Invariant Theory of Artin-Schelter Regular Algebras: The Shephard-Todd-Chevalley Theorem

Ellen Kirkman

WAKE FOREST UNIVERSITY

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Collaborators

- Jacque Alev
- Kenneth Chan
- James Kuzmanovich
- Chelsea Walton
- James Zhang

Goal and Rationale:

Extend "Classical Invariant Theory" to an appropriate noncommutative context.

"Classical Invariant Theory": Group G acts on $k[x_1, \cdots, x_n]$.
f is invariant under G if $g \cdot f = f$ for all g in G.

Invariant theory important in the theory of commutative rings.

Productive context for using homological techniques.

Further the study of Artin-Schelter Regular Algebras A and other non-commutative algebras.

Extend from group G action to Hopf algebra H action.

Linear Group Actions on $k[x_1, \cdots, x_n]$

Let G be a finite group of $n \times n$ matrices acting on $k[x_1, \cdots, x_n]$

$$g = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$g \cdot x_j = \sum_{i=1}^{n} a_{ij} x_i$$

Extend to an automorphism of $k[x_1, \cdots, x_n]$.

Invariants under S_n
Permutations of $x_1, \cdots, x_n$.

(Painter: Christian Albrecht Jensen) (Wikepedia)

The subring of invariants under S_n is a polynomial ring

$$k[x_1, \cdots, x_n]^{S_n} = k[\sigma_1, \cdots, \sigma_n]$$

where σ_k are the n elementary symmetric functions for $k = 1, \ldots, n$:

$$\sigma_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k} = \mathcal{O}_{S_n}(x_1 x_2 \cdots x_k)$$

or the n power functions:

$$P_k = \sum x_i^k = \mathcal{O}_{S_n}(x_1^k).$$

Question: When is $k[x_1, \cdots, x_n]^G$ a polynomial ring?

Shephard-Todd-Chevalley Theorem

Let k be a field of characteristic zero.

Theorem (1954). The ring of invariants $k[x_1, \cdots, x_n]^G$ under a finite group G is a polynomial ring if and only if G is generated by reflections.

A linear map g on V is called a reflection of V if all but one of the eigenvalues of g are 1, i.e. $\dim V^g = \dim V - 1$.

Example: Transposition permutation matrices are reflections, and S_n is generated by reflections.

Noncommutative Generalizations?

Replace $k[x_1, \cdots, x_n]$ by a "polynomial-like" noncommutative algebra A.

Let A be Artin-Schelter regular algebra. A commutative Artin-Schelter regular ring is a commutative polynomial ring.

Consider groups G of graded automorphisms acting on A.
Note that not all linear maps act on A.

More generally, consider finite dimensional semi-simple Hopf algebras H acting on A.

Artin-Schelter Gorenstein/Regular

Noetherian connected graded algebra A is Artin-Schelter Gorenstein if:

- A has graded injective dimension $d < \infty$ on the left and on the right,
- $\text{Ext}^i_A(k, A) = \text{Ext}^i_{A^{op}}(k, A) = 0$ for all $i \neq d$, and
- $\text{Ext}^d_A(k, A) \cong \text{Ext}^d_{A^{op}}(k, A) \cong k(\ell)$ for some ℓ.

If in addition,

- A has finite (graded) global dimension, and
- A has finite Gelfand-Kirillov dimension,

then A is called Artin-Schelter regular of dimension d.

An Artin-Schelter regular graded domain A is called a quantum polynomial ring of dimension n if $H_A(t) = (1 - t)^{-n}$.

Linear automorphisms of $\mathbb{C}_q[x, y]$

If $q \neq \pm 1$ there are only diagonal automorphisms:

$$g = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

When $q = \pm 1$ there also are automorphisms of the form:

$$g = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}:$$

$$yx = qxy$$

$$g(yx) = g(qxy)$$

$$axby = qbyax$$

$$abxy = q^2 abxy$$

$$q^2 = 1.$$

Noncommutative Shephard-Todd-Chevalley Theorem

1. A^G is a polynomial ring $\rightsquigarrow$??? $A^G \cong A$??

Example (a): Let

$$g = \begin{pmatrix} \epsilon_n & 0 \\ 0 & 1 \end{pmatrix}$$

act on $A = \mathbb{C}_{-1}[x, y]$. Then $A^G = \mathbb{C}\langle x^n, y\rangle$.

