$ for all $x\in E$, and $u(x)\le\mathbf{E}_x[u(X_{\tau_U})]$ holds for all $x\in U$, then u is called *subharmonic in D in the weak sense without exceptional set*. The *subharmonicity/harmonicity in D in the weak sense without exceptional set* is analogously defined.

Clearly $\mathbf{1}_D$ is superharmonic in D in the weak sense.

Remark 2.3 Our definition on the subharmonicity or superharmonicity in the weak sense is different from what is defined in the Dynkin's textbook [11] and is weaker than it when X is an

m-irreducible diffusion process satisfying (2.1) below. Actually, superharmonicity of u in [11] requires u be locally bounded from below instead of the $\mathbf{P}_x$-integrability of $u(X_{\tau_U})$ for any relatively compact open U with $\overline{U} \subset D$. Indeed, suppose that X is a diffusion process and u is a superharmonic function in D in the sense of [11]. Then for U as above, we have

$$\mathbf{E}_x[|u(X_{\tau_U})|] \leq \mathbf{E}_x[u(X_{\tau_U})] + 2\mathbf{E}_x[(-u)^+(X_{\tau_U})] \leq u(x) + 2(-\inf_{\partial U} u)^+ < \infty$$

for q.e. $x \in E$. □

@ We introduce the following condition:

$$\text{For any relatively compact open set } U \text{ with } \overline{U} \subsetneq D,\ \mathbf{P}_x(\tau_U < \infty) > 0 \ \text{ for q.e. } x \in U. \tag{2.1}$$

Condition (2.1) is satisfied if $(\mathcal{E}, \mathcal{F})$ is *m-irreducible*, that is, any (T_t)-invariant set B is trivial in the sense that $m(B) = 0$ or $m(B^c) = 0$.

It will be shown in Lemma 3.9 that under condition (2.1), every subharmonic function in D is a subharmonic function in D in the weak sense.

In what follows, all functions denoted by u or u_i, $(i = 1, 2)$ are defined on E and are (nearly) Borel measurable and finite quasi everywhere.

For an open set $D \subset E$, we consider the following conditions for a (nearly) Borel function u on E that are introduced in [3]. For any relatively compact open sets U, V with $\overline{U} \subset V \subset \overline{V} \subset D$,

$$\int_{U \times (E \setminus V)} |u(y)| J(dxdy) < \infty \tag{2.2}$$

and

$$\mathbf{1}_U \mathbf{E}_\cdot[(1 - \phi_V)|u|(X_{\tau_U})] \in (\mathcal{F}_U)_e, \tag{2.3}$$

where $\phi_V \in \mathcal{F} \cap C_c(E)$ with $0 \leq \phi_V \leq 1$ and $\phi_V = 1$ on V.

As is noted in [3], in many concrete cases such as in Examples 2.12-2.14 in [3] (see also Examples 4.1-4.4 below), one can show that condition (2.2) implies condition (2.3).

Remark 2.4 (i) By [4, Lemma 6.7.6], condition (2.3) is equivalent to

$$\int_{U \times (E \setminus V)} \mathbf{E}_x[(1 - \phi_V)|u|(X_{\tau_U})](1 - \phi_V(y))|u(y)| J(dxdy) < \infty. \tag{2.4}$$

(ii) In view of [3, Lemma 2.3], every nearly Borel bounded function u on E satisfies both (2.2) and (2.3).

(iii) If $u \in \mathcal{F}_{D,\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(D;m)$, then u is bounded q.e. on any relatively compact open U with $\overline{U} \subset D$, so for any U, V as above, (2.2) is equivalent to

$$\int_{U\times(E\setminus V)} |u(y)-u(x)|J(dxdy) < \infty \tag{2.5}$$

for such u. Clearly, any $u \in \mathcal{F}_e$ satisfies

$$\int_{U\times(E\setminus V)} |u(y)-u(x)|J(dxdy) \le J(U\times V^c)^{1/2}\left(\int_{E\times E}|u(y)-u(x)|^2J(dxdy)\right)^{1/2} < \infty;$$

that is, (2.5) is satisfied by $u \in \mathcal{F}_e$. Furthermore, by Lemma 2.5 of [3], both (2.2) and (2.3) hold for every $u \in \mathcal{F}_e \cap L^\infty_{\mathrm{loc}}(D;m)$. □

The following is proved in [3].

Lemma 2.5 (cf. Lemma 2.6 in [3]) *Let D be an open set of E. Suppose that u is a locally bounded function on D such that u belongs to $\mathcal{F}_{D,\mathrm{loc}}$ and it satisfies condition* (2.2)*. Then for every $v \in \mathcal{F}\cap C_c(D)$, each term in the following expression*

$$\frac{1}{2}\mu^c_{\langle u,v\rangle}(D) + \frac{1}{2}\int_{E\times E}(u(x)-u(y))(v(x)-v(y))J(dxdy) + \int_D u(x)v(x)\kappa(dx)$$

is well-defined and finite; the sum will still be denoted as $\mathcal{E}(u,v)$.

If u is a function on E that can be decomposed as u_1+u_2, where $u_1 \in \mathcal{F}_e$ and u_2 is a locally bounded function on D such that u belongs to $\mathcal{F}_{D,\mathrm{loc}}$ and it satisfies condition (2.2), then we define for every $v \in \mathcal{F}\cap C_c(D)$, $\mathcal{E}(u,v) = \mathcal{E}(u_1,v)+\mathcal{E}(u_2,v)$. It is easy to see that such $\mathcal{E}(u,v)$ is well-defined, whose value is independent of a particular decomposition of u into u_1+u_2.

Definition 2.6 ($\mathcal{E}$-sub/super-harmonicity) *Let $u \in \mathcal{F}_{D,\mathrm{loc}}\cap L^\infty_{\mathrm{loc}}(D;m)$ be a function satisfying the condition* (2.2)*. We say that u is $\mathcal{E}$-subharmonic (resp. $\mathcal{E}$-superharmonic) in D if and only if $\mathcal{E}(u,v) \le 0$ (resp. $\mathcal{E}(u,v) \ge 0$) for every non-negative $v \in \mathcal{F}\cap C_c(D)$. A function $u \in \mathcal{F}_{D,\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(D;m)$ satisfying condition* (2.2) *is said to be $\mathcal{E}$-harmonic in D if u is both $\mathcal{E}$-superharmonic and $\mathcal{E}$-subharmonic in D. When $D = E$, we omit the phrase 'in D'.*

Note that $\mathbf{1}_D \in \mathcal{F}_{D,\mathrm{loc}}$ satisfies (2.2) and is $\mathcal{E}$-superharmonic in D. It is $\mathcal{E}$-harmonic in D provided $\kappa(D)=0$ and $J(D,D^c)=0$.

Definition 2.7 *For a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(E;m)$, a linear subspace $\mathcal{C}$ of $\mathcal{F}\cap C_c(E)$ is said to be a special standard core of $(\mathcal{E},\mathcal{F})$ if the following holds:*

(i) *$\mathcal{C}$ is dense both in $(\mathcal{F}, \|\cdot\|_{\mathcal{E}_1})$ and in $(C_c(E), \|\cdot\|_\infty)$;*

(ii) *for every $\varepsilon > 0$, there is a a function $\varphi_\varepsilon : \mathbb{R}\to[-\varepsilon, 1+\varepsilon]$ satisfying $\varphi_\varepsilon(t) = t$ for $t \in [0,1]$ and $0 \le \varphi_\varepsilon(t)-\varphi_\varepsilon(s) \le t-s$ for every $s < t$ so that $\varphi_\varepsilon(\mathcal{C}) \subset \mathcal{C}$;*

(iii) *for any compact set K and relatively compact open set U with $K \subset U$, there exists a non-negative function $f \in \mathcal{C}$ such that $f = 1$ on K and $f = 0$ on $E \setminus U$.* □

Clearly $\mathcal{F} \cap C_c(E)$ is a special standard core of a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$. In the remaining of this paper, we fix a special standard core $\mathcal{C}$ of $(\mathcal{E}, \mathcal{F})$. Let D be an open subset of E. It is known that (cf. [4, Theorem 3.3.9]) the space $\mathcal{C}_D := \{f \in \mathcal{C} : \text{supp}[f] \subset D\}$ is a core of $(\mathcal{E}^D, \mathcal{F}_D)$.

Theorem 2.8 *Let $u \in \mathcal{F}_{D,\text{loc}} \cap L^\infty_{\text{loc}}(D; m)$ be a function satisfying the condition* (2.2)*. Then u is $\mathcal{E}$-subharmonic (resp. $\mathcal{E}$-superharmonic) in D if and only if there is a special standard core $\mathcal{C}$ of $(\mathcal{E}, \mathcal{F})$ so that $\mathcal{E}(u, v) \le 0$ (resp. $\mathcal{E}(u, v) \ge 0$) for every non-negative $v \in \mathcal{C}_D$.*

Proof. We only prove it for the $\mathcal{E}$-subharmonic case, as the $\mathcal{E}$-superharmonic case is similar. It suffices to prove the 'if' part.

