

MODULAR REPRESENTATIONS AND INDICATORS FOR BISMASH PRODUCTS

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ABSTRACT. We introduce Brauer characters for representations of the bismash products of groups in characteristic $p > 0$, $p \neq 2$ and study their properties analogous to the classical case of finite groups. We then use our results to extend to bismash products a theorem of Thompson on lifting Frobenius-Schur indicators from characteristic p to characteristic 0.

1. INTRODUCTION

In this paper we study the representations of bismash products $H_{\mathbb{k}} = \mathbb{k}^G \# \mathbb{k}F$, coming from a factorizable group of the form $Q = FG$ over an algebraically closed field \mathbb{k} of characteristic $p > 0$, $p \neq 2$. Our general approach is to reduce the problem to a corresponding Hopf algebra in characteristic 0.

In the first part of the paper, we extend many of the classical facts about Brauer characters of groups in char $p > 0$ to the case of our bismash products; our Brauer characters are defined on a special subset of H of non-nilpotent elements, using the classical Brauer characters of certain stabilizer subgroups F_x of the group F . In particular we relate the decomposition matrix of a character for the bismash product in char 0 with respect to our new Brauer characters, to the ordinary decomposition matrices for the group algebras of the F_x with respect to their Brauer characters. As a consequence we are able to extend a theorem of Brauer saying that the determinant of the Cartan matrix for the above decomposition is a power of p (Theorem 4.14).

These results about Brauer characters may be useful for other work on modular representations. We remark that the only other work on lifting from characteristic p to characteristic 0 of which we are aware is that of [EG], and they work only in the semisimple case.

In the second part, we first extend known facts on Witt kernels for G -invariant forms to the case of a Hopf algebra H , as well as some facts about G -lattices. We then use these results and Brauer characters to extend a theorem of J. Thompson [Th] on Frobenius-Schur indicators for representations of finite groups to the case of bismash product Hopf algebras. In particular we show that if $H_{\mathbb{C}} = \mathbb{C}^G \# \mathbb{C}F$ is a bismash product over \mathbb{C} and $H_{\mathbb{k}} = \mathbb{k}^G \# \mathbb{k}F$ is the corresponding bismash product over an algebraically closed field \mathbb{k} of characteristic $p > 0$, and if $H_{\mathbb{C}}$ is totally orthogonal (that is, all Frobenius-Schur indicators are +1), then the same is true for $H_{\mathbb{k}}$ (Corollary 6.6).

The first author was supported by NSF grant DMS 0701291 and the second author by DMS 1001547.

This paper is organized as follows. Section 2 reviews known facts about bismash products and their representations, and Section 3 summarizes some basic facts about Brauer characters for representations of finite groups. In Section 4 we prove our main results about Brauer characters for the case of bismash products.

In Section 5 we extend the facts we will need on Witt kernels and lattices, and in Section 6 we combine all these results to prove our extension of Thompson's theorem. Finally in Section 7 we give some applications and raise some questions.

Throughout \mathbb{E} will be an arbitrary field and H will be a finite dimensional Hopf algebra over \mathbb{E} , with comultiplication $\Delta : H \rightarrow H \otimes H$ given by $\Delta(h) = \sum h_1 \otimes h_2$, counit $\epsilon : H \rightarrow \mathbb{E}$ and antipode $S : H \rightarrow H$.

2. EXTENSIONS ARISING FROM FACTORIZABLE GROUPS AND THEIR REPRESENTATIONS

The Hopf algebras we consider here were first described by G. Kac [Ka] in the setting of C^* -algebras, in which case $\mathbb{E} = \mathbb{C}$, and in general by Takeuchi [Ta], constructed from what he called a matched pair of groups. These Hopf algebras can also be constructed from a factorizable group, and that is the approach we use here. Throughout, we assume that F and G are finite groups.

Definition 2.1. A group Q is called *factorizable* into subgroups $F, G \subset Q$ if $FG = Q$ and $F \cap G = 1$; equivalently, every element $q \in Q$ may be written uniquely as a product $q = ax$ with $a \in F$ and $x \in G$.

