Invariant Theory of Artin-Schelter Regular Algebras: Gorenstein Invariant Subrings

Ellen Kirkman

WAKE FOREST UNIVERSITY

University of Washington: May 27, 2012

AS-regular algebra A with m generators.

G finite group of $m \times m$ matrices (or H a finite dimensional Hopf algebra) acting on it.

When is A^G (A^H) AS-regular?

Conjecture (proof in some cases): A^G AS-regular if and only if G is generated by elements with

$$Tr_A(g, t) = \frac{1}{(1-t)^{n-1} q(t)}$$

and n=GKdim(A).

A^H can be AS-regular.

A may not have any such group or Hopf actions (e.g the (non-PI) 3 dim Sklyanin algebra, Jordan plane and down-up algebras have no "reflections" so A^G is not regular, and hence $A^G \ncong A$.)

More is known about when A^G (A^H) is AS-Gorenstein.

Watanabe's Theorem

Watanabe's Theorem. (1974) If G is a finite subgroup of $\mathrm{SL}(V)$ acting naturally on the commutative polynomial algebra $k[V]$, then the fixed subring $k[V]^G$ is Gorenstein.

Example: Let $g = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ act on $k[x, y]$

$$k[x, y]^g = k\langle x^2, xy, y^2 \rangle \cong \frac{k[a, c][b]}{\langle b^2 - ac \rangle}$$

$$H_{A^g}(t) = \frac{1 + t^2}{(1 - t^2)^2}$$

The Homological Determinant

A AS-regular and G a finite group acting linearly on A.

If A is AS-regular of dimension n, then when the trace is written as a Laurent series in t^{-1}

$$Tr_A(g, t) = (-1)^n (\text{hdet } g)^{-1} t^{-\ell} + \text{higher terms}$$

hdet is a homomorphism from G into $\mathbb{C}$.

Noncommutative Watanabe's Theorem

Jørgensen-Zhang's Theorem (2000). If G is a finite subgroup of $\mathrm{GL}(V)$ acting linearly on an Artin-Schelter regular algebra A, if the homological determinant of each g in G is 1, then the fixed subring A^G is AS-Gorenstein.

Example. Let $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ act on $\mathbb{C}_{-1}[x, y]$.

Then

$$Tr_A(g, t) = \frac{1}{1 + t^2} = \frac{1}{t^2} + \text{ higher degree terms},$$

so $\mathrm{hdet}(g) = 1$. Then $\mathbb{C}_{-1}[x, y]^g$ is AS-Gorenstein, $\mathbb{C}_{-1}[x, y]^g$ is generated by $\sigma_1 = x + y$ and $\sigma_2 = x^3 + y^3$.

Noncommutative Stanley's Theorem

Jørgensen-Zhang's Theorem (2000). Let A be AS-regular, and G be a finite group of graded automorphisms of A. Then A^G is AS-Gorenstein if and only its Hilbert series satisfies the functional equation

$$H_{A^G}(t^{-1}) = \pm t^m H_{A^G}(t) \text{ for some integer } m.$$

Example. Let $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ act on $\mathbb{C}_{-1}[x, y]$. Then $H_{A^g}(t) = \dfrac{1 - t + t^2}{(1-t)^2(1+t^2)}$ and $H_{A^g}(t^{-1}) = t^2 H_{A^g}(t)$.

Invariants of $A = \mathbb{C}_{-1}[x_1 \ldots, x_n]$ under Permutations

If g is a 2-cycle and $A = \mathbb{C}_{-1}[x_1 \ldots, x_n]$ then

$$Tr_A(g) = \frac{1}{(1+t^2)(1-t)^{n-2}}$$

$$= (-1)^n \frac{1}{t^n} + \text{ higher terms}$$

so hdet $g = 1$, and for ALL groups G of $n \times n$ permutation matrices, A^G is AS-Gorenstein. Not true for commutative polynomial ring – e.g.

$$\mathbb{C}[x_1, x_2, x_3, x_4]^{\langle(1,2,3,4)\rangle}$$

is not Gorenstein, while

$$\mathbb{C}_{-1}[x_1, x_2, x_3, x_4]^{\langle(1,2,3,4)\rangle}$$

is AS-Gorenstein.

Example: Down-Up Algebra

Let A be AS-regular algebra generated by x, y with $x^2y = yx^2$ and $y^2x = xy^2$.