Suppose that $u \in \mathcal{F}_{D,\text{loc}} \cap L^\infty_{\text{loc}}(D; m)$ is a function satisfying the condition (2.2) and that there is a special standard core $\mathcal{C}$ of $(\mathcal{E}, \mathcal{F})$ so that $\mathcal{E}(u, v) \le 0$ for every $v \in \mathcal{C}_D$. Let U be a relatively compact open subset of D with $\overline{U} \subset D$. Take $\phi \in \mathcal{F} \cap \mathcal{C}_D$ such that $0 \le \phi \le 1$ and $\phi = 1$ on a relatively compact open neighborhood V of $\overline{U}$ with $\overline{V} \subset D$. Define $h_0(x) := \mathbf{E}_x[u(X_{\tau_U})]$. It is known from the proof of [3, Theorem 2.7] (more specifically, from (2.11) to (2.13) there) that h_0 is well-defined with $u - h_0 \in (\mathcal{F}_U)_e$, $h_0 - (1 - \phi)u \in \mathcal{F}_e$ and that

$$\mathcal{E}(h_0, v) = 0 \qquad \text{for every } v \in \mathcal{F} \cap C_c(U). \tag{2.6}$$

Consequently, $\mathcal{E}(u - h_0, v) \le 0$ for every non-negative $v \in \mathcal{C}_U$. We claim that

$$\mathcal{E}(u - h_0, v^+) \le 0 \qquad \text{for every } v \in \mathcal{C}_U. \tag{2.7}$$

For $\varepsilon > 0$, let φ_ε be the contraction function appeared in the definition of special standard core $\mathcal{C}$. For $v \in \mathcal{C}_U$ with $\|v\|_\infty \le 1$, note that $\varphi_\varepsilon(v) \in \mathcal{C}_U$ and that $\text{supp}[\varphi_\varepsilon(v)] \subset \text{supp}[v]$ for every $\varepsilon > 0$. Let $\psi \in \mathcal{C}_U$ so that $0 \le \psi \le 1$ and $\psi = 1$ on $\text{supp}[v]$. For $k \ge 1$, $f_k := \varphi_{1/k}(v) + (1/k)\psi$. Then $f_k \in \mathcal{C}_D$ is non-negative, and the sequence $\{f_k, k \ge 1\}$ is $\mathcal{E}$-bounded and converges to v^+ pointwise on U. It follows from Banach-Saks theorem that there is a subsequence of $\{f_k, k \ge 1\}$ whose Cesàro mean sequence is $\mathcal{E}_1$-convergent to v^+. It follows that (2.7) holds for every $v \in \mathcal{C}_U$ with $\|v\|_\infty \le 1$ and hence for every $v \in \mathcal{C}_U$.

Since $\mathcal{C}_U$ is a core of $(\mathcal{E}^U, \mathcal{F}_U)$, for every non-negative $v \in (\mathcal{F}_U)_e$, there is an $\mathcal{E}$-Cauchy sequence $\{v_n, n \ge 1\}$ in $\mathcal{C}_U$ that converges to v m-a.e. on U. Then $\{v_n^+, n \ge 1\}$ is $\mathcal{E}$-bounded and converges to v m-a.e. on U. Applying Banach-Saks theorem again, there is a subsequence of $\{v_n^+, n \ge 1\}$ whose Cesàro mean sequence is $\mathcal{E}_1$-convergent to v. It follows from (2.7) that $\mathcal{E}(u - h_0, v) \le 0$. This combined with (2.6) implies that $\mathcal{E}(u, v) = 0$ for every $v \in \mathcal{F} \cap C_c(U)$ and hence for every $v \in \mathcal{F} \cap C_c(D)$. In other words, u is $\mathcal{E}$-subharmonic on D. The proof of the theorem is now complete. □

The harmonic version of the above theorem has been established earlier in [4, Section 6.7].

We say that X satisfies *the absolute continuity condition with respect to m* if the transition kernel $P_t(x, dy)$ of X is absolutely continuous with respect to $m(dy)$ for any $t > 0$ and $x \in E$.

Our main theorem below is an analogy of Theorem 2.11 in [3] for subharmonic functions.

Theorem 2.9 *Let D be an open subset of E. Suppose that a nearly Borel $u \in L^\infty_{\rm loc}(D; m)$ satisfies conditions* (2.2) *and* (2.3)*. Then*

(i) *u is subharmonic in D if and only if $u \in (\mathcal{F}_D)_{\rm loc}$ and it is $\mathcal{E}$-subharmonic in D.*

(ii) *Assume that* (2.1) *holds. Then u is subharmonic in D if and only if u is subharmonic in D in the weak sense, that is, for any relatively compact open set U with $\overline{U} \subsetneq D$, $u(X_{\tau_U})$ is $\mathbf{P}_x$-integrable and $u(x) \le \mathbf{E}_x[u(X_{\tau_U})]$ for q.e. $x \in E$.*

Moreover, if we assume that X satisfies the absolute continuity condition with respect to m and u is (nearly) Borel finely continuous, then the assertion in (i) *(resp.* (ii)*) can be strengthened to the subharmonicity in D without exceptional set (resp. subharmonicity in D in the weak sense without exceptional set).*

Theorem 2.9 will be established through Lemma 3.9 and Theorems 3.10-3.12. As an application of Theorem 2.9, we have the following.

Corollary 2.10 (i) *Let $\eta \in C^1(\mathbb{R})$ be a convex function and $u \in \mathcal{F}_{D,{\rm loc}} \cap L^\infty_{\rm loc}(D; m)$ be an $\mathcal{E}$-harmonic function in D satisfying conditions* (2.2)–(2.3)*. Suppose that η has bounded first derivative or u is bounded on E. Then $\eta(u) \in \mathcal{F}_{D,{\rm loc}}$ and is $\mathcal{E}$-subharmonic in D satisfying conditions* (2.2)–(2.3)*.*

(ii) *The conclusion of* (i) *remains true if $\eta \in C^1(\mathbb{R})$ is an increasing convex function and $u \in \mathcal{F}_{D,{\rm loc}} \cap L^\infty_{\rm loc}(D; m)$ is an $\mathcal{E}$-subharmonic function in D satisfying conditions* (2.2)–(2.3)*.*

(iii) *Let $p \ge 1$ and $u \in \mathcal{F}_{D,{\rm loc}}$ be an $\mathcal{E}$-harmonic function in D that is locally bounded in D and satisfies conditions* (2.2)–(2.3)*. Suppose that $|u|^p$ satisfies conditions* (2.2) *and* (2.3)*, and that* (2.1) *holds. Then $|u|^p \in \mathcal{F}_{D,{\rm loc}}$ and is $\mathcal{E}$-subharmonic in D.*

(iv) *Let $u_1, u_2 \in \mathcal{F}_{D,{\rm loc}} \cap L^\infty_{\rm loc}(D; m)$ be $\mathcal{E}$-subharmonic functions in D satisfying conditions* (2.2)–(2.3)*. Then $u_1 \vee u_2 \in \mathcal{F}_{D,{\rm loc}}$ satisfies* (2.2)–(2.3) *and is $\mathcal{E}$-subharmonic in D.*

As a consequence of Corollary 2.10(iv), we have the following strong maximum principle.

Theorem 2.11 (Strong maximum principle) *Assume that D is an open subset of E, X satisfies the absolute continuity condition with respect to m and $(\mathcal{E}^D, \mathcal{F}_D)$ is m-irreducible. Suppose that $u \in \mathcal{F}_{D,{\rm loc}}$ satisfying conditions* (2.2)-(2.3) *is a locally bounded finely continuous $\mathcal{E}$-subharmonic function in D. If u attains a maximum at a point $x_0 \in D$. Then $u^+ \equiv u^+(x_0)$ on D. If in addition $\kappa(D) = 0$, then $u \equiv u(x_0)$ on D.*

3 Proofs

In this section, we present proofs for Theorem 2.9, Corollary 2.10 and Theorem 2.11. First we prepare a lemma.

Lemma 3.1 *For $u \in \mathcal{F}$, the following are equivalent.*

(i) $\mathcal{E}(u, v) \le 0$ *for every* $v \in \mathcal{F}^+$.

(ii) $T_t u \ge u$ *m-a.e. on* E *for every* $t \ge 0$.

Proof. Clearly (ii) implies (i). The proof of (i)⇒(ii) is quite similar to the proof of Lemma 2.2 in [16]. So it is omitted. Note that we do not assert that $u \le 0$ m-a.e. on E. □

Lemma 3.2 *For $u_1, u_2 \in \mathcal{F}_e$, if u_1 and u_2 are $\mathcal{E}$-subharmonic, then so is $u_1 \vee u_2$.*

Proof. Let $g \in L^1(E; m)$ be such that $0 < g \le 1$ m-a.e. on E and that $u_1, u_2 \in L^2(E; gm)$. The measure gm has full quasi-support with respect to $(\mathcal{E}, \mathcal{F})$ by Corollary 4.6.1 in [12]. Denote by $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ the Dirichlet form of the process X time-changed by the inverse of $A_t := \int_0^t g(X_s)ds$. Then by (6.2.22)-(6.2.23) of [12], $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}_e) = (\mathcal{E}, \mathcal{F}_e)$ and $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_e \cap L^2(E; gm)$. By Theorem 6.2.1 of [12], $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is a regular Dirichlet form on $L^2(E; gm)$ with core $\widetilde{\mathcal{F}} \cap C_c(E) = \mathcal{F} \cap C_c(E)$. So u_1 and u_2 are $\widetilde{\mathcal{E}}$-subharmonic functions in $\widetilde{\mathcal{F}}$. Let $\{\widetilde{T}_t, t \ge 0\}$ be the semigroup associated with $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$. From Lemma 3.1, we see $u_1 \le \widetilde{T}_t u_1$ and $u_2 \le \widetilde{T}_t u_2$ m-a.e. on E, which implies $u_1 \vee u_2 \le \widetilde{T}_t(u_1 \vee u_2)$. By Lemma 3.1 again, $u_1 \vee u_2$ is an $\widetilde{\mathcal{E}}$-subharmonic function in $\widetilde{\mathcal{F}} \subset \widetilde{\mathcal{F}}_e = \mathcal{F}_e$. The conclusion of the lemma now follows. □

Lemma 3.3 *Let v_1 be an excessive function of X and $v_2 \in \mathcal{F}_e$ such that $v_1 \le v_2$ m-a.e. on E. Then $v_1 \in \mathcal{F}_e$ with $\mathcal{E}(v_1, v_1) \le \mathcal{E}(v_2, v_2)$.*

Proof. As in the proof of Lemma 3.2, let $g \in L^1(E; m)$ be such that $0 < g \le 1$ m-a.e. on E and that $v_1, v_2 \in L^2(E; gm)$. Let $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ be the time-changed Dirichlet form with semigroup $\{\widetilde{T}_t, t \ge 0\}$ as in the proof of Lemma 3.2. Note that $v_2 \in \widetilde{\mathcal{F}}_e \cap L^2(E; gm) = \widetilde{\mathcal{F}}$. By Proposition 2.8 in [1], we have $\mathbf{E}_x[v_1(X_{\tau_t})] \le v_1(x)$, where $\tau_t := \inf\{s > 0 \mid \int_0^s g(X_u)du > t\}$. That is, $\widetilde{T}_t v_1 \le v_1$. Observe that since $\widetilde{T}_t$ is a contraction operator on $L^2(E; gm)$ for each $t > 0$, $(f, g) \mapsto (f, g - \widetilde{T}_t g)_{gm}$ is a non-negative symmetric quadratic form on $L^2(E; gm)$. Hence

$$|(f,\, g - \widetilde{T}_t g)_{gm}| \le (f,\, f - \widetilde{T}_t f)_{gm}^{1/2} \cdot (g,\, g - \widetilde{T}_t g)_{gm}^{1/2}.$$