A factorizable group gives rise to actions of each subgroup on the other. That is, we have

$$\triangleright : G \times F \rightarrow F \text{ and } \triangleleft : G \times F \rightarrow G,$$

where for all $x \in G$, $a \in F$, the images $x \triangleright a \in F$ and $x \triangleleft a \in G$ are the (necessarily unique) elements of F and G such that $xa = (x \triangleright a)(x \triangleleft a)$.

Although these actions \triangleright and \triangleleft of F and G on each other are not group automorphisms, they induce actions of F and G as automorphisms of the dual algebras \mathbb{E}^G and \mathbb{E}^F . Let $\{p_x \mid x \in G\}$ be the basis of \mathbb{E}^G dual to the basis G of $\mathbb{E}G$ and let $\{p_a \mid a \in F\}$ be the basis of \mathbb{E}^F dual to the basis F of $\mathbb{E}F$. Then the induced actions are given by

$$(2.2) \quad a \cdot p_x := p_{x \triangleleft a^{-1}} \text{ and } x \cdot p_a := p_{x \triangleright a},$$

for all $a \in F$, $x \in G$. We let F_x denote the stabilizer in F of x under the action \triangleleft .

The *bismash product* Hopf algebra $H_{\mathbb{E}} := \mathbb{E}^G \# \mathbb{E}^F$ associated to $Q = FG$ uses the actions above. As a vector space, $H_{\mathbb{E}} = \mathbb{E}^G \otimes \mathbb{E}^F$, with \mathbb{E} -basis $\{p_x \# a \mid x \in G, a \in F\}$. The algebra structure is the usual smash product, given by

$$(2.3) \quad (p_x \# a)(p_y \# b) = p_x(a \cdot p_y) \# ab = p_x p_{y \triangleleft a^{-1}} \# ab = \delta_{y, x \triangleleft a} p_x \# ab.$$

The coalgebra structure may be obtained by dualizing the algebra structure of $H_{\mathbb{E}}^*$, although we will only need here that $H_{\mathbb{E}}$ has counit $\epsilon(p_x \# a) = \delta_{x,1}$. Finally the antipode of H is given by $S(p_x \# a) = p_{(x \triangleleft a)^{-1}} \# (x \triangleright a)^{-1}$. One may check that $S^2 = id$.

For other facts about bismash products, including the alternate approach of matched pairs of groups, see [Ma2], [Ma3]. We will consider the explicit example of a factorization of the symmetric group in Section 7.

Observe that for any field \mathbb{E} , a *distinguished basis* of $H_{\mathbb{E}}$ over \mathbb{E} is the set

$$(2.4) \quad \mathcal{B} := \{p_y \# a \mid y \in G, a \in F\},$$

and that \mathcal{B} has the property that if $b, b' \in \mathcal{B}$, then $bb' \in \mathcal{B} \cup \{0\}$. In particular, if $w = p_y \# a$, then (2.3) implies that for all $k \geq 2$,

$$(2.5) \quad w^k = \begin{cases} p_y \# a^k & \text{if } a \in F_y \\ 0 & \text{if } a \notin F_y. \end{cases}$$

Thus if $a \in F_y$ and has order m , the minimum polynomial of $p_y \# a$ is

$$f(Z) = Z^{m+1} - Z,$$

and so the characteristic roots of $p_y \# a$ are $\{0\} \cup \{m^{\text{th}} \text{ roots of } 1\}$.

Lemma 2.6. (1) \mathcal{B} is closed under the antipode S .

(2) The set $\mathcal{B}' := \{p_y \# a \in \mathcal{B} \mid a \in F_y\}$ is also closed under S .

(3) If $w = p_y \# a \in \mathcal{B}'$, then $S(w) = p_{y^{-1}} \# ya^{-1}y^{-1}$.

Proof. (1) is clear from the formula for S above. For (2), formula (2.5) shows that \mathcal{B}' is exactly the set of non-nilpotent elements of \mathcal{B} , so it is also closed under S .

For (3), $w \in \mathcal{B}'$ implies that $a \in F_y$, and thus $y \triangleleft a = y$. Then

$$ya = (y \triangleright a)(y \triangleleft a) = (y \triangleright a)y$$

and so $y \triangleright a = yay^{-1}$. Substituting in the formula for S , we see $S(w) = p_{y^{-1}} \# ya^{-1}y^{-1}$. □

We review the description of the simple modules over a bismash product.