$\mathrm{GL}(2, \mathbb{C})$ acts on A.

Then A^G is Gorenstein if and only if all elements of G have determinant 1 or -1.

Example. Let $A = A(a, b, c)$ be an AS-regular 3-dim Sklyanin algebra, i.e. the algebra generated by x, y, z with relations

$$ax^2 + byz + czy = 0$$

$$ay^2 + bzx + cxz = 0$$

$$az^2 + bxy + cyx = 0.$$

Then the fixed ring under each of the following elements g of order 3 is AS-Gorenstein:

$$g = \begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{bmatrix}, \begin{bmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

where ω satisfies $\omega^3 = 1$.

$k[x_1, \cdots, x_n]^G$ a complete intersection:

Groups G with $k[x_1, \cdots, x_n]^G$ a complete intersection were classified by Nakajima (1984) and Gordeev (1986).

Theorem: (Kac and Watanabe – Gordeev) (1982). If $\mathbb{C}[x_1, \ldots, x_n]^G$ is a complete intersection then G is generated by bi-reflections (all but two eigenvalues are 1).

Example: Let $g = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ act on $k[x, y]$

$$k[x, y]^g = k\langle x^2, xy, y^2\rangle \cong \frac{k[a, b, c]}{\langle b^2 - ac\rangle}$$

Bi-reflection of A

For an AS-regular algebra A a graded automorphism g is a "bi-reflection" of A if

$$Tr_A(g, t) = \sum_{k=0}^{\infty} trace(g|A_k)t^k$$

$$= \frac{1}{(1-t)^{n-2}q(t)},$$

n $=$ GKdim A, and $q(1) \neq 0$.

Questions

Is there a version of Kac-Watanabe-Gordeev Theorem that is true in the noncommutative setting?

If A^G is a complete intersection must G be generated by bi-reflections?

What is a "noncommutative complete intersection"?

Example: A^G a complete intersection

$A = \mathbb{C}_{-1}[x, y, z]$ is AS-regular of dimension 3, and

$$g = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

acts on it. The eigenvalues of g are $-1, i, -i$
so g is not a bi-reflection of A_1.
However, $Tr_A(g, t) = 1/((1+t)^2(1-t)) = -1/t^3+$ higher degree terms and g is a "bi-reflection" with hdet $g = 1$.

$$A^g \cong \frac{k[X, Y, Z, W]}{\langle W^2 - (X^2 + 4Y^2)Z \rangle},$$

a commutative complete intersection.

Example (b): $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ acting on $\mathbb{C}_{-1}[x, y]$.

Then $\mathbb{C}_{-1}[x, y]^{S_2}$ is generated by

$$P_1 = x + y \text{ and } P_2 = x^3 + y^3$$

with

$$P_1 P_2^2 = P_2^2 P_1 \text{ and } P_1^2 P_2 = P_2 P_1^2,$$

$$\text{and } 2P_1^6 - 3P_1^3 P_2 - 3P_2 P_1^3 + 4P_2^2 = 0.$$

Let $A = \langle x, y \rangle$ with $xy^2 = y^2 x$ and $x^2 y = yx^2$, then A is AS-regular and

$$\mathbb{C}_{-1}[x, y]^{S_2} \cong \frac{A}{\langle 2x^6 - 3x^3 y - 3yx^3 + 4y^2 \rangle}.$$

Commutative Complete Intersections

Theorem (Y. Félix, S. Halperin and J.-C. Thomas)(1991):
Let A be a connected graded noetherian commutative algebra. Then the following are equivalent.

1. A is isomorphic to $k[x_1, x_2, \ldots, x_n]/(d_1, \ldots, d_m)$ for a homogeneous regular sequence.
2. The Ext-algebra $\mathrm{Ext}^*_A(k, k)$ is noetherian.
3. The Ext-algebra $\mathrm{Ext}^*_A(k, k)$ has finite GK-dimension.

Noncommutative Complete Intersections

Let A be a connected graded noetherian algebra.

1. We say A is a *classical complete intersection ring* if there is a connected graded noetherian AS regular algebra R and a sequence of normal regular homogeneous elements $\{d_1, \cdots, d_n\}$ of positive degree such that A is isomorphic to $R/(d_1, \cdots, d_n)$.
2. We say A is a *complete intersection ring of type NP* if the Ext-algebra $\mathrm{Ext}^*_A(k, k)$ is noetherian.
3. We say A is a *complete intersection ring of type GK* if the Ext-algebra $\mathrm{Ext}^*_A(k, k)$ has finite Gelfand-Kirillov dimension.
4. We say A is a *weak complete intersection ring* if the Ext-algebra $\mathrm{Ext}^*_A(k, k)$ has subexponential growth.