When n odd, $A^G \cong A$. When n even $A^G \cong \mathbb{C}[x, y]$.

Replace "A^G is a polynomial ring" with "A^G is AS-regular".

When A commutative $A^G \cong A$ equivalent to A^G AS-regular.

Noncommutative Shephard-Todd-Chevalley Theorem

1. A^G is a polynomial ring $\rightsquigarrow$ A^G is AS-regular.

2. Definition of "reflection":

All but one eigenvalue of g is 1 $\rightsquigarrow$???

Examples $G = < g >$ on $A = \mathbb{C}_{-1}[x, y]$ $(yx = -xy)$:

Example (b): $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. A^{S_2} is generated by

$$P_1 = x + y \text{ and } P_2 = x^3 + y^3$$

$(x^2 + y^2 = (x + y)^2$ and $g \cdot xy = yx = -xy$ so no generators in degree 2); alternatively, generators are

$$\sigma_1 = x + y \text{ and } \sigma_2 = x^2 y + xy^2.$$

The generators are NOT algebraically independent. A^{S_2} is AS-regular (but it is a hyperplane in an AS-regular algebra). The transposition $(1, 2)$ is NOT a "reflection".

Examples $G =< g >$ on $A = \mathbb{C}_{-1}[x, y]$ $(yx = -xy)$:

Example (c): $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Now $\sigma_1 = x^2 + y^2$ and $\sigma_2 = xy$ are invariant and

$A^g \cong \mathbb{C}[\sigma_1, \sigma_2]$ is AS-regular.

g is a “mystic reflection”.

2. Definition of "reflection":

All but one eigenvalue of g is 1 $\rightsquigarrow$

The trace function of g acting on A of dimension n has a pole of order $n-1$ at $t=1$, where

$$Tr_A(g,t) = \sum_{k=0}^{\infty} trace(g|A_k)t^k = \frac{1}{(t-1)^{n-1}q(t)} \text{ for } q(1) \neq 0.$$

Examples $G =< g >$ on $A = \mathbb{C}_{-1}[x, y]$ $(yx = -xy)$:

(a) $g = \begin{bmatrix} \epsilon_n & 0 \\ 0 & 1 \end{bmatrix}$, $Tr(g, t) = \dfrac{1}{(1-t)(1-\epsilon_n t)}$, A^g AS-regular.

(b) $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Tr(g, t) = \dfrac{1}{1+t^2}$, A^g not AS-regular.

(c) $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $Tr(g, t) = \dfrac{1}{(1-t)(1+t)}$, A^g AS-regular.

For $A = \mathbb{C}_{q_{ij}}[x_1, \cdots, x_n]$ the groups generated by "reflections" are exactly the groups whose fixed rings are AS-regular rings.

Noncommutative Shephard-Todd-Chevalley Theorem

If G is a finite group of graded automorphisms of an AS-regular algebra A of dimension n then A^G is AS-regular if and only if G is generated by elements whose trace function

$$Tr_A(g, t) = \sum_{k=0}^{\infty} trace(g|A_k)t^k = \frac{1}{(t-1)^{n-1}q(t)},$$

i.e. has a pole of order $n-1$ at t=1.

Proven for cases:

1. G abelian and A a "quantum polynomial algebra".
2. $A = \mathbb{C}_{q_{ij}}[x_1 \cdots, x_n]$, skew polynomial ring.
3. A is an AS-regular graded Clifford algebra.

Molien's Theorem: Using trace functions

Jørgensen-Zhang: $H_{A^G}(t) = \frac{1}{|G|} \sum_{g \in G} Tr_A(g, t)$

.

Example (c) $A = \mathbb{C}_{-1}[x, y]$ and $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\sigma_1 = x^2 + y^2$, $\sigma_2 = xy$ and $A^g \cong \mathbb{C}[\sigma_1, \sigma_2]$.

$$H_{A^G}(t) = \frac{1}{4(1-t)^2} + \frac{2}{4(1-t^2)} + \frac{1}{4(1+t)^2} = \frac{1}{(1-t^2)^2}.$$

Bounds on Degrees of Generators: Commutative Polynomial Algebras

Noether's Bound (1916):
For k of characteristic zero, generators of $k[x_1, \cdots, x_n]^G$ can be chosen of degree $\leq |G|$.

Göbel's Bound (1995):
For subgroups G of permutations in S_n, generators of $k[x_1, \cdots, x_n]^G$ can be chosen of degree $\leq \max\{n, \binom{n}{2}\}$.