Proposition 2.7. Let $H = \mathbb{E}^G \# \mathbb{E}F$ be a bismash product, as above, where now \mathbb{E} is algebraically closed. For the action \triangleleft of F on G , fix one element x in each F -orbit \mathcal{O} of G , and let F_x be its stabilizer in F , as above. Let $V = V_x$ be a simple left F_x -module and let $\hat{V}_x = \mathbb{E}F \otimes_{\mathbb{E}F_x} V_x$ denote the induced $\mathbb{E}F$ -module.

\hat{V}_x becomes an H -module in the following way: for any $y \in G$, $a, b \in F$, and $v \in V_x$,

$$(p_y \# a)[b \otimes v] = \delta_{y \triangleleft (ab), x}(ab \otimes v).$$

Then \hat{V}_x is a simple H -module under this action, and every simple H -module arises in this way.

Proof. In the case of characteristic 0, this was first proved for the Drinfel'd double $D(G)$ of a finite group G over \mathbb{C} by [DPR] and [M]. The case of characteristic $p > 0$ was done by [?].

For bismash products, extending the results for $D(G)$, the characteristic 0 case was done in [KMM, Lemma 2.2 and Theorem 3.3]. The case of characteristic $p > 0$ follows by extending the arguments of [?] for $D(G)$; see also [MoW]. □

Remark 2.8. The arguments for Proposition 2.7 also show that if we begin with an indecomposable module V_x of F_x , then \hat{V}_x is an indecomposable module for $H_{\mathbb{E}}$, and all indecomposable $H_{\mathbb{E}}$ -modules arise in this way. This fact is discussed in [W2] after Proposition 4.4; it could also be obtained from [?], using the methods of Theorem 2.2 and Corollary 2.3 in that paper.

Now fix an irreducible $H_{\mathbb{E}}$ -module $\hat{V} = \hat{V}_x = \mathbb{E}F \otimes_{\mathbb{E}F_x} V_x$ as in Proposition 2.7. To compute the values of the character for \hat{V} , we use a formula from [JM]; it is a simpler version of [N2, Proposition 5.5] and is similar to the formula in [KMM, p 898]:

Lemma 2.9. [JM, Lemma 4.5] *Fix a set T_x of representatives for the right cosets of F_x in F . Let χ_x be the character of V_x . Then the character $\hat{\chi}_x$ of \hat{V}_x may be computed as follows:*

$$\hat{\chi}_x(p_y \# a) = \sum_{t \in T_x \text{ and } t^{-1}at \in F_x} \delta_{y \triangleleft t, x} \chi_x(t^{-1}at),$$

for any $y \in G$, $a \in F$.

Next we review some known facts about Frobenius-Schur indicators for representations of Hopf algebras. For a representation V of H , recall that a bilinear form $\langle -, - \rangle : V \otimes_{\mathbb{E}} V \rightarrow \mathbb{E}$ is H -invariant if for all $h \in H$ and $v, w \in V$,

$$\sum \langle h_1 \cdot v, h_2 \cdot w \rangle = \varepsilon(h) 1_{\mathbb{E}}.$$

It follows that the antipode is the adjoint of the form; that is, for all $h, l \in H$, $v, w \in V$,

$$\langle S(h) \cdot v, l \cdot w \rangle = \langle h \cdot v, \bar{S}(l) \cdot w \rangle = \langle h \cdot v, S(l) \cdot w \rangle,$$

using that $S^2 = id$.

Theorem 2.10. [GM] *Let H be a finite-dimensional Hopf algebra over \mathbb{E} such that $S^2 = id$ and \mathbb{E} splits H . Let V be an irreducible representation of H . Then V has a well-defined Frobenius-Schur indicator $\nu(V) \in \{0, 1, -1\}$. Moreover*

$$(1) \nu(V) \neq 0 \iff V^* \cong V.$$

(2) $\nu(V) = +1$ (respectively -1) $\iff V$ admits a non-degenerate H -invariant symmetric (resp, skew-symmetric) bilinear form.

If in addition H is semisimple and cosemisimple, then in fact $\nu(V)$ can be computed by the formula $\nu(V) = \chi(\Lambda_1 \Lambda_2)$, where χ is the character belonging to V and Λ is a normalized integral of H [LM]. This formula does not work in general, but still Theorem 2.10 applies to any bismash product since as noted above, it is always true that $S^2 = id$. Sometimes the indicator is called the *type* of V .