Noncommutative case:

classical complete intersection ring $\Rightarrow$
complete intersection ring of type GK

complete intersection ring of type NP (GK) $\Rightarrow$
weak complete intersection ring

complete intersection ring of type GK $\not\Rightarrow$
complete intersection ring of type NP

Example: $A = k\langle x, y\rangle/(x^2, xy, y^2)$ is a Koszul algebra with Ext-algebra $E := k\langle x, y\rangle/(yx)$; $\operatorname{GKdim} E = 2$ but E is not noetherian.

Examples of noncommutative complete intersections of type NP (GK) include noetherian Koszul algebras that have Ext-algebras that are Noetherian (finite GK) for example

$$A = \frac{\mathbb{C}_{-1}[x, y]}{\langle x^2 - y^2 \rangle} \text{ with } \mathrm{Ext}_A^*(k, k) = A^! = \frac{\mathbb{C}[x, y]}{\langle x^2 + y^2 \rangle}$$

or

$$A = \frac{\mathbb{C}\langle x, y \rangle}{\langle x^2, y^2 \rangle} \text{ with } \mathrm{Ext}_A^*(k, k) = A^! = \frac{\mathbb{C}[x, y]}{\langle xy \rangle};$$

in second case

$$A \cong \frac{B}{\langle x^2, y^2 \rangle}$$

where B is the AS-regular algebra generated by x, y with $yx^2 = x^2y$ and $y^2x = xy^2$.

Let A be a connected graded Noetherian ring. We say A is cyclotomic Gorenstein if the following conditions hold:

(i) A is AS-Gorenstein;

(ii) $H_A(t)$, the Hilbert series of A, is a rational function $p(t)/q(t)$ for some relatively prime polynomials $p(t), q(t) \in \mathbb{Z}[t]$ where all roots of $p(t)$ are roots of unity.

Suppose that A is isomorphic to R^G for some Auslander regular algebra R and a finite group $G \subseteq \mathrm{Aut}(R)$. If $\mathrm{Ext}_A^*(k,k)$ has subexponential growth, then A is cyclotomic Gorenstein.

Hence if A not cyclotomic Gorenstein, then A is not a complete intersection of any type.

Veronese Subrings

For a graded algebra A the rth Veronese $A^{\langle r \rangle}$ is the subring generated by all monomials of degree r.

If A is AS-Gorenstein of dimension d, then $A^{\langle r \rangle}$ is AS-Gorenstein if and only if r divides ℓ where $\mathrm{Ext}^d_A(k, A) = k(\ell)$ (Jørgensen-Zhang).

Let $g = \mathrm{diag}(\lambda, \cdots, \lambda)$ for λ a primitive rth root of unity; $G = (g)$ acts on A with $A^{\langle r \rangle} = A^G$.

If the Hilbert series of A is $(1-t)^{-d}$ then

$$Tr_A(g^i, t) = \frac{1}{(1 - \lambda^i t)^d}.$$

For $d \geq 3$ the group $G = (g)$ contains no "bi-reflections", so $A^G = A^{\langle r \rangle}$ should not be a complete intersection.

Theorem:

Let A be noetherian connected graded algebra.

1. Suppose the Hilbert series of A is $(1-t)^{-d}$. If $r \geq 3$ or $d \geq 3$, then $H_{A^{\langle r\rangle}}(t)$ is not cyclotomic. Consequently, $A^{\langle r\rangle}$ is not a complete intersection of any type.

2. Suppose A is a quantum polynomial ring of dimension 2 (and $H_A(t) = (1-t)^{-2}$). If $r = 2$, then $H_{A^{\langle r\rangle}}(t)$ is cyclotomic and $A^{\langle r\rangle}$ is a classical complete intersection.

Permutation Actions on $A = \mathbb{C}_{-1}[x_1, \cdots, x_n]$

If g is a 2-cycle then hdet $g = 1$, and all A^G are AS-Gorenstein.

A permutation matrix g is a "bi-reflection" of A if and only if it is a 2-cycle or a 3-cycle.