Since $v_1 \in L^2(E; gm)$ and

$$(v_1, v_1 - \widetilde{T}_t v_1)_{gm} \le (v_2, v_1 - \widetilde{T}_t v_1)_{gm} = (v_1, v_2 - \widetilde{T}_t v_2)_{gm} \le (v_1, v_1 - \widetilde{T}_t v_1)_{gm}^{1/2} \cdot (v_2, v_2 - \widetilde{T}_t v_2)_{gm}^{1/2},$$

we have

$$\lim_{t \to 0} \frac{1}{t}(v_1,\, v_1 - \widetilde{T}_t v_1)_{gm} \le \lim_{t \to 0} \frac{1}{t}(v_2,\, v_2 - \widetilde{T}_t v_2)_{gm} = \widetilde{\mathcal{E}}(v_2, v_2) < \infty.$$

It follows that $v_1 \in \widetilde{\mathcal{F}} \subset \widetilde{\mathcal{F}}_e = \mathcal{F}_e$ with $\mathcal{E}(v_1, v_1) \le \mathcal{E}(v_2, v_2)$. □

Lemma 3.4 *Let D be an open set of E. Suppose that $|u_1| \leq |u_2|$ q.e. on D and u_2 satisfies* (2.3)*. Then u_1 satisfies* (2.3)*.*

Proof. Let U, V be relatively compact open sets such that $\overline{U} \subset V \subset \overline{V} \subsetneq D$. Note that $\mathbf{E}_x[u(X_{\tau_U})]$ is excessive with respect to X^U for any non-negative nearly Borel function u. For $i = 1, 2$ and $v_i(x) := \mathbf{E}_x[(1-\phi_V)|u_i|(X_{\tau_U})]$, by assumption, $v_2 \in (\mathcal{F}_U)_e$ and $|v_1| \leq |v_2|$ q.e. on U. It follows from Lemma 3.3 that $v_1 \in (\mathcal{F}_U)_e$, namely u_1 satisfies (2.3). □

The following lemma is needed for the proof of Corollary 3.7 below.

Lemma 3.5 *Let $\phi \in L^2(E;m) \cap L^1(E;m)$ and $v \in \mathcal{F}_e \cap L^\infty(E;m)$. Then $\mathcal{E}_\alpha(G_\alpha \phi, v) = \int_E \phi\, v\, dm$ for any $\alpha > 0$.*

Proof. Let $\{v_n, n \geq 1\}$ be an $\mathcal{E}$-Cauchy sequence of $\mathcal{F}$ such that $\lim_{n\to\infty} v_n = v$ m-a.e. For each $n \geq 1$, define $w_n = (-\|v\|_\infty) \vee v_n \wedge \|v\|_\infty$. Then $\{w_n, n \geq 1\}$ is an $\mathcal{E}$-bounded sequence in $\mathcal{F}$ and it converges boundedly to v m-a.s. By Banach-Saks theorem, there is a subsequence of $\{w_n, n \geq 1\}$ whose Cesàro mean sequence $\{u_k, k \geq 1\}$ is $\mathcal{E}$-Cauchy and u_k converges boundedly to v m-a.s. Clearly, for each $k \geq 1$,

$$\mathcal{E}_\alpha(G_\alpha \phi, u_k) = \int_E \phi\, u_k\, dm. \tag{3.1}$$

Since $G_\alpha \phi \in L^1(E;m)$ for every $\alpha > 0$, letting $k \to \infty$ in (3.1), we have by Lebesgue dominated convergence theorem that $\mathcal{E}_\alpha(G_\alpha \phi, v) = \int_E \phi\, v\, dm$. □

Lemma 3.6 (Riesz decomposition) *Suppose that u is a non-negative $\mathcal{E}$-superharmonic function in $\mathcal{F}_e$. Then there exist an $\mathcal{E}$-harmonic function $h \in \mathcal{F}_e$ and a PCAF A so that $u(x) = \mathbf{E}_x[A_\zeta] + h(x)$ q.e. $x \in E$. Moreover, $t \mapsto u(X_t)$ is a uniformly integrable $\mathbf{P}_x$-supermartingale for q.e. $x \in E$.*

Proof. There is a bounded strictly positive $g \in L^1(E;m)$ such that $u \in L^1(E;gm) \cap L^2(E;gm)$. As in the proof of Lemma 3.2, let $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ be time-changed Dirichlet form of $(\mathcal{E}, \mathcal{F})$ by the inverse of PCAF $A_t := \int_0^t g(X_s)ds$. It is known (cf. [12]) that $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}_e) = (\mathcal{E}, \mathcal{F}_e)$ and so $u \in \widetilde{\mathcal{F}}_e \cap L^2(E;gm) = \widetilde{\mathcal{F}}$. Since

$$\mathcal{E}(u, \phi) + \int_E u(x)\phi(x)g(x)m(dx) \geq 0 \qquad \text{for every } \phi \in \widetilde{\mathcal{F}}^+ \cap C_c(E),$$

by Theorem 2.2.1 of [12], there is a Radon measure ν so that

$$\mathcal{E}(u, \phi) + \int_E u(x)\phi(x)g(x)m(dx) = \int_E \phi(x)\nu(dx) \qquad \text{for every } \phi \in \widetilde{\mathcal{F}} \cap C_c(E).$$

Define $\mu(dx) := \nu(dx) - u(x)g(x)m(dx)$. As $\mathcal{F} = \mathcal{F}_e \cap L^2(E;m) \subset \widetilde{\mathcal{F}}_e \cap L^2(E;gm) = \widetilde{\mathcal{F}}$, we have $\mathcal{E}(u, \phi) = \langle \mu, \phi \rangle$ for every $\phi \in \mathcal{F} \cap C_c(E)$ and hence for every $\phi \in \mathcal{F}_e$. Since $u \in \mathcal{F}_e$ is $\mathcal{E}$-superharmonic, the right hand side of the above display is non-negative. It follows that μ is a non-negative Radon measure and consequently it is of finite energy integral with respect to

$(\mathcal{E},\mathcal{F})$. Hence there exists a PCAF A corresponding to μ such that for each $\alpha>0$, u_α defined by $u_\alpha(x):=\mathbf{E}_x[\int_0^\infty e^{-\alpha t}dA_t]$ is an element of $\mathcal{F}$ and

$$\mathcal{E}_\alpha(u_\alpha,\phi)=\langle\mu,\phi\rangle \qquad \text{for every } \phi\in\mathcal{F}\cap C_c(E),$$

(see Theorem 2.2.1 and Lemma 5.1.3 of [12]). Note that for $\alpha>0$,

$$\mathcal{E}_\alpha(u_\alpha,u_\alpha)=\langle\mu,\,u_\alpha\rangle=\mathcal{E}(u,u_\alpha)\le\sqrt{\mathcal{E}(u,u)}\sqrt{\mathcal{E}_\alpha(u_\alpha,u_\alpha)}.$$

Hence we have

$$\mathcal{E}(u_\alpha,\,u_\alpha)\le\mathcal{E}_\alpha(u_\alpha,\,u_\alpha)\le\mathcal{E}(u,u) \qquad \text{for every } \alpha>0. \tag{3.2}$$

Let $\{\alpha_k,k\ge 1\}$ be a decreasing sequence of positive numbers that converges to 0. Since $u_{\alpha_k}(x)$ increases to $u_0(x):=\mathbf{E}_x[A_\zeta]$ for q.e. $x\in E$, we have $u_0\in\mathcal{F}_e$. Hence by [4, Theorem 6.7.4],

$$\mathcal{E}(u_0,\phi)=\langle\mu,\phi\rangle \qquad \text{for every } \phi\in\mathcal{F}_e.$$

Consequently, $h:=u-u_0\in\mathcal{F}_e$ and $\mathcal{E}(h,\phi)=0$ for every $\phi\in\mathcal{F}_e$. This in particular implies that h is $\mathcal{E}$-harmonic with $\mathcal{E}(h,h)=0$. By Lemma 2.2 of [3], $t\mapsto h(X_t)$ is a bounded $\mathbf{P}_x$-martingale for q.e. $x\in E$. Observe that

$$u_0(X_t)=\mathbf{E}_x[A_\zeta|\mathcal{F}_t]-A_t$$

is a uniformly integrable $\mathbf{P}_x$-supermartingale for those $x\in E$ such that $u_0(x)=\mathbf{E}_x[A_\zeta]<\infty$. It follows that $u(X_t)=u_0(X_t)+h(X_t)$ is a uniformly integrable $\mathbf{P}_x$-supermartingale for q.e. $x\in E$. □

Corollary 3.7 (Refined Riesz decomposition) *Assume that X satisfies the absolute continuity condition with respect to m. Suppose that u is a non-negative finely continuous (nearly) Borel $\mathcal{E}$-superharmonic function in $\mathcal{F}_e$. Then there exist a finely continuous (nearly) Borel $\mathcal{E}$-harmonic function $h\in\mathcal{F}_e$ and a PCAF A admitting no exceptional set so that $u(x)=\mathbf{E}_x[A_\zeta]+h(x)$ for all $x\in E$. Moreover, $t\mapsto u(X_t)$ is a uniformly integrable $\mathbf{P}_x$-supermartingale for all $x\in E$.*

Proof. Let μ be the non-negative Radon measure of finite energy integral appeared in the proof of Lemma 3.6. Let O be a finely open (nearly) Borel set such that $m(O)+\mu(O)<\infty$ and u is bounded on O. Set $u_\alpha^O(x):=\mathbf{E}_x[\int_0^\infty \mathbf{1}_O(X_t)dA_t]$, where A is the PCAF specified in the proof of Lemma 3.6. We prove that $u_\alpha^O\in L^\infty(E;m)$ under the boundedness of u on O. Since $u\in\mathcal{F}_e$ is non-negative $\mathcal{E}$-superharmonic, $u-H_{O^c}u\in(\mathcal{F}_e)_O=(\mathcal{F}_O)_e$ is $\mathcal{E}$-superharmonic in O by Theorem 4.6.5 in [12], where $H_{O^c}u(x):=\mathbf{E}_x[u(X_{\tau_O})]$. We may write $H_{O^c}u(x)=\mathbf{E}_x[u(X_{\sigma_{O^c}})]$. Here $\sigma_{O^c}:=\inf\{t>0\mid X_t\in E\setminus O\}$ is the first hitting time to $E\setminus O$. We now confirm that $u-H_{O^c}u$ is non-negative and bounded q.e. on O. Set $Y:=\{x\in E\mid \mathbf{P}_x(\sigma_{O^c}<\infty)>0\}$. Then $Y\setminus N$ and $Y^c\setminus N$ are X-invariant for an adequate properly exceptional set N by Lemma 4.6.4 in [12]. We see that $u-H_{O^c}u$ is non-negative and bounded on $O\cap(Y\setminus N)$ by use of Lemma 2.2 in

[3]. Lemma 2.1 in [18] shows that u is strictly $\mathcal{E}$-quasi-continuous with $u(\partial) = 0$. Hence $H_{O^c}u = 0$ on $O \cap (Y^c \setminus N)$, consequently $u - H_{O^c}u$ is non-negative and bounded on $O \setminus N$.