We remark that, unlike the case for groups, $\nu(V) = +1$ does not imply that the character χ_V is real-valued, even when $\mathbb{E} = \mathbb{C}$. However it is still true that if $V^* \cong V$, then $\chi^* = \chi$, that is, $\chi \circ S = \chi$.

Finally we fix the following notation:

Definition 2.11. [CR, p 402]. A p -modular system $(\mathbb{K}, R, \mathbb{k})$ consists of a discrete valuation ring R with quotient field \mathbb{K} , maximal ideal $\mathfrak{p} = \pi R$ containing the rational prime p , and residue class field $\mathbb{k} = R/\mathfrak{p}$ of characteristic p .

We will mainly be interested in the following special case, as in [Th] with a slight change in notation.

Example 2.12. Let $H_{\mathbb{Q}}$ be a Hopf algebra over \mathbb{Q} and let \mathbb{L} be an algebraic number field which is a splitting field for $H_{\mathbb{Q}}$. Let \mathcal{P} be a prime ideal of the ring of integers of \mathbb{L} containing the rational prime $p \neq 2$, let R be the completion of the ring of \mathcal{P} -integers of \mathbb{L} , \mathbb{K} be the field of fractions of R , π be a generator for the maximal ideal \mathfrak{p} of R , and $\mathbb{k} = R/\pi R$.

Then $(\mathbb{K}, R, \mathbb{k})$ is a p -modular system.

3. BRAUER CHARACTERS FOR G

In this section we review the definition of Brauer characters for a finite group [CR], [Nv] and summarize some of the classical results.

We fix the following notation, for a given finite group G . Let $|G|$ denote the order of G , and let $|G|_p$ denote the largest power of p in $|G|$; thus $|G| = |G|_p m$ where $p \nmid m$. Since \mathbb{K} splits G , it contains a primitive m^{th} root of 1, say ω , which in fact is in R . Under the natural map $f : R \rightarrow \mathbb{k}$, $\bar{\omega} = f(\omega)$ is a primitive m^{th} root of 1 in \mathbb{k} .

Let $G_{p'}$ denote the set of p -regular elements of G , that is elements of G whose order is prime to p . Thus for each $x \in G_{p'}$, all of the eigenvalues of x on any (left) $\mathbb{k}G$ -module W are m^{th} roots of 1, and hence may be expressed as a power of $\bar{\omega}$. Denote the eigenvalues of x by $\{\bar{\omega}^{i_1}, \dots, \bar{\omega}^{i_t}\}$.

Definition 3.1. For each (left) $\mathbb{k}G$ -module W , the \mathbb{K} -valued function $\phi : G_{p'} \rightarrow \mathbb{K}$ defined for each $x \in G_{p'}$ by

$$\phi(x) = \omega^{i_1} + \dots + \omega^{i_t} = \sum_{j=1}^t f^{-1}(\bar{\omega}^{i_j}).$$

is called the *Brauer character of G afforded by W* .

Remark 3.2. Note that ϕ is a class function on the conjugacy classes of p -regular elements of G . ϕ can be extended to $\phi^{\#}$, a class function on all of G , by defining $\phi^{\#}(x) = 0$ for any x in the complement of $G_{p'}$. It follows that $\phi^{\#}$ is a \mathbb{K} -linear combination of the ordinary irreducible characters χ_i of $\mathbb{K}G$ [CR, p 423]. Thus ϕ is a \mathbb{K} -linear combination of the $\chi_i|_{G_{p'}}$.

We record some facts about Brauer characters of groups. See [CR, 17.5], [I, Chapter 15].

- Proposition 3.3.**
- (1) λ takes values in R and $\overline{\lambda(x)} = \text{Tr}(x, V)$, all $x \in G_{p'}$.
 - (2) Let $W_0 \supset W_1 \supset 0$ be $\mathbb{k}G$ -modules, let ϕ be the Brauer character afforded by W_0/W_1 , ϕ_1 the Brauer character afforded by W_1 and ϕ_0 the Brauer character of W_0 . Then $\phi_0 = \phi + \phi_1$.
 - (3) Let V be a $\mathbb{K}G$ -module with \mathbb{K} -character χ . Then for each R -lattice M in V , the restriction $\chi|_{G_{p'}}$ is the Brauer character of the $\mathbb{k}G$ -module $\overline{M} := M/\mathfrak{p}M$.

