Seattle — 0 —

Some really old "birds"

(A) Herstein: R left Noeth, $\lambda \supset \mu$ left ideals. Suppose $x \in \lambda \Rightarrow x^{n(x)} \in \mu$. Does $\exists$ $N > 0$ such that $\lambda^N \subset \mu$. — Herstein, Stafford

Yes $\begin{cases} R \text{ PI} \\ n(x) \text{ bounded} & \text{Herstein} \\ |R| = 1 & \text{A. Covington} \\ R, \text{ simple}, & \text{C. Dean.} \end{cases}$

Lacks a "module" approach.

(B) Jacobson Problem
R r+l Noeth, is $\bigcap_{n=1}^{\infty} J^n(R) = (0)$

Yes, if R is PI. of course, no for R v.n.

(C) Some affine alg. questions.
$A = k\langle V\rangle$, $\dim V < \infty$, A Noeth.

(1) If k is uncountable, then $J(A)$ is nil $\Rightarrow$ nilp. for A Noeth.

(2) What about k countable? Is $J(A)$ nil? (of course, there are no countable fields)

If $\mathfrak{N}(A)$ is Noeth $\Rightarrow J(A)$ is nil.

Irving ~~result~~ example on 0-sets.

(3) Is A Jacobsen.

(4) Is $GK(A)$ integral?

Seattle 5/12 - Albatrosses

(0) "Rime of Ancient Mariner" Coleridge

(1) Lie-Kolchin over Div. Rings

@ Any elt $x \in R$ is unipotent if $x = 1+n$, n nilp

Th (L-K) Let G be a group of unipotent matrices in $M_n(F)$, F field $\Rightarrow$ G can be simultaneously Δ-ized (Thus, G is a nilpotent group)

Here's an easy proof (Zassenhaus?)

$g_1 = 1+n_1$, $g_2 = 1+n_2 \in G \Rightarrow 1 + \overbrace{g_1 + g_2 + g_1 g_2}^{g_1 \circ g_2} \in G$

$\Rightarrow g_1 \circ g_2$ is nilp. &, thus, $g_1 g_2$ is a sum of nilp.

Let A = subalgebra of $M_n(F)$ spanned by the nilp parts of the elts of G. A is fd/F & ~~is a sum~~ every a sum of nilp. So by Wedderburn, A is nilp $\Rightarrow$ A can be Δ-ized & so can G.

In ch 0 + trace arg Shows every elt is nilp

A trace argument does the job. (ch 0)

In ch 0, Mochizuki (1978) showed L-K holds for $M_n(D)$ using a result of Heinelsen & true for ch=2! (Derakhshan & Wagner, 2006)

Consequences for Burnside: skew linear, p groups in ch p are locally-finite

Other cases due to Lichtman, et al.

(2) Is the enveloping alg of the Witt alg. (centerless Virasoro) Noetherian?

W = Lie algebra with basis e_i, $i \in \mathbb{Z}$, over $\mathbb{C}$
$[e_i e_j] = (j-i)\, e_{i+j}$

(a) GK $W = 1 \Rightarrow U(W)$ has subexpon. growth.
$\Rightarrow Q(U_{\mathbb{C}}(W))$ exists.

(b) $U(W)$ is primitive (Wallach) $\Rightarrow$ a count. sep. of elts & center of $Q(W)$ is also $\mathbb{C}$.

(c) Let $W^+ = \{e_1, \ldots, e_n, \ldots\}$.
$U(W)$ noeth $\Rightarrow$ $U(W^+)$ noeth

(d) Let I_j = ideal generated by all e_k $k \leq j$ in $U(W^+)$. $U(W^+)/I_j \simeq U$(fin. dim nilp Lie alg) — a domain.
So if $U(W^+)$ noeth $\Rightarrow$ we have a descending chain of prime ideals that doesn't stop.

(e) Naturally, $U(W^+)$ is graded — So $U(W^+)$ noeth $\Rightarrow$ c-x ple to Stephenson–Zhang question:
Is a noeth graded ring of polynomial growth.

(f) Old Question: Can $U(L)$, L an inf. dim Lie alg, be Noeth?

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III Infinite Dimensional Div. Algebras and the ~~Koethe~~ Kurosh conjecture

(a) Hilbert, 1903

(i) Twisted Laurent Series order

$\mathrm{ch}\, k = 0$. $k(x) \xrightarrow{\sigma} k(x)$ $\sigma(x) = x+1$

Writing $k = k(x)$. $D = \{ \sum_{i \geq -n}^{\infty} a_i z^i \mid za = \sigma(a) z \}$

Inf. dim'l over center $= k$.

(b) $k = \mathbb{Q}$. $E_n = D_1 \otimes \cdots \otimes D_n$

f.d. / of coprime dim over center $\mathbb{Q}$

Koethe ~1931 E_n is a div. ring $(D_1, \ldots, D_n, \ldots)$

$D = \varinjlim E_i$

D is locally finite dim'l / $\mathbb{Q}$

(c) McConnell (1982). Constructed a division ring that is locally-PI of GK=1. Possible to construct such locally-PI of arbitrary integral GK. (Orig. example is a localization of a twisted poly. ring an aut. (fixed point) of an infinite alg. Galois ext)

(d) FOSS conjecture: If a div ring "can" contain a free subalg., it does

E.g., Makar-Limanov showed that $D_1 = Q(A_1)$ Weyl alg

Orig. on more variables contains a free

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Conj (Bell-Rogalski) If a div. alg contains a non-triv free $\Longleftrightarrow$ it's not locally-PI (algebraic).

Kurosh (a) Do there exist affine div. alg (Kurosh) that are not finite dim'l?

(b) Is an affine Div. Alg algebraic over its center?

Kurosh Conj, in gen'l asks: Are algebraic algebras locally-finite? Analog to Burnside for groups

Ans. No Golod-Shafarevich 1964.
~~Cou~~ C-xple to orig Burnside

G-S example is nil & has exponential growth.

Recently, examples of affine, ~~∞~~ dim nil alg. $Z \in K=3$ have been const. by B, Smolk, and Young. r

~~So~~ how to construct inf-dim'l (affine) div. alg.?

(1) Hom. image
(2) Localization
(3) Direct limit

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Typical Results

(A) A affine Noeth/k with $gr(A)$ also Noetherian. If A is algebraic over k
$\Rightarrow$ A is finite diml,
Proof $gr(A) = k + nil$ (alg). aug generated by
Aug.$(gr(A))$ is generated by a nil semigroup — nilpotent $\Rightarrow$ $gr(A)$ fin.

(B) Now assume base field is uncountable & D affine over its uncount. center, k. By Amitsur D is alg. $/k$. $D = k\langle V\rangle$

If $gr(A)$ noeth $\Rightarrow$ D is finite diml

(C) Suppose D affine$/Z$ = center & A affine over k (uncount). Can $Q(A) = D$? (with $gr(A)$ Noeth)
alg closed

No, if A is a hom. image$/k$ filtered Noeth. ring with assoc. graded Noeth.

like AZ — $Q(A)$ — affine$/Z$

$Z(Q(A))$ — alg

A

k

$AZ = Q(A)$ previous case