Applying Lemma 3.5 to X^O, we have

$$(u - H_{O^c}u - u_\alpha^O, \phi)_m = \mathcal{E}_\alpha(u - H_{O^c}u - u_\alpha^O, G_\alpha^O\phi) \quad \text{for } \phi \in \mathcal{F}_O \cap L^\infty(O; m).$$

Thus, for any $\phi \in \mathcal{F}_O^+ \cap L^\infty(O; m)$

$$\begin{aligned}
(u - H_{O^c}u - u_\alpha^O, \phi) &= \mathcal{E}_\alpha(u - H_{O^c}u - u_\alpha^O, G_\alpha^O\phi) \\
&= \mathcal{E}(u - u_\alpha^O, G_\alpha^O\phi) + \alpha(u - H_{O^c}u - u_\alpha^O, G_\alpha^O\phi) \\
&= \langle \mathbf{1}_O\mu, \widetilde{G_\alpha^O\phi}\rangle - \mathcal{E}(u_\alpha^O, G_\alpha^O\phi) + \alpha(u - H_{O^c}u - u_\alpha^O, G_\alpha^O\phi)_m \\
&= \alpha(u_\alpha^O, G_\alpha^O\phi)_m + \alpha(u - H_{O^c}u - u_\alpha^O, G_\alpha^O\phi)_m \\
&= \alpha(u - H_{O^c}u, G_\alpha^O\phi)_m \geq 0,
\end{aligned}$$

which implies $u_\alpha^O + H_{O^c}u \leq u$ q.e. on O. Hence in view of Lemma 2.2.4 in [12], $u_\alpha^O \in L^\infty(E; m)$. From this, we can conclude that the measure μ is a smooth measure in the strict sense. So we can take A as a PCAF admitting no exceptional set such that $u_\alpha(x) = \mathbf{E}_x[\int_0^\infty e^{-\alpha t} dA_t]$ and $u_0(x) := \mathbf{E}_x[A_\zeta]$ can be redefined for all $x \in E$. We can get a finer assertion than Lemma 2.2 of [3], that is, $t \mapsto h(X_t)$ is a bounded $\mathbf{P}_x$-martingale for all $x \in E$, because $h := u - u_0$ is a (nearly) Borel finely continuous function in the present setting. Therefore, we can obtain the assertion as seen in the proof of the previous lemma. □

Remark 3.8 The assertion of Lemma 3.6 also holds in the quasi-regular Dirichlet form setting. In this case, the definition of $\mathcal{E}$-superharmonicity of $u \in \mathcal{F}_e$ should be taken to be that $\mathcal{E}(u, \phi) \geq 0$ for any $\phi \in \mathcal{F}_e^+$. □

Lemma 3.9 *Let D be an open set and u a nearly Borel function on E.*

(i) *Assume that condition* (2.1) *holds. If u is a subharmonic function in D and is in $L^2_{\rm loc}(D; m)$ then u is subharmonic in D in the weak sense. Moreover, if X satisfies the absolute continuity condition with respect to m, then the same assertion holds with respect to the subharmonicity in D (in the weak sense) admitting no exceptional set.*

(ii) *If u is a nearly Borel q.e. finely continuous function on E such that u is subharmonic in D in the weak sense, then for each relatively compact open set U with $\overline{U} \subsetneq D$, $\{u(X_{t\wedge\tau_U}),\, t \geq 0\}$ is a (not necessarily uniformly integrable) $\mathbf{P}_x$-submartingale for q.e. $x \in E$. Moreover, if X satisfies the absolute continuity condition with respect to m and u is a finely continuous subharmonic function in D in the weak sense without exceptional set, then for U above, $\{u(X_{t\wedge\tau_U}),\, t \geq 0\}$ is a (not necessarily uniformly integrable) $\mathbf{P}_x$-submartingale for all $x \in U$.*

Proof. (i): Suppose that $u \in L^2_{\rm loc}(D;m)$ is subharmonic in D. For any relatively compact open set U with $\overline{U} \subsetneq D$, by assumption, $\{u(X_{t\wedge\tau_U}), t \geq 0\}$ is a uniformly integrable $\mathbf{P}_x$-submartingale for q.e. $x \in E$. Then as $t \to \infty$, $u(X_{t\wedge\tau_U})$ converges in $L^1(\mathbf{P}_x)$ as well as $\mathbf{P}_x$-a.s. to some random variable ξ for q.e. $x \in E$. Set $Y_t := u(X_{t\wedge\tau_U})$ for $t \in [0,\infty)$ and $Y_\infty := \xi$. Then $\{Y_t, t \in [0,\infty]\}$ is a right-closed $\mathbf{P}_x$-submartingale for q.e. $x \in E$. Applying the optional sampling theorem (see Theorem 2.59 in [13]) to $\{Y_t, t \in [0,\infty]\}$, we have $\mathbf{E}_x[|u|(X_{\tau_U})] < \infty$ and $u(x) \leq \mathbf{E}_x[Y_{\tau_U}]$ for q.e. $x \in E$. Note that $Y_{\tau_U}\mathbf{1}_{\{\tau_U<\infty\}} = u(X_{\tau_U})$ and $Y_{\tau_U} = u(X_{\tau_U}) + \xi\mathbf{1}_{\{\tau_U=\infty\}}$ $\mathbf{P}_x$-a.s. for q.e. $x \in E$. Set $u_2(x) := \mathbf{E}_x[\xi\mathbf{1}_{\{\tau_U=\infty\}}]$. We now show that $u_2 = 0$ q.e. on E if $\mathbf{P}_x(\tau_U < \infty) > 0$ for q.e. $x \in E$. It is easy to see that for each $t > 0$ $P_t^U u_2(x) = u_2(x)$ for q.e. $x \in U$. Note that

$$u_2(x) = \lim_{t\to\infty} \mathbf{E}_x[u(X_t)\mathbf{1}_U(X_t)\mathbf{1}_{\{\tau_U=\infty\}}]$$

for q.e. $x \in E$. It follows from Schwarz inequality that

$$\int_U u_2(x)^2 m(dx) \leq \varliminf_{n\to\infty} \int_U P_n(\mathbf{1}_U u^2)(x)m(dx) \leq \int_U u(x)^2 m(dx) < \infty.$$

Thus $u_2 \in \mathcal{F}_U$ and $\mathcal{E}(u_2,u_2) = 0$. Applying Lemma 2.2 in [3] to u_2, we have that $u_2 = 0$ q.e. on U if $\mathbf{P}_x(\tau_U < \infty) > 0$ for q.e. $x \in U$. Therefore we obtain that $u(x) \leq \mathbf{E}_x[u(X_{\tau_U})]$ for q.e. $x \in U$ if $\mathbf{P}_x(\tau_U < \infty) > 0$ for q.e. $x \in U$. That is, under condition (2.1), u is subharmonic in D in the weak sense. Next suppose that X satisfies the absolute continuity condition with respect to m. Noting that u_2 defined above is α-excessive with respect to X^U for any $\alpha > 0$, we see $u_2 = 0$ on U. The rest of the proof is quite similar with the argument above.

(ii): Suppose that a nearly Borel q.e. finely continuous u is subharmonic in D in the weak sense. Then for any relatively compact open set U with $\overline{U} \subsetneq D$, $|u(X_{\tau_U})|$ is $\mathbf{P}_x$-integrable for q.e. $x \in E$ and for each $t > 0$,

$$\mathbf{E}_x[u(X_{\tau_U})|\mathcal{F}_{t\wedge\tau_U}] \geq u(X_{t\wedge\tau_U}) \qquad \mathbf{P}_x\text{-a.s.} \tag{3.3}$$

for q.e. $x \in E$. Set $h_0(x) := \mathbf{E}_x[u(X_{\tau_U})]$. Then $u_0 := h_0 - u \geq 0$ q.e. on U, $u_0 = 0$ q.e. on U^c, and has the property that for any relatively compact open subset O with $\overline{O} \subset U$, $\mathbf{E}_x[u_0(X_{\tau_O})] \leq u_0(x)$ for q.e. on U. By taking a property exceptional set N of X and restricting the process X^U to $U \setminus N$ if necessary, we have from Theorem 12.4 in [11] that the function u_0 is excessive with respect to X^U. In particular, $t \mapsto u_0(X_t)\mathbf{1}_{\{t<\tau_U\}} = u_0(X_{t\wedge\tau_U})$ is a $\mathbf{P}_x$-supermartingale for q.e. $x \in U$. On the other hand, we see that $\{h_0(X_{t\wedge\tau_U}), t \geq 0\}$ is a uniformly integrable $\mathbf{P}_x$-martingale for q.e. $x \in U$. Therefore $\{u(X_{t\wedge\tau_U}),\, t \geq 0\}$ is a $\mathbf{P}_x$-submartingale for q.e. $x \in U$. The proof for the situation when X satisfies the absolute continuity condition with respect to m is also quite similar so it is omitted. □

The following theorem is an extension of Theorem 2.7 in [3] to subharmonic functions.