We fix the following notation, as in [CR]:

- (1) $\text{Irr}(G) = \{\chi_1, \dots, \chi_n\}$ denotes the irreducible characters of $\mathbb{K}G$;

- (2) $\text{Irr}_{\mathbb{k}}(G) = \{\psi_1, \dots, \psi_d\}$ denotes the irreducible characters of $\mathbb{k}G$;
(3) $\text{IBr}(G) = \{\phi_1, \dots, \phi_d\}$ denotes the Brauer characters corresponding to $\{\psi_1, \dots, \psi_d\}$.

By [CR, 16.7 and 16.20], there exists a homomorphism of abelian groups

$$(3.4) \quad d : G_0(\mathbb{k}G) \rightarrow G_0(\mathbb{k}G),$$

called the *decomposition map*, such that for any class $[V]$ in $G_0(\mathbb{k}G)$, $d([V]) = [\overline{M}] \in G_0(\mathbb{k}G)$, where M is any RG -lattice in V and $\overline{M} := M/\mathfrak{p}M$.

Using Proposition 3.3(3), it follows that for any χ_i , there are integers d_{ij} such that

$$(3.5) \quad \chi_i|_{G_{p'}} = \sum_j d_{ij} \phi_j.$$

The multiplicities $d_{ij} = d(\chi_i|_{G_{p'}}, \phi_j)$ are called *decomposition numbers*, and the matrix $D = [d_{ij}]$ is called the *decomposition matrix*. The matrix $C = D^t D$ is called the *Cartan matrix*.

From [CR], 17.12 - 17.15, the set $Bch(\mathbb{k}G)$ of *virtual Brauer characters*, that is \mathbb{Z} -linear combinations of Brauer characters of $\mathbb{k}G$ -modules, is a ring under addition and multiplication of functions, and $Bch(\mathbb{k}G) \cong G_0(\mathbb{k}G)$. Using that $G_0(\mathbb{k}G) \cong ch(\mathbb{k}G)$, the ring of virtual characters of $\mathbb{k}G$, it follows that the decomposition map d induces a map

$$d' : ch(\mathbb{k}G) \rightarrow Bch(\mathbb{k}G),$$

where d' is the restriction map $\psi \rightarrow \psi|_{G_{p'}}$.

Consequently Equation (3.5) implies that if χ is the character for V and $\chi|_{G_{p'}} = \sum_j \alpha_j \phi_j$, where ϕ_j is the Brauer character of the simple $\mathbb{k}G$ -module W_j , and $d([V]) = [\overline{M}] \in G_0(\mathbb{k}G)$, then

$$(3.5) \quad [\overline{M}] = \sum_j \alpha_j [W_j].$$

We will need the following theorem in our more general situation:

Theorem 3.6. (Brauer) [CR, 18.25][I, Ex (15.3)] *Det(C) is a power of p.*

We will also need the analog of the following:

Theorem 3.7. [CR, (17.9)] *The irreducible Brauer characters IBr(G) form a \mathbb{K} -basis of the space of \mathbb{K} -valued class functions of $G_{p'}$.*

One consequence of this theorem is:

Corollary 3.8. *Let \mathbb{E} be a splitting field for G of char $p > 0$. Then the number of simple $\mathbb{E}G$ -modules is equal to the number of p -regular conjugacy classes of G .*

A crucial ingredient of the proof of the theorem is the following elementary lemma.

Lemma 3.9. *Let $\rho : G \rightarrow GL_n(\mathbb{k})$ be a matrix representation of G over \mathbb{k} . For any $x \in G$, we may write $x = us$, where u is a p -element of G and s is a p' -element. Then x and s have the same eigenvalues, counting multiplicities.*

The lemma follows since $su = us$, and all eigenvalues of u will equal 1.

4. BRAUER CHARACTERS FOR $H_{\mathbb{k}}$

In this section we define Brauer characters for our bismash products and show that they have properties analogous to those for finite groups discussed in Section 3.