Both A^{S_n} and A^{A_n} are classical complete intersections.

$$A = \mathbb{C}_{-1}[x_1, \cdots, x_n]$$

Example:

$$g = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Then $A^{(g)}$ has Hilbert series

$$\frac{1 - 2t + 4t^2 - 2t^3 + t^4}{(1 + t^2)^2(1 - t)^4}$$

whose numerator is not a product of cyclotomic polynomials, so $A^{(g)}$ is not any of our types of complete intersection.

Toward a Kac-Watanabe-Gordeev Theorem
Examples in Dimension 3:

Consider AS-Gorenstein fixed rings of AS-regular algebras of dimension 3 (e.g. 3-dimensional Sklyanin, down-up algebras, $\mathbb{C}_{-1}[x, y, z]$).

Thus far all our examples are either classical complete intersections or not cyclotomic (so none of our types of complete intersection).

In all the cases where A^G is a complete intersection, G is generated by "bi-reflections" of A.

Down-up algebra examples

Let A be generated by x, y with relations

$$y^2x = xy^2 \text{ and } yx^2 = x^2y.$$

Represent the automorphism $g(x) = ax + cy$ and $g(y) = bx + dy$ by the 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Any invertible matrix induces a graded automorphism of A.
The homological determinant of a graded automorphism g with eigenvalues λ_1 and λ_2 is $(\lambda_1\lambda_2)^2$.
A^G is AS-Gorenstein if and only if the $hdet(g) = (\lambda_1\lambda_2)^2 = 1$ for all $g \in G$.

“bi-reflections”

The trace of a graded automorphism g of A with eigenvalues λ_1 and λ_2 is

$$Tr_A(g,t) = \frac{1}{(1-\lambda_1 t)(1-\lambda_2 t)(1-\lambda_1\lambda_2 t^2)}.$$

Assuming $(\lambda_1\lambda_2)^2 = 1$ for all $g \in G$, “bi-reflections” are:

Classical Reflections: One eigenvalue of g is 1 and the other eigenvalue is a root of unity; since $(\lambda_1\lambda_2)^2 = 1$ the other eigenvalue must be -1.

In $SL_2(\mathbb{C})$: The eigenvalues of g are λ and λ^{-1} for $\lambda \neq 1$ (which forces the (homological) determinant to be 1).

Abelian Groups of Graded Automorphisms of A

Example:

$G = \langle g_1, g_2 \rangle$ for $g_1 =$ diag $[\epsilon_n, \epsilon_n^{-1}]$ and $g_2 =$ diag $[1, -1]$.

The group $G = \langle g_1, g_2 \rangle$ is a "bi-reflection" group of order 2n and $G \cong \mathbb{Z}_n \times \mathbb{Z}_2$.

When n is even, A^G is a classical complete intersection, and when n is odd A^G is not cyclotomic Gorenstein (so no kind of complete intersection).

n even

For n=2 G is a classical reflection group – the Klein-4 group.
$A^G = k\langle x^2, y^2, (yx)^2, (xy)^2\rangle$, the commutative hypersurface:

$$\frac{k[X, Y, Z, W]}{\langle ZW - X^2Y^2\rangle}.$$

For $n \geq 4$ G is a "bi-reflection" group.
$A^G = k\langle x^n, y^n, (xy)^2, (yx)^2, x^2y^2\rangle$, the commutative complete intersection:

$$A^G \cong \frac{k[X, Y, Z, W, V]}{(XY - V^{n/2},\ \ ZW - V^2)}.$$

n odd

$G = \langle g \rangle$ is generated by $g =$ diag $[\epsilon_n, -\epsilon_n^{-1}]$.

The numerator of the Hilbert series for A^G is

$$= 1 + t^4 + 2t^{n+2} - 2t^{2n+2} - t^{3n} - t^{3n+4}$$

$$= (1 - t^n)(1 + t^4 + t^n + 2t^{n+2} + t^{n+4} + t^{2n} + t^{2n+4}),$$

which we showed is NOT a product of cyclotomic polynomials for $n > 1$.