Theorem 3.10 *Let D be an open set of E. Suppose that $u \in \mathcal{F}_{D,\rm loc} \cap L^\infty_{\rm loc}(D;m)$ satisfying conditions* (2.2)-(2.3) *is $\mathcal{E}$-subharmonic in D. Then u is subharmonic in D. Moreover, if X satisfies the absolute continuity condition with respect to m and u is finely continuous and (nearly) Borel measurable, then u is subharmonic in D without exceptional set.*

Proof. Let U be a relatively compact open subset of D with $\overline{U} \subset D$. Take $\phi \in \mathcal{F} \cap C_c(D)$ such that $0 \le \phi \le 1$ and $\phi = 1$ on a relatively compact open neighborhood V of $\overline{U}$ with $\overline{V} \subset D$. Define $h_0(x) := \mathbf{E}_x[u(X_{\tau_U})]$. As we saw from the first part of the proof of Theorem 2.8, $u - h_0 \in (\mathcal{F}_U)_e$, $h_0 - (1-\phi)u \in \mathcal{F}_e$ and that (2.6) holds. Consequently,

$$\mathcal{E}(u - h_0, v) \le 0 \qquad \text{for every } v \in \mathcal{F}^+ \cap C_c(U).$$

This in particular implies that $u - h_0$ is $\mathcal{E}$-subharmonic in U. Note that $(u-h_0)^+ \in (\mathcal{F}_U)_e^+$ and, by Lemma 3.2, $(u-h_0)^+$ is $\mathcal{E}$-subharmonic in U; that is,

$$\mathcal{E}((u-h_0)^+,\, v) \le 0 \qquad \text{for every } v \in \mathcal{F}^+ \cap C_c(U). \tag{3.4}$$

Since $\mathcal{F} \cap C_c(U)$ is $\mathcal{E}$-dense in $(\mathcal{F}_U)_e$, the above display holds for every non-negative $v \in (\mathcal{F}_U)_e$. Indeed, since $(\mathcal{E}, \mathcal{F}_U)$ is a regular Dirichlet form on $L^2(U;m)$, for $v \in (\mathcal{F}_U^+)_e$, there is an $\mathcal{E}$-Cauchy sequence $\{v_n, n \ge 1\}$ in $\mathcal{F}_U \cap C_c(U)$ that converges to v m-a.e. on U. By the normal contraction property, $\{v_n^+, n \ge 1\} \subset \mathcal{F}^+ \cap C_c(U)$ is $\mathcal{E}$-bounded. Thus in view of the Banach-Saks theorem, there is a subsequence $\{v_{n_k}^+, n \ge 1\}$ whose Cesàro mean sequence is $\mathcal{E}$-Cauchy and converges to v m-a.e. on E. From it we deduce that (3.4) holds for every $v \in (\mathcal{F}_U)_e^+$. We have in particular

$$0 \le \mathcal{E}((u-h_0)^+, (u-h_0)^+) \le 0. \tag{3.5}$$

Thus by Lemma 2.2 in [3], we get $(u-h_0)^+(X_t) = (u-h_0)^+(x)$ for all $t \ge 0$ $\mathbf{P}_x$-a.s. for q.e. $x \in E$. Consequently, $(u-h_0)^+(X_t)$ is a bounded $\mathbf{P}_x$-martingale for q.e. $x \in E$. From this fact, the sets $A := \{u > h_0\}$ and $A^c = \{u \le h_0\}$ are X-invariant. So after taking out a proper exceptional set of X if needed, we may and do assume that h_0 is finely continuous and that either $A = E$ or $A^c = E$.

Suppose $A = E$ and take $x \in A$. Then $u(x) \ge h_0(x) + \varepsilon$ for some $\varepsilon > 0$. We fix such an $\varepsilon > 0$. We then have that $u(X_t) \ge h_0(X_t) + \varepsilon$ for all $t \ge 0$ $\mathbf{P}_x$-a.s. Consequently,

$$u(X_{t\wedge\tau_U}) \ge h_0(X_{t\wedge\tau_U}) + \varepsilon = \mathbf{E}_x[u(X_{\tau_U})|\mathcal{F}_{t\wedge\tau_U}] + \varepsilon \qquad \mathbf{P}_x\text{-a.s.}$$

Since $\bigvee_{t\ge0} \mathcal{F}_{t\wedge\tau_U} = \mathcal{F}_{\tau_U}$ (see (47.7) in [23]), we have $u(X_{\tau_U}) \ge u(X_{\tau_U}) + \varepsilon$ $\mathbf{P}_x$-a.s. on $\{\tau_U < \infty\}$ by letting $t \to \infty$. This implies that $\mathbf{P}_x(\tau_U < \infty) = 0$ for every $x \in A$. Consequently $h_0 = 0$ q.e. on E. As $u \ge h_0 \ge 0$ on $A = E$, we have from above that $u(X_t) = u(X_0)$ for all $t \ge 0$ $\mathbf{P}_x$-a.s. for q.e. $x \in E$. This in particular implies that $t \mapsto u(X_{t\wedge\tau_U})$ is a uniformly integrable $\mathbf{P}_x$-martingale for q.e. $x \in E$.

Next suppose $A^c = E$. Then $h_0 - u \in (\mathcal{F}_U)_e$ is a non-negative $\mathcal{E}$-superharmonic function in U. By Lemma 3.6 and Remark 3.8, $t \mapsto (u-h_0)(X_{t\wedge\tau_U})$ is a uniformly integrable $\mathbf{P}_x$-submartingale. By (2.3), $\mathbf{E}_x[|u(X_{\tau_U})|] < \infty$ for q.e. $x \in U$, and so $t \mapsto h_0(X_{t\wedge\tau_U})$ is also a uniformly integrable $\mathbf{P}_x$-martingale for q.e. $x \in E$. This proves that $t \mapsto u(X_{t\wedge\tau_U})$ is a uniformly integrable $\mathbf{P}_x$-martingale. The proof of the last statement is quite similar with the above argument by replacing the use of Lemma 3.6 with the use of Corollary 3.7. So it is omitted. □

The following two theorems are the subharmonic counterpart of Theorem 2.9 in [3].

Theorem 3.11 *Let D be an open subset of E and u a nearly Borel measurable function on E that is locally bounded in D. Suppose one of the following holds:*

(i) *u is subharmonic in D.*

(ii) *u is subharmonic in D in the weak sense and* (2.1) *holds.*

Then $u \in (\mathcal{F}_D)_{\rm loc}$.

Proof. Take a relatively compact open set U with $\overline{U} \subsetneq D$. Set $M := \|u\|_{L^\infty(U;m)}$. Then $0 \leq M - u \leq 2M$ q.e. on U. If (i) (resp. (ii)) holds, then $\{(M-u)(X_{t\wedge\tau_U}), t \geq 0\}$ is a uniformly integrable (resp., by Lemma 3.9(ii), a (not necessarily uniformly integrable)) $\mathbf{P}_x$-supermartingale for q.e. $x \in E$. Hence for each $t > 0$

$$P_t^U(M-u) \leq M - u \quad \text{q.e. on } U.$$

By the same argument as that after (2.17) in the proof of Theorem 2.9 in [3], we conclude that $M - u \in \mathcal{F}_{U,\rm loc}$ and so $u \in \mathcal{F}_{U,\rm loc}$. Since U is arbitrary, we obtain $u \in \mathcal{F}_{D,\rm loc}$. Since u is locally bounded on D, this implies that $u \in (\mathcal{F}_D)_{\rm loc}$. □

Theorem 3.12 *Let D be an open subset of E and u be a nearly Borel function on E that is in $\mathcal{F}_{D,\rm loc} \cap L^\infty_{\rm loc}(D;m)$ and satisfies conditions* (2.2) *and* (2.3)*. Suppose one of the following holds:*

(i) *u is subharmonic in D.*

(ii) *u is subharmonic in D in the weak sense and* (2.1) *holds.*

Then u is $\mathcal{E}$-subharmonic in D.

Proof. Note that u is automatically q.e. finely continuous in D. In either case, by the assumption and Lemma 3.9(ii), for any relatively compact open set U with $\overline{U} \subsetneq D$, we have $\mathbf{E}_x[|u(X_{\tau_U})|] < \infty$ for q.e. $x \in E$. Take $\phi \in \mathcal{F} \cap C_c(D)$ with $0 \leq \phi \leq 1$ and $\phi = 1$ on a relatively compact open set V with $\overline{U} \subset V \subset \overline{V} \subsetneq D$. Set $h_1(x) := \mathbf{E}_x[(\phi u)(X_{\tau_U})]$ and $h_2(x) := \mathbf{E}_x[((1-\phi)u)(X_{\tau_U})]$, which is q.e. well-defined as $\mathbf{E}_x[|u|(X_{\tau_U})] < \infty$ for q.e. $x \in E$. By the same argument as that for Theorems 2.9 and 2.7 in [3], we see that $\phi u \in \mathcal{F}_D$, $h_1 \in (\mathcal{F}_D)_e$, $\mathbf{1}_U h_2 \in (\mathcal{F}_U)_e$, $h_2 = \mathbf{1}_U h_2 + u - \phi u \in \mathcal{F}_{U,\rm loc}$ and $\mathcal{E}(h_1, v) = \mathcal{E}(h_2, v) = 0$ for any $v \in (\mathcal{F}_U)_e$. Therefore $h_0(x) := h_1(x) + h_2(x) = \mathbf{E}_x[u(X_{\tau_U})]$ satisfies $u_0 := h_0 - u = \mathbf{1}_U h_2 + h_1 - \phi u \in (\mathcal{F}_U)_e$. For the case (ii), as in the proof of Lemma 3.9 we see u_0 is excessive with respect to the subprocess X^U. For the case (i), we have the same conclusion easily. Then for each $n \in \mathbb{N}$, we have

$$P_t^U(u_0 \wedge n)(x) \leq (u_0 \wedge n)(x) \qquad \text{for q.e. } x \in U.$$

Since $u_0 \wedge n \in \mathcal{F}_U$ because $m(U) < \infty$, Lemma 3.1 leads us to

$$\mathcal{E}(u_0 \wedge n, \phi) \geq 0 \qquad \text{for every } \phi \in \mathcal{F}^+ \cap C_c(U).$$

On the other hand, $\{u_0 \wedge n\}$ is an $\mathcal{E}$-bounded sequence. There is a subsequence of $\{u_0 \wedge n\}$ whose Cesàro mean sequence is $\mathcal{E}$-Cauchy, and so is $\mathcal{E}$-convergent to u_0. We thus have $\mathcal{E}(u_0, \phi) \leq 0$ for every $\phi \in \mathcal{F}^+ \cap C_c(U)$, and so

$$\mathcal{E}(u, \phi) \leq \mathcal{E}(h_0, \phi) = \mathcal{E}(h_1 + h_2, \phi) = 0 \qquad \text{for every } \phi \in \mathcal{F}^+ \cap C_c(U).$$