Assume that $\mathbb{L}, \mathbb{K}, \pi, R$ and \mathbb{k} are as Example 2.12, with $\mathbb{k} = R/\pi R$.

Fix an irreducible $H_{\mathbb{L}}$ -module $V_{\mathbb{L}}$ whose indicator is non-zero. Since \mathbb{L} is a splitting field for $H_{\mathbb{Q}}$, so is \mathbb{K} , and thus

$$V := V_{\mathbb{L}} \otimes_{\mathbb{L}} \mathbb{K}$$

is an irreducible $H_{\mathbb{K}}$ -module. Moreover the bilinear form on $V_{\mathbb{L}}$ extends to a bilinear form on V , and thus there is a non-singular $H_{\mathbb{K}}$ -invariant bilinear form $\langle \cdot, \cdot \rangle$ on V , with values in \mathbb{K} , which is symmetric or skew-symmetric by 2.10.

Recall the basis \mathcal{B} of $H_{\mathbb{K}}$ from Section 2.

Definition 4.1. Let $V = V_{\mathbb{L}} \otimes_{\mathbb{L}} \mathbb{K}$ be as above. Then an $R\mathcal{B}$ -lattice in V is a finitely generated $R\mathcal{B}$ -submodule L of V such that $\mathbb{K}L = V$.

From now on we also assume that \mathbb{L} denotes an algebraic number field which is a splitting field for $H_{\mathbb{Q}} = \mathbb{Q}^G \# \mathbb{Q}F$. Then \mathbb{k} is a splitting field for $H_{\mathbb{k}}$.

Let $\hat{W} = \hat{W}_x$ be a simple $H_{\mathbb{k}}$ module, as in Proposition 2.7. That is, for a given F -orbit \mathcal{O} of G and fixed $x \in \mathcal{O} = \mathcal{O}_x$, with F_x the stabilizer of x in F and $W = W_x$ a simple $\mathbb{k}F_x$ -module, $\hat{W} = \mathbb{k}F \otimes_{\mathbb{k}F_x} W$. Recall \hat{W} becomes an H -module via

$$(p_y \# a)[b \otimes w] = \delta_{y \triangleleft (ab), x}[ab \otimes w],$$

for any $y \in G$, $a, b \in F$, and $w \in W$.

As in Lemma 2.9, fix a set T_x of representatives of the right cosets of F_x in F .

Lemma 4.2. Consider the action of $p_y \# a$ on $\hat{W} = \hat{W}_x$ as above.

(1) If $(p_y \# a)\hat{W} \neq 0$, then there exists $t \in T_x$ and $w \in W$ such that

$$(p_y \# a)[t \otimes w] = \delta_{y \triangleleft (at), x}[at \otimes w] \neq 0.$$

Thus $y = x \triangleleft (at)^{-1} \in \mathcal{O}_x$.

(2) If $p_y \# a$ has non-zero eigenvalues on \hat{W}_x , then $a \in F_y$ and $x = y \triangleleft t$, where t is as in (1).

(3) For t as in (1) and (2), $t^{-1}at \in F_x$ and so $at \otimes w = t \otimes (t^{-1}at)w$.

Proof. (1) By the formula for the action of $p_y \# a$ on \hat{W} , there exists $b \in F$ and $w \in W$ such that $(p_y \# a)[b \otimes w] = \delta_{y \triangleleft (ab), x}[ab \otimes w] \neq 0$. Thus $y = x \triangleleft (ab)^{-1} \in \mathcal{O}_x$. Now for some $t \in T_x$, $b \in tF_x$. It is easy to see that t satisfies the same properties as b .

(2) If $p_y \# a$ has non-zero eigenvalues on \hat{W} , then $(p_y \# a)^2 \neq 0$, and so $a \in F_y$ by (2.5). Now using (1), $x = y \triangleleft (at) = (y \triangleleft a) \triangleleft t = y \triangleleft t$.

(3) Since $y = x \triangleleft t^{-1}$ and $a \in F_y$, it follows that $t^{-1}at \in F_x$. Thus we can write $at \otimes w = t \otimes (t^{-1}at)w$. \square

We next prove an analog of Lemma 3.9, although in our case the two factors do not necessarily commute in $H_{\mathbb{k}}$.