Dihedral Groups $G = \langle g_1, g_2 \rangle$

$$g_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } g_2 = \begin{bmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{bmatrix}$$

n even:

$$A^G = \frac{k[X, Y, Z, W]}{\langle W^2 - XYW - 4Z^{\frac{n+2}{2}} + Y^2 Z + X^2 Z^{\frac{n}{2}} \rangle}$$

n odd:

$$A^G = \frac{k[X, Y, Z][W; \sigma, \delta]}{\langle W^2 - Y^2 Z \rangle}.$$

Sklyanin Example

$$ax^2 + yz + zy = 0$$

$$ay^2 + zx + xz = 0$$

$$az^2 + xy + yx = 0$$

with $a^3 \neq 1$ and

$$g = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for ω a primitive cubed root of unity. hdet $g = 1$ and $Tr_A(g, t) = 1/(1 - t^3)$, so g is a "bi-reflection".

$$A^g \cong \frac{\mathbb{C}_{-1}[x, x^3 - y^3][xy; \sigma, \delta][x^3; \sigma', \delta']}{\langle f \rangle}.$$

Let

$$h = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

then g, h generate a group G of order 27 with

$$H_{A^G}(t) = \frac{1 - t^{18}}{(1-t^3)^2(1-t^6)(1-t^9)},$$

so possibly a complete intersection!

Questions:

If A is a classical complete intersection, is $\mathrm{Ext}_A^*(k,k)$ Noetherian?

Are there algebras A with $\mathrm{Ext}_A^*(k,k)$ Noetherian and finite GKdim that are not classical complete intersections?

What does $\mathrm{Ext}_A^*(k,k)$ finite GKdim say about A?

Do "complete intersections" have small numbers of generators? (2n-1 in commutative case).

Is there a version of the Kac-Watanabe-Gordeev Theorem in our context?

Classify the groups that give "complete intersections".

Hopf Actions on A

Hypotheses:

1. *H is a finite dimensional semisimple Hopf algebra,*
2. *A is a connected graded noetherian Artin-Schelter Gorenstein algebra of dimension d, and*
3. *A is a left H-module algebra and each A_i is a left H-module for each i.*

Since $\mathrm{Ext}_A^d(k, A)$ is 1-dimensional, the left H-action on $\mathrm{Ext}_A^d(k, A)$ defines an algebra map $\eta' : H \to k$ such that $h \cdot \mathbf{e} = \eta'(h)\mathbf{e}$ for all $h \in H$.

The homological determinant hdet is equal to $\eta' \circ S$, where S is the antipode of H.

The homological determinant is trivial if $\mathrm{hdet} = \epsilon$.

Theorem. If the homological determinant hdet of the H-action on A is trivial, then the invariant subring A^H is Artin-Schelter Gorenstein.

Theorem. Assume in addition that A is PI. Then A^H is Artin-Schelter Gorenstein if and only if its Hilbert series satisfies the functional equation

$$H_{A^H}(t^{-1}) = \pm t^{-m} H_{A^H}(t)$$

for some integer m.

Example:

Let $A = \langle x, y \rangle$ be the AS-regular algebra with relations:

$$xy^2 - y^2x = 0, \quad \text{and} \quad x^2y - yx^2 = 0.$$

$A = kS$ where S is semigroup generated by a, b subject to the relations

$$a^2b = ba^2, \quad ab^2 = b^2a.$$

Let $G = S/\langle a^2 = 1,\ b^2 = 1,\ (ab)^4 = 1 \rangle$ is the dihedral group $G = \{\mathrm{u} = 1, a, b, ab, ba, aba, bab, abab\}$.
Since G is a quotient group of S, A is a $\mathbb{Z} \times G$-graded algebra, and hence A is a $K = kG$-comodule algebra. Let $H = (kG)^{\circ}$.
Then A is a left H-module algebra. H and K are semisimple Hopf algebras.

A is $\mathbb{Z} \times G$-graded with $\deg x = (1, a)$ and $\deg y = (1, b)$, we have a $\mathbb{Z} \times G$-graded resolution of the trivial A-module k:

$$0 \to A(-4, \mathrm{u}) \to A(-3, a) \oplus A(-3, b)$$

$$\to A(-1, a) \oplus A(-1, b) \to A \to k \to 0.$$

Can show that $\mathrm{Ext}^3_A(k, A) \cong k(4, \mathrm{u})$ as a $\mathbb{Z} \times G$-graded vector space and hence the K-comodule action maps a basis element $\mathfrak{e} \in \mathrm{Ext}^3_A(k, A)$ to $\mathfrak{e} \otimes \mathrm{u} = \mathfrak{e} \otimes 1_K$. It can be shown that hdet is trivial and hence A^H is Artin-Schelter Gorenstein. Also $A^H \neq A^G$ for all groups G.