Since U is arbitrary, we obtain the $\mathcal{E}$-subharmonicity of u in D. □

Proof of Theorem 2.9. Theorem 2.9 is an easy consequence of Lemma 3.9, Theorems 3.10, 3.11 and 3.12. □

Proof of Corollary 2.10. (i): By Theorem 3.10, for each relatively compact open set U with $\overline{U} \subsetneq D$, $\{u(X_{t\wedge\tau_U}), t \geq 0\}$ is a uniformly integrable $\mathbf{P}_x$-martingale for q.e. $x \in E$. First assume that η has bounded first derivative. Since $|\eta(t) - \eta(s)| \leq \sup_{\ell\in\mathbb{R}} |\eta'(\ell)| \cdot |t - s|$ for $t, s \in \mathbb{R}$, $\eta(u) \in \mathcal{F}_{D,\text{loc}}$. Meanwhile, $|\eta(u)| \leq \sup_{\ell\in\mathbb{R}} |\eta'(\ell)||u| + |\eta(0)|$ yields that $\{\eta(u)(X_{t\wedge\tau_U}), t \geq 0\}$ is uniformly integrable under $\mathbf{P}_x$ for q.e. $x \in U$ and $\eta(u)$ satisfies (2.2)–(2.3) by Lemma 3.4. (Recall that any bounded function satisfies (2.2)–(2.3).) By Jensen's inequality $\{\eta(u)(X_{t\wedge\tau_U}), t \geq 0\}$ is a $\mathbf{P}_x$-submartingale for q.e. $x \in U$. The $\mathcal{E}$-subharmonicity of $\eta(u)$ in D now follows from Theorem 3.12. Next we assume the boundedness of u on E. Then $\eta(u) \in \mathcal{F}_{D,\text{loc}}$ is bounded on E and it satisfies (2.2)–(2.3). The rest of the proof is similar as above.

(ii): The proof is the same as that for (i).

(iii): By Theorem 2.9, $\mathbf{E}_x[|u(X_{\tau_U})|] < \infty$ and $u(x) = \mathbf{E}_x[u(X_{\tau_U})]$ for q.e. $x \in E$, and consequently $u(X_{t\wedge\tau_U}) = \mathbf{E}_x[u(X_{\tau_U})|\mathcal{F}_{t\wedge\tau_U}]$ for q.e. $x \in E$. Since $u \in L^\infty_{\text{loc}}(D; m)$, $|u|^p \in \mathcal{F}_{D,\text{loc}} \cap L^\infty_{\text{loc}}(D; m)$. Therefore for every $\phi \in \mathcal{F} \cap C_c(D)$ with $0 \leq \phi \leq 1$ and $\phi = 1$ on an open neighborhood V of $\overline{U}$ with $\overline{V} \subsetneq D$, $\mathbf{E}_x[\phi|u|^p(X_{\tau_U})] < \infty$ for q.e. $x \in E$. By assumption, $\mathbf{E}_x[(1-\phi)|u|^p(X_{\tau_U})] < \infty$ for q.e. $x \in E$. Therefore $\mathbf{E}_x[|u|^p(X_{\tau_U})] < \infty$ for q.e. $x \in E$. By Jensen's inequality, $|u|^p$ is subharmonic in D in the weak sense. The $\mathcal{E}$-subharmonicity of $|u|^p$ in D now follows from Theorem 3.12.

(iv): Note that $|u_1 \vee u_2| \leq |u_1| + |u_2|$. So by Lemma 3.4, $u_1 \vee u_2$ satisfies conditions (2.2)–(2.3). The conclusion follows from Theorem 2.9. □

Proof of Theorem 2.11. Since $u^+(x_0) \geq 0$ and $\mathbf{1}_D \in \mathcal{F}_{D,\text{loc}}$ is $\mathcal{E}$-superharmonic in D, $u^+(x_0) - u \in \mathcal{F}_{D,\text{loc}}$ is a finely continuous Borel measurable non-negative $\mathcal{E}$-superharmonic function in D. Hence so is $v := u^+(x_0) - u^+ = (u^+(x_0) - u) \wedge u^+(x_0)$ by Corollary 2.10(iv). We set $Y := \{x \in D \mid v(x) > 0\}$. By Theorem 3.10, v is also excessive with respect to X^D, so is $\mathbf{1}_Y$ (cf. [15]). In particular, $\mathbf{1}_Y$ is finely continuous with respect to X^D. By Theorem 5.3 in [16], the irreducibility of $(\mathcal{E}^D, \mathcal{F}_D)$ implies the connectedness of the fine topology on D induced by the part process X^D. Thus either $Y = \emptyset$ or $D \setminus Y = \emptyset$. Since $x_0 \in D \setminus Y$, we have $Y = \emptyset$. So $u^+ \equiv u^+(x_0)$ on D. The proof for the case $\kappa(D) = 0$ is quite similar, so it is omitted. □

4 Examples

Example 4.1 (Stable-like process on $\mathbb{R}^d$) Consider the following Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathbb{R}^d)$, where

$$\begin{cases} \mathcal{F}=W^{\alpha/2,2}(\mathbb{R}^d)=\left\{u\in L^2(\mathbb{R}^d)\;\middle|\;\int_{\mathbb{R}^d\times\mathbb{R}^d}(u(x)-u(y))^2|x-y|^{d+\alpha}dxdy<\infty\right\}, \\ \mathcal{E}(u,v)=\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}(u(x)-u(y))(v(x)-v(y)|x-y|^{d+\alpha}c(x,y)dxdy \text{ for } u,v\in\mathcal{F}. \end{cases}$$

Here $d\geq 1$, $\alpha\in]0,2[$, and $c(x,y)$ is a symmetric function in (x,y) that is bounded between two positive constants. In literature, $W^{\alpha/2,2}(\mathbb{R}^d)$ is called the Sobolev space on $\mathbb{R}^d$ of fractional order $(\alpha/2,2)$. For an open set $D\subset\mathbb{R}^d$, $W^{\alpha/2,2}(D)$ is similarly defined as above but with D in place of $\mathbb{R}^d$. It is easy to check that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$ and its associated symmetric Hunt process X is called symmetric α-stable-like process on $\mathbb{R}^d$, which is studied in [6]. When $c(x,y)\equiv A(d,-\alpha):=\frac{\alpha 2^{d+\alpha}\Gamma(\frac{d+\alpha}{2})}{2^{d+1}\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}$, the process X is nothing but the rotationally symmetric α-stable process on $\mathbb{R}^d$. It is shown in [6] that the symmetric α-stable-like process X has strictly positive jointly continuous transition density function $p_t(x,y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$ and hence is irreducible. Moreover, there is constant $c>0$ such that

$$p_t(x,y)\leq ct^{-d/\alpha}\qquad\text{for } t>0 \text{ and } x,y\in\mathbb{R}^d. \tag{4.1}$$

Consequently, by [10, Theorem],

$$\sup_{x\in U}\mathbf{E}_x[\tau_U]<\infty. \tag{4.2}$$

for any open set U having finite Lebesgue measure. Note that in this example, the jumping measure

$$J(dxdy)=\frac{c(x,y)}{|x-y|^{d+\alpha}}dxdy$$

Hence for any non-empty open set $D\subset\mathbb{R}^d$, condition (2.2) is satisfied if and only if $(1\wedge|x|^{-d-\alpha})u(x)\in L^1(\mathbb{R}^d)$ (or equivalently, $u(x)/(1+|x|)^{d+\alpha}\in L^1(\mathbb{R}^d)$). As is shown in [3, Example 2.12], condition (2.3) is automatically satisfied for such u. When $\alpha\in]1,2[$, every (globally) Lipschitz function u on $\mathbb{R}^d$ satisfies the condition (2.2), that is, $(1\wedge|x|^{-d-\alpha})u(x)\in L^1(\mathbb{R}^d)$ holds. Consequently (2.3) holds for any Lipschitz function u provided $\alpha\in]1,2[$. Indeed, for any relatively compact open sets U, V with $\overline{U}\subset V\subset\overline{V}\subset D$,

$$\begin{aligned}\int_{U\times V^c}\frac{|u(y)-u(x)|}{|x-y|^{d+\alpha}}dxdy &\leq \|u\|_{\mathrm{Lip}}\int_{U\times V^c}\frac{|x-y|}{|x-y|^{d+\alpha}}dxdy \\ &\leq \|u\|_{\mathrm{Lip}}\sigma(\mathbb{S}^{d-1})\int_U\int_{d(x,V^c)}^{\infty}r^{-\alpha}drdx \\ &\leq \|u\|_{\mathrm{Lip}}|U|\sigma(\mathbb{S}^{d-1})\frac{d(U,V^c)^{1-\alpha}}{\alpha-1}<\infty,\end{aligned}$$

and so by Remark 2.3, (2.2) holds. Here $\|u\|_{\mathrm{Lip}}:=\sup_{x,y\in\mathbb{R}^d}\frac{|u(x)-u(y)|}{|x-y|}$, $|U|$ denotes the volume of U and $\sigma(\mathbb{S}^{d-1})$ is the $(d-1)$-dimensional volume of the unit sphere $\mathbb{S}^{d-1}$.

Observe that $C_c^\infty(\mathbb{R}^d)$ is a special standard core of $(\mathcal{E},\mathcal{F})$. By Theorems 2.8 and 2.9, for an open set D and a nearly Borel function u on $\mathbb{R}^d$ that is locally bounded on D with $(1\wedge|x|^{-d-\alpha})u(x)\in L^1(\mathbb{R}^d)$, the following are equivalent.

(i) u is subharmonic in D;

(ii) For every relatively compact open subset U of D, $u(X_{\tau_U})\in L^1(\mathbf{P}_x)$ and $u(x)\le \mathbf{E}_x[u(X_{\tau_U})]$ for q.e. $x\in U$;

(iii) $u\in\mathcal{F}_{D,\mathrm{loc}}=W^{\alpha/2,2}_{\mathrm{loc}}(D)$ and

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}(u(x)-u(y))(v(x)-v(y))\frac{c(x,y)}{|x-y|^{d+\alpha}}dxdy\le 0$$

for every non-negative $v\in C_c^\infty(D)$.