Lemma 4.3. Consider $\rho : H_{\mathbb{k}} \rightarrow \text{End}_{\mathbb{k}}(\hat{W}) \cong M_n(\mathbb{k})$. For $a \in F$ write $a = su$, with s the p -regular part and u the p -part of a . Then $\rho(p_y \# a)$ and $\rho(p_y \# s)$ have the same eigenvalues, counting multiplicities.

Proof. Even though $p_y \# s$ and $1 \# u$ do not commute, their actions on \hat{W} do commute: suppose $b \otimes w$ is such that $(p_y \# a) \cdot [b \otimes w] \neq 0$. By Lemma 4.2, $y \triangleleft b = x$ and $a \in F_y$. Thus $s \in F_y$ since s is a power of a . Then

$$\begin{aligned} (p_y \# s)(1 \# u) \cdot [b \otimes w] &= (p_y \# a) \cdot [b \otimes w] \\ &= ab \otimes v \end{aligned}$$

and

$$\begin{aligned} (1 \# u)(p_y \# s) \cdot [b \otimes w] &= \delta_{y \triangleleft sb, x} (1 \# u) \cdot [sb \otimes w] \\ &= (1 \# u) \cdot [sb \otimes w] \\ &= usb \otimes w \\ &= ab \otimes w. \end{aligned}$$

Since the eigenvalues of $1 \# u$ are all 1, the eigenvalues of $p_y \# a$ are the same as those of $p_y \# s$. \square

The lemma shows that to find the character of some $p_y \# a$, it suffices to look at the character of $p_y \# s$, where s is the p' -part of a . Moreover, by Lemma 4.2, the character of $p_y \# a$ will be non-zero only if $a \in F_y$.

Thus, as a replacement for the p' -elements of the group in the classical case, we consider the subset of the basis \mathcal{B} defined in (2.4) of those elements which are non-nilpotent element and have group element in $F_{p'}$. That is, we define

$$(4.4) \quad \mathcal{B}_{p'} := \{p_y \# a \in \mathcal{B}' \mid a \in F_{p'}\} = \{p_y \# a \in \mathcal{B} \mid a \in F_y \cap F_{p'}\},$$

where $F_{p'}$ is the set of p -regular elements in F . By Lemma 2.6, $\mathcal{B}_{p'}$ is also closed under the antipode S , since if $a \in F_{p'}$ and $w = p_y \# a$ is non-nilpotent, then by Lemma 2.6(3), $S(w) = p_{y^{-1}} \# ya^{-1}y^{-1}$. Since $ya^{-1}y^{-1}$ has the same order as a , $S(w)$ is also in $\mathcal{B}_{p'}$.

The above remarks motivate our definition of Brauer characters for $H_{\mathbb{k}}$, by using the formula in Lemma 2.9. That is, if $W = W_x$ is a simple $\mathbb{k}F_x$ -module with character ψ , then the character of the simple $H_{\mathbb{k}}$ -module \hat{W} is given by

$$(4.5) \quad \hat{\psi}(p_y \# a) = \sum_{t \in T_x \text{ and } t^{-1}at \in F_x} \delta_{y \triangleleft t, x} \psi(t^{-1}at).$$

Definition 4.6. Let $W = W_x$ be a simple $\mathbb{k}F_x$ -module with character ψ , and let ϕ be the classical Brauer character of W constructed from ψ . Then the *Brauer character* of \hat{W} is the function

$$\hat{\phi} : \mathcal{B}_{p'} \rightarrow \mathbb{K}$$

defined on any $p_y \# a \in \mathcal{B}_{p'}$ by

$$\hat{\phi}(p_y \# a) = \sum_{t \in T_x \text{ and } t^{-1}at \in F_x} \delta_{y \triangleleft t, x} \phi(t^{-1}at).$$

Remark 4.7. If $\hat{\phi}$ is a Brauer character, then also $\hat{\phi}^* = \hat{\phi} \circ S$ is a Brauer character: namely if $\hat{\phi}$ is the Brauer character for $\hat{\psi}$, then $\hat{\phi}^*$ is the Brauer character of $\hat{\psi}^*$, using the fact that $\mathcal{B}_{p'}$ is stable under S .