Invariants of $A = \mathbb{C}_{-1}[x_1, \dots, x_n]$ under the full Symmetric Group S_n

Example (b): $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ acts on A.

Both bounds fail for A^{S_2}, which required generators of degree $3 > |S_2| = 2 = \max\{2, \binom{2}{2}\}$: Generating sets

$$P_1 = x + y = \mathcal{O}_{S_2}(x) \text{ and } P_2 = x^3 + y^3 = \mathcal{O}_{S_2}(x^3)$$

or

$$\sigma_1 = x + y = \mathcal{O}_{S_2}(x) \text{ and } \sigma_2 = x^2y + xy^2 = \mathcal{O}_{S_2}(x^2y).$$

Invaríants of $A = \mathbb{C}_{-1}[x_1, \ldots, x_n]$ under the full Symmetric Group S_n

Invariants are generated by sums over S_n-orbits
$\mathcal{O}_{S_n}(X^I) =$ the sum of the S_n-orbit of a monomial X^I.
$\mathcal{O}_{S_n}(X^I)$ can be represented by X^I, where I is a partition:

$$X^{(i_1, \cdots, i_n)} \text{ where } i_1 \geq i_2 \geq \ldots \geq i_n$$

$\mathcal{O}_{S_n}(X^I) = 0$ if and only if I is a partition with repeated odd parts (e.g. $\mathcal{O}_{S_n}(x_1^5 x_2^3 x_3^3) = 0$ it corresponds to the partition $5 + 3 + 3$).

A^{S_n} is generated by the n odd power sums

$$P_k = \sum x_i^{2k-1}$$

or the n invariants

$$\sigma_k = \mathcal{O}_{S_n}(x_1^2 \ldots x_{k-1}^2 x_k)$$

for $k = 1, \ldots, n$.

Bound on degrees of generators of A^{S_n} is $2n-1$.

Invariants under the Alternating Group A_n: Commutative Case

$\mathbb{C}[x_1, \ldots, x_n]^{A_n}$ is generated by the symmetric polynomials (or power functions) and

$$D = \prod_{i<j} (x_i - x_j),$$

which has degree $\binom{n}{2}$. The Göbel bound is sharp.

Invariants of $A = \mathbb{C}_{-1}[x_1, \ldots, x_n]$ under the Alternating Group:

A^{A_n} is generated by $\mathcal{O}_{A_n}(x_1 x_2 \cdots x_{n-1})$,

and the n-1 polynomials $\sigma_1, \ldots, \sigma_{n-1}$

(or the power functions $P_1, \ldots, P_{n-1}$),

An upper bound on the degrees of generators of A^{A_n} is $2n-3$.

Questions

For A an Artin-Schelter regular algebra, find an upper bound on the degrees of generators of A^G.

Find an analogue of Göbel bound
(for $A = \mathbb{C}_{-1}[x_1, \cdots, x_n]$ we proved n^2, but probably not sharp).

Find an analogue of Noether bound
(consider cyclic groups?).

What are the "reflection groups"?

Shephard-Todd classified the reflection groups (finite groups G where $\mathbb{C}[x_1, \cdots, x_n]^G$ is a polynomial ring) – 3 infinite families and 34 exceptional groups.

If A is a quantum polynomial ring, a "reflection" of A must be a classical reflection, or a mystic reflection $\tau_{i,j,\lambda}$ where

$$\tau_{s,t,\lambda}(x_i) = \begin{cases} x_i & i \neq s, t \\ \lambda x_t & i = s \\ -\lambda^{-1} x_s & i = t. \end{cases}$$

Question: Do other AS-regular algebras have other kinds of "reflections"?

The Groups $M(n, \alpha, \beta)$

Let $A = \mathbb{C}_{-1}[x_1, \cdots, x_n]$, $\alpha, \beta \in \mathbb{N}$ with $\alpha | \beta$ and $2 | \beta$.
Let $\theta_{s,\lambda}$ be the classical reflection

$$\theta_{s,\lambda}(x_i) = \begin{cases} x_i & i \neq s \\ \lambda x_s & i = s. \end{cases}$$

$M(n, \alpha, \beta)$ is the subgroup of graded automorphisms of A generated by

$$\{\theta_{i,\lambda} | \lambda^\alpha = 1\} \cup \{\tau_{i,j,\lambda} | \lambda^\beta = 1\}.$$

Then $M(n, \alpha, \beta)$ is a "reflection group".