Moreover, if u is (finely) continuous, the above equivalence can be formulated without exceptional sets.

Example 4.2 (Symmetric Relativistic α-stable Process) Take $\alpha\in]0,2[$ and $m\ge 0$. Let $\mathrm{X}^{\mathrm{R},\alpha}=(\Omega,X_t,\mathbf{P}_x)_{x\in\mathbb{R}^d}$ be a Lévy process on $\mathbb{R}^d$ with

$$\mathbf{E}_0\big[e^{i\langle\xi,X_t\rangle}\big]=e^{-t((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m)}.$$

If $m>0$, it is called the *relativistic α-stable process with mass m* (see [22]). In particular, if $\alpha=1$ and $m>0$, it is called the *relativistic free Hamiltonian process* (see [14]). When $m=0$, $\mathrm{X}^{\mathrm{R},\alpha}$ is nothing but the usual *symmetric α-stable process*. Let $(\mathcal{E}^{\mathrm{R},\alpha},\mathcal{F}^{\mathrm{R},\alpha})$ be the Dirichlet form on $L^2(\mathbb{R}^d)$ associated with $\mathbf{X}^{\mathrm{R},\alpha}$. Using Fourier transform $\widehat{f}(x):=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}e^{i\langle x,y\rangle}f(y)dy$, it follows from Example 1.4.1 of [12] that

$$\begin{cases}\mathcal{F}^{\mathrm{R},\alpha}:=\left\{f\in L^2(\mathbb{R}^d)\;\Big|\;\int_{\mathbb{R}^d}|\widehat{f}(\xi)|^2\left((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m\right)d\xi<\infty\right\}=W^{\alpha/2,s}(\mathbb{R}^d),\\ \mathcal{E}^{\mathrm{R},\alpha}(f,g):=\int_{\mathbb{R}^d}\widehat{f}(\xi)\bar{\widehat{g}}(\xi)\left((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m\right)d\xi\quad\text{for }f,g\in\mathcal{F}^{\mathrm{R},\alpha}.\end{cases}$$

Global sharp two-sided estimates on the transition density function $p_t(x,y)$ of $\mathbf{X}^{\mathrm{R},\alpha}$ has recently been obtained in [5]. In particular, it implies (see also [22, Lemma 3]) that there exists $C(d,m)>0$ depending only on m and d so that

$$\sup_{x,y\in\mathbb{R}^d}p_t(x,y)\le C(d,m)\left(m^{d/\alpha-d/2}t^{-d/2}+t^{-d/\alpha}\right)\qquad\text{for any }t>0.$$

This yields by [10, Theorem 1] that (4.2) holds for any open set U having finite Lebesgue measure. It is shown in [9] that the corresponding jumping measure satisfies

$$J(dxdy)=\frac{c(x,y)}{|x-y|^{d+\alpha}}dxdy\quad\text{with}\quad c(x,y):=\frac{A(d,-\alpha)}{2}\Psi(m^{1/\alpha}|x-y|),$$

where $A(d,-\alpha)=\frac{\alpha 2^{d+\alpha}\Gamma(\frac{d+\alpha}{2})}{2^{d+1}\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}$, and the function Ψ on $[0,\infty[$ is given by $\Psi(r):=I(r)/I(0)$ with $I(r):=\int_0^\infty s^{\frac{d+\alpha}{2}-1}e^{-\frac{s}{4}-\frac{r^2}{s}}ds$. Note that Ψ is decreasing and satisfies $\Psi(r)\asymp e^{-r}(1+r^{(d+\alpha-1)/2})$ near $r=\infty$, and $\Psi(r)=1+\Psi''(0)r^2/2+o(r^4)$ near $r=0$. In particular, we have

$$\begin{cases} \mathcal{F}^{\mathrm{R},\alpha} &= \left\{f\in L^2(\mathbb{R}^d)\;\Big|\; \int_{\mathbb{R}^d\times\mathbb{R}^d}|f(x)-f(y)|^2\frac{c(x,y)}{|x-y|^{d+\alpha}}dxdy<\infty\right\},\\ \mathcal{E}^{\mathrm{R},\alpha}(f,g) &= \int_{\mathbb{R}^d\times\mathbb{R}^d}(f(x)-f(y))(g(x)-g(y))\frac{c(x,y)}{|x-y|^{d+\alpha}}dxdy \quad \text{for } f,g\in\mathcal{F}^{\mathrm{R},\alpha}. \end{cases}$$

It is easy to see that for any relatively compact open sets U,V and $\overline{U}\subset V\subset\overline{V}\subset D$, condition (2.2) is satisfied if and only if $\Psi(m^{1/\alpha}|x|)(1\wedge|x|^{-d-\alpha})u(x)\in L^1(\mathbb{R}^d)$ (equivalently $\Psi(m^{1/\alpha}|x|)u(x)/(1+|x|)^{d+\alpha}\in L^1(\mathbb{R}^d)$). Similarly, any function u with $\Psi(m^{1/\alpha}|x|)(1\wedge|x|^{-d-\alpha})u(x)\in L^1(\mathbb{R}^d)$ also satisfies the condition (2.3) in the same way as in Example 4.1. Moreover, any (globally) Lipschitz function u satisfies (2.2) and, consequently, satisfies condition (2.3). Indeed, for any relatively compact open sets U, V with $\overline{U}\subset V$,

$$\begin{aligned}\int_{U\times V^c}\frac{|u(y)-u(x)|}{|x-y|^{d+\alpha}}c(x,y)dxdy &\le \frac{A(d,-\alpha)}{2}\|u\|_{\mathrm{Lip}}\int_{U\times V^c}\frac{|x-y|\Psi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}dxdy\\ &\le \frac{A(d,-\alpha)}{2}\|u\|_{\mathrm{Lip}}\sigma(\mathbb{S}^{d-1})\int_U\int_{d(x,V^c)}^\infty \Psi(m^{1/\alpha}r)r^{-\alpha}drdx\\ &\le C\int_{d(U,V^c)}^\infty e^{-m^{1/\alpha}r}(1+m^{\frac{d+\alpha-1}{2\alpha}}r^{\frac{d+\alpha-1}{2}})r^{-\alpha}dr<\infty,\end{aligned}$$

and so (2.2) holds by Remark 2.3. Here C is a positive constant.

Observe that $C_c^\infty(\mathbb{R}^d)$ is a special standard core of $(\mathcal{E}^{\mathrm{R},\alpha},\mathcal{F}^{\mathrm{R},\alpha})$. By Theorems 2.8 and 2.9, for an open set D and a nearly Borel function u on $\mathbb{R}^d$ that is locally bounded on D with $\Psi(m^{1/\alpha}|x|)(1\wedge|x|^{-d-\alpha})u(x)\in L^1(\mathbb{R}^d)$, the following are equivalent.

(i) u is subharmonic in D;

(ii) For every relatively compact open subset U of D, $u(X_{\tau_U})\in L^1(\mathbf{P}_x)$ and $u(x)\le\mathbf{E}_x[u(X_{\tau_U})]$ for q.e. $x\in U$;

(iii) $u\in\mathcal{F}^{\mathrm{R},\alpha}_{D,\mathrm{loc}}$ and

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}(u(x)-u(y))(v(x)-v(y))\frac{\Psi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}dxdy\le 0$$

for every non-negative $v\in C_c^\infty(D)$.

Moreover, if u is (finely) continuous, the above equivalence can be formulated without exceptional sets.

One may ask concrete examples of $\mathcal{E}$-(sub/super)-harmonicity on D. To answer this question, in what follows, we assume $d>2$ ($d>\alpha$ if $m=0$). Applying Theorems 3.1 and 3.3 in [21] to

$\phi(\lambda) := (\lambda + m^{2/\alpha})^{\alpha/2} - m$, $\lambda > 0$, we can obtain that the Green kernel $r(x,y) := \int_0^\infty p_t(x,y)dt$ of X satisfies $r(x,y) \asymp (K_\alpha(x,y) + K_2(x,y))$, $x, y \in \mathbb{R}^d$, where $K_\beta(x,y) := A(d,\beta)/|x-y|^{d-\beta}$ for $\beta \in]0,2]$. In particular, X is transient and $r(x,x) = \infty$ for $x \in \mathbb{R}^d$. Note that $r(x,y) = K_\alpha(x,y)$ provided $m = 0$. Let u be a Borel function satisfying $u(x)\Psi(m^{1/\alpha}|x|)/(1+|x|)^{d+\alpha} \in L^1(\mathbb{R}^d)$. For $\varepsilon > 0$ and $x \in \mathbb{R}^d$, we define the approximate fractional Laplacian by

$$\Delta_\varepsilon^{\alpha/2,m} u(x) := A(d,-\alpha) \int_{|x-y|>\varepsilon} \frac{u(y)-u(x)}{|x-y|^{d+\alpha}} \Psi(m^{1/\alpha}|x-y|)dy,$$

and put $\Delta^{\alpha/2,m} u(x) := \lim_{\varepsilon \to 0} \Delta_\varepsilon^{\alpha/2,m} u(x)$ whenever the limit exists. It is essentially shown in Lemma 3.5 in [2] (resp. the remark after Definition 3.7 in [2]) that for any $u \in C_c^2(D)$ (resp. $u \in C^2(D)$ satisfying $u(x)\Psi(m^{1/\alpha}|x|)/(1+|x|)^{d+\alpha} \in L^1(\mathbb{R}^d)$), $\Delta^{\alpha/2,m} u$ always exists in $C(\mathbb{R}^d)$ (resp. in $C(D)$). Recall that for $u \in C^2(\mathbb{R}^d)$ with $u(x)\Psi(m^{1/\alpha}|x|)/(1+|x|)^{d+\alpha} \in L^1(\mathbb{R}^d)$, u satisfies (2.2) and (2.3). Hence, for such u and $\varphi \in C_c^2(D)$, $\mathcal{E}(u,\varphi)$ is well-defined and the proof of Lemma 2.6 in [3] shows

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |u(x)-u(y)||\varphi(x)-\varphi(y)| \frac{\Psi(m^{1/\alpha}|x-y|)dxdy}{|x-y|^{d+\alpha}} < \infty,$$

which implies $\mathcal{E}(u,\varphi) = (-\Delta^{\alpha/2,m} u, \varphi)$ and the $\mathcal{E}$-subharmonicity in D of u is equivalent to $\Delta^{\alpha/2,m} u \leq 0$ on D.