Rotation group of cube is generated by

$$g_1 := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } g_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

that act on $A = \mathbb{C}_{-1}[x, y, z]$ as the mystic reflections $g_1 = \tau_{1,2,1}$ and $g_2 = \tau_{2,3,1}$, respectively, and generate $G = M(3, 1, 2)$.

The mystic reflection groups $M(2, 1, 2\ell)$, for $\ell \gg 0$, are not isomorphic to classical reflection groups as abstract groups. They are the "dicyclic groups" of order 4ℓ generated by

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for λ a primitive 2ℓth root of unity.

Let $A = \mathbb{C}_{q_{ij}}[x_1, \cdots, x_n]$ and G be a finite subgroup of graded automorphisms of A.

If G is generated by "reflections" of A, then G as an abstract group is isomorphic to a direct product of classical reflection groups and groups of the form $M(n, \alpha, \beta)$.

Invariants under Hopf Algebra Actions

Let (H, Δ, ϵ, S) be a Hopf algebra and A be a Hopf-module algebra so

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b) \quad \text{and} \quad h \cdot 1_A = \epsilon(h) 1_A$$

for all $h \in H$, and all $a, b \in A$.
The invariants of H on A are

$$A^H := \{a \in A \mid h \cdot a = \epsilon(h) a \text{ for all } h \in H\}.$$

When $H = k[G]$ and $\Delta(g) = g \otimes g$ then $g \cdot (ab) = g(a) g(b)$.

Kac/Masuoka's 8-dimensional semisimple Hopf algebra

H_8 is generated by x, y, z with the following relations:

$$x^2 = y^2 = 1, \;\; xy = yx, \;\; zx = yz,$$

$$zy = xz, \;\; z^2 = \frac{1}{2}(1 + x + y - xy).$$

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y,$$

$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

$$\epsilon(x) = \epsilon(y) = \epsilon(z) = 1, \quad S(x) = x^{-1}, \; S(y) = y^{-1}, \; S(z) = z.$$

Hopf Action of H_8 on $A = \mathbb{C}_{-1}[u, v]$

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$A = \mathbb{C}_{-1}[u, v]$ is a left H_8-module algebra.

Let $a = u^3 v - uv^3$ and $b = u^2 + v^2$, then $A^{H_8} = \mathbb{C}[a, b]$, so H_8 is a "reflection quantum group".

Hopf Action of H_8 on $A = \mathbb{C}_i[u, v]$ $(vu = iuv)$

$$x \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$A = \mathbb{C}_i[u, v]$ is an H_8-module algebra

$$z \cdot (uv) = -vu, \quad z \cdot (vu) = uv,$$
$$z \cdot (u^2) = v^2, \quad z \cdot (v^2) = u^2.$$

$A^{H_8} = \mathbb{C}[u^2 v^2, u^2 + v^2]$, so H_8 is a "reflection quantum group".

Furthermore $A^{H_8} \neq A^G$ for any finite group G.

Molien's Theorem

When H is a finite dimensional semisimple Hopf algebra acting on A.

Then $H_{A^H}(t) = Tr(\int, t)$, where $\int$ has $\epsilon(\int) = 1$.

E.g. for H_8

$$\int = \frac{1 + x + y + xy + z + xz + yz + xyz}{8}.$$

Questions

When is $k[x_1, \cdots, x_n]^H$ a polynomial ring?
Must H be a group algebra or the dual of a group algebra?

If H is a semisimple Hopf algebra and $A = \mathbb{C}[u, v]$ then if A is an inner faithful H-module algebra then H is a group algebra (Chan-Walton-Zhang).

If A is Artin-Schelter regular, when is A^H regular?

What happpens when G (or H) is infinite?

What happens when H is not semisimple?

H not semisimple

Consider the Sweedler algebra $H(-1)$ generated by g and x

$$g^2 = 1, \;\; x^2 = 0, \;\; xg = -gx$$

$$\Delta(g) = g \otimes g \quad \Delta(x) = g \otimes x + x \otimes 1,$$

$$\epsilon(g) = 1, \epsilon(x) = 0 \quad S(g) = g, \; S(x) = -gx.$$

Then $H(-1)$ acts on $k[u, v]$ as

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$k[u, v]^{H(-1)} = k[u, v^2]$.