For $\varphi \in C_c(\mathbb{R}^d)$, we set

$$R^{(\alpha)}\varphi(x) := \int_{\mathbb{R}^d} r(x,y)\varphi(y)dy \quad x \in \mathbb{R}^d.$$

Then, we see $R^{(\alpha)}\varphi$ is locally bounded on $\mathbb{R}^d$ and $(R^{(\alpha)}\varphi)(x)\Psi(m^{1/\alpha}|x|)/(1+|x|)^{d+\alpha} \in L^1(\mathbb{R}^d)$ for such φ, because of $r(x,y) \asymp (K_\alpha(x,y) + K_2(x,y))$. Moreover, we see $R^{(\alpha)}\varphi \in \mathcal{F}_{\rm loc}$ for such φ. Indeed, for any relatively compact open set D with $\overline{D} \subset \mathbb{R}^d$, $R^{(\alpha)}\varphi$ is a difference of excessive functions with respect to X^D and bounded on D, so $R^{(\alpha)}\varphi \in \mathcal{F}_{D,\rm loc}$ by Theorem 3.11. Since D is arbitrary, $R^{(\alpha)}\varphi \in \mathcal{F}_{\rm loc}$. Thus $R^{(\alpha)}\varphi$ satisfies (2.2) and (2.3) for U, V with $\overline{U} \subset V \subset \overline{V} \subset \mathbb{R}^d$. Similarly, $r(a,\cdot) \in L^\infty_{\rm loc}(\mathbb{R}^d \setminus \{a\})$ satisfies $\int_{\mathbb{R}^d} \frac{r(a,x)\Psi(m^{1/\alpha}|x|)}{(1+|x|)^{d+\alpha}}dx < \infty$. We can obtain $r(a,\cdot) \in \mathcal{F}_{\mathbb{R}^d \setminus \{a\},\rm loc}$ in a similar way as above. Hence $r(a,\cdot)$ satisfies (2.2) and (2.3) for U, V with $\overline{U} \subset V \subset \overline{V} \subset \mathbb{R}^d \setminus \{a\}$. Note that for $\varphi \in C_c^\infty(D)$, $\Delta^{\alpha/2,m}\varphi = L^{\alpha,m}\varphi$ a.e. on $\mathbb{R}^d$ and $R^{(\alpha)}\Delta^{\alpha/2,m}\varphi = -\varphi$ on $\mathbb{R}^d$. Here $L^{\alpha,m}$ is the L^2-generator of $(\mathcal{E}^{\mathrm{R},\alpha}, \mathcal{F}^{\mathrm{R},\alpha})$.

For $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{a\})$, we then have

$$\begin{aligned}\mathcal{E}(r(a,\cdot),\varphi) &= -\int_{\mathbb{R}^d} r(a,x)\Delta^{\alpha/2,m}\varphi(x)dx \\ &= -(R^{(\alpha)}\Delta^{\alpha/2,m}\varphi)(a) = \varphi(a) = 0.\end{aligned}$$

This means the $\mathcal{E}$-harmonicity in $\mathbb{R}^d \setminus \{a\}$ of $r(a,\cdot)$. Similarly, for non-negative $\psi, \varphi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\mathcal{E}(R^{(\alpha)}\psi, \varphi) = (\psi, -R^{(\alpha)}\Delta^{\alpha/2,m}\varphi) = (\psi,\varphi) \geq 0,$$

which implies the $\mathcal{E}$-superharmonicity of $R^{(\alpha)}\psi$ for non-negative $\psi \in C_c^\infty(\mathbb{R}^d)$.

Example 4.3 (Diffusion process on a locally compact separable metric space) Let $(\mathcal{E},\mathcal{F})$ be a local regular Dirichlet form on $L^2(E;m)$, where E is a locally compact separable metric space, and X is its associated Hunt process. In this case, X has continuous sample paths and so the jumping measure J is null (cf. [12]). Hence conditions (2.2) and (2.3) are automatically satisfied. Let D be an open subset of E and u be a nearly Borel function on E that is locally bounded in D. Then by Theorem 2.9, u is subharmonic in D if and only if u is $\mathcal{E}$-subharmonic in D.

Now consider the following special case: $E=\mathbb{R}^d$ with $d\ge 1$, $m(dx)$ is the Lebesgue measure dx on $\mathbb{R}^d$, $\mathcal{F}=W^{1,2}(\mathbb{R}^d):=\{u\in L^2(\mathbb{R}^d)\mid \nabla u\in L^2(\mathbb{R}^d)\}$ and

$$\mathcal{E}(u,v)=\frac{1}{2}\sum_{i,j=1}^d\int_{\mathbb{R}^d}a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}dx\qquad\text{for } u,v\in W^{1,2}(\mathbb{R}^d),$$

where $(a_{i,j}(x))_{1\le i,j\le d}$ is a $d\times d$-matrix valued measurable function on $\mathbb{R}^d$ that is uniformly elliptic and bounded. In literature, $W^{1,2}(\mathbb{R}^d)$ is the Sobolev space on $\mathbb{R}^d$ of order (1.2). For an open set $D\subset\mathbb{R}^d$, $W^{1,2}(D)$ is similarly defined as above but with D in lace of $\mathbb{R}^d$. Then $(\mathcal{E},\mathcal{F})$ becomes a regular local Dirichlet form on $L^2(\mathbb{R}^d)$ and its associated Hunt process X is a conservative diffusion on $\mathbb{R}^d$ having jointly continuous transition density function. Let D be an open set in $\mathbb{R}^d$. Observe that $C_c^\infty(\mathbb{R}^d)$ is a special standard core of $(\mathcal{E},W^{1,2}(\mathbb{R}^d))$. By Theorem 2.8 and Theorem 2.9, the following are equivalent for a locally bounded nearly Borel measurable function u on D.

(i) u is subharmonic in D;

(ii) For every relatively compact open subset U of D, $u(X_{\tau_U})\in L^1(\mathbf{P}_x)$ and $u(x)\le\mathbf{E}_x[u(X_{\tau_U})]$ for q.e. $x\in U$; u is subharmonic in D in the weak sense;

(iii) $u\in\mathcal{F}_{D,\mathrm{loc}}=W^{1,2}_{\mathrm{loc}}(D)$ and

$$\sum_{i,j=1}^d\int_{\mathbb{R}^d}a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}dx\le 0\qquad\text{for every non-negative } v\in C_c^\infty(D).$$

Moreover, if u is (finely) continuous, the above equivalence can be formulated without exceptional sets.

Example 4.4 (Diffusions with jumps on $\mathbb{R}^d$) Consider the following Dirichlet form $(\mathcal{E},\mathcal{F})$, where $\mathcal{F}=W^{1,2}(\mathbb{R}^d)$ and for $u,v\in W^{1,2}(\mathbb{R}^d)$

$$\begin{aligned}\mathcal{E}(u,v):\;=\;&\frac{1}{2}\sum_{i,j=1}^d\int_{\mathbb{R}^d}a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}dx\\&+\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}(u(x)-u(y))(v(x)-v(y))\frac{c(x,y)}{|x-y|^{d+\alpha}}dxdy.\end{aligned}\tag{4.3}$$

Here $d\ge 1$ and $(a_{i,j}(x))_{1\le i,j\le d}$ is a $d\times d$-matrix valued measurable function on $\mathbb{R}^d$ that is uniformly elliptic and bounded, $\alpha\in]0,2[$ and $c(x,y)$ is a symmetric function in (x,y) that is bounded between two positive constants. It is easy to check that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$. Its associated symmetric Hunt process X has both the diffusion and jumping components. Such a process has recently been studied in [7]. Note that when $(a_{i,j}(x))_{1\le i,j\le d}$ is the identity matrix and $c(x,y)$ is constant, the process X is nothing but the symmetric Lévy process that is the independent sum of a Brownian motion and a rotationally symmetric α-stable process on $\mathbb{R}^d$. It is shown in [7] that the Hunt process X associated with the Dirichlet form $(\mathcal{E}, W^{1,2}(\mathbb{R}^d))$ given by (4.3) has strictly positive jointly continuous transition density function $p_t(x,y)$ and hence is irreducible. Moreover, a sharp two-sided estimate is obtained in [7] for $p_t(x,y)$. In particular, there is a constant $c>0$ such that

$$p_t(x,y)\le c\left(t^{-d/\alpha}\wedge t^{-d/2}\right)\qquad \text{for any } t>0 \text{ and } x,y\in\mathbb{R}^d.$$

In this example, the jumping measure

$$J(dxdy)=\frac{c(x,y)}{|x-y|^{d+\alpha}}dxdy.$$

Hence for any non-empty open set $D\subset\mathbb{R}^d$, condition (2.2) is satisfied if and only if $(1\wedge|x|^{-d-\alpha})u(x)\in L^1(\mathbb{R}^d)$. By [3, Example 2.14], for this example, condition (2.3) is implied by condition (2.2). Again note that $C_c^\infty(\mathbb{R}^d)$ is a special standard core of $(\mathcal{E}^{\mathrm{R},\alpha},\mathcal{F}^{\mathrm{R},\alpha})$. So Theorem 2.8 and Theorem 2.9 imply that for an open set D and a nearly Borel measurable function u on $\mathbb{R}^d$ that is locally bounded on D with $(1\wedge|x|^{-d-\alpha})u(x)\in L^1(\mathbb{R}^d)$, the following are equivalent

(i) u is subharmonic in D;

(ii) For every relatively compact open subset U of D, $u(X_{\tau_U})\in L^1(\mathbf{P}_x)$ and $u(x)\le\mathbf{E}_x[u(X_{\tau_U})]$ for q.e. $x\in U$; u is subharmonic in D in the weak sense;

(iii) $u\in\mathcal{F}_{D,\mathrm{loc}}=W^{1,2}_{\mathrm{loc}}(D)$ and

$$\sum_{i,j=1}^d\int_{\mathbb{R}^d}a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}dx+\int_{\mathbb{R}^d\times\mathbb{R}^d}(u(x)-u(y))(v(x)-v(y))\frac{c(x,y)}{|x-y|^{d+\alpha}}dxdy\le 0$$

for every non-negative $v\in C_c^\infty(D)$.

Moreover, if u is (finely) continuous, the above equivalence can be formulated without exceptional sets.

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