We fix the following notation, as for groups:

- (1) $Irr(H_{\mathbb{K}}) = \{\hat{\chi}_1, \dots, \hat{\chi}_n\}$ denotes the irreducible characters of $H_{\mathbb{K}}$;
- (2) $Irr_{\mathbb{k}}(H_{\mathbb{k}}) = \{\hat{\psi}_1, \dots, \hat{\psi}_d\}$ denotes the irreducible characters of $H_{\mathbb{k}}$;
- (3) $IBr(H_{\mathbb{k}}) = \{\hat{\phi}_1, \dots, \hat{\phi}_d\}$ denotes the Brauer characters corresponding to $\{\hat{\psi}_1, \dots, \hat{\psi}_d\}$. As for groups, the elements of $IBr(H_{\mathbb{k}})$ are called *irreducible* Brauer characters.
- (4) $Bch(H_{\mathbb{k}})$ denotes the ring of *virtual* Brauer characters of $H_{\mathbb{k}}$, that is, the \mathbb{Z} -linear span of the irreducible Brauer characters.

Lemma 4.8. $\hat{\phi}_j$ is a \mathbb{K} -linear combination of the $\hat{\chi}_i|_{\mathcal{B}_{p'}}$. Consequently if all $\hat{\chi}_i$ are self-dual, then all $\hat{\phi}_j$ are also self-dual, and so are all $\hat{\psi}_j$.

Proof. By Remark 3.2 applied to F_x , the Brauer character ϕ_j may be written as $\phi_j = \sum_i \alpha_i \chi_i|_{(F_x)_{p'}}$, for $\alpha_i \in \mathbb{K}$. Lifting this equation through induction up to $F_{p'}$ (and so to $\mathcal{B}_{p'}$) as in Lemma 2.9, we obtain the first statement in the lemma.

Now if all $\hat{\chi}_i$ are self-dual, then the same property holds for the $\hat{\phi}_j$ since they are linear combinations of the $\hat{\chi}_i|_{\mathcal{B}_{p'}}$. Fix one of the $\hat{\psi}_j$ and its Brauer character $\hat{\phi}_j$. Since $\hat{\phi}_j^* = \hat{\phi}_j \circ S$ and $\hat{\psi}_j^* = \hat{\psi}_j \circ S$, using the formula for S as well as (4.5) and the formula in 4.6, we see that $\hat{\phi}_j^* = \hat{\phi}_j$ if and only if $\hat{\psi}_j^* = \hat{\psi}_j$. \square

We may follow exactly the proof of Proposition 3.3, that is [CR, 17.5, (2) - (4)], to show the following:

Proposition 4.9. (1) $\hat{\phi}$ takes values in R and $\overline{\hat{\phi}(p_y \# a)} = Tr(p_y \# a, \hat{W})$, for $a \in F_{p'}$.

(2) Given $H_{\mathbb{k}}$ -modules $\hat{W}_0 \supset \hat{W}_1 \supset 0$, let $\hat{\phi}$ be the Brauer character afforded by \hat{W}_0 / \hat{W}_1 , $\hat{\phi}_1$ the Brauer character afforded by \hat{W}_1 , and $\hat{\phi}_0$ the Brauer character of \hat{W}_0 . Then $\hat{\phi}_0 = \hat{\phi} + \hat{\phi}_1$.

(3) Let V be a $\mathbb{K}\mathcal{B}$ -module with \mathbb{K} -character χ . Then for each $R\mathcal{B}$ -lattice M in V , the restriction $\chi|_{R\mathcal{B}_{p'}}$ is the Brauer character of the $H_{\mathbb{k}}$ -module $\overline{M} := M/\mathfrak{p}M$.

Similarly, one may follow the first part of the proof of Theorem 3.7 [CR, 17.9], replacing Lemma 3.9 with Lemma 4.3, to show

Theorem 4.10. The irreducible Brauer characters $IBr(H_{\mathbb{k}})$ are \mathbb{K} -linearly independent.

We may also extend the decomposition map d in Section 3 to obtain a homomorphism of abelian groups

$$(4.11) \quad \hat{d} : G_0(H_{\mathbb{K}}) \rightarrow G_0(H_{\mathbb{k}}),$$

called the *decomposition map*, such that for any class $[\hat{V}]$ in $G_0(H_{\mathbb{K}})$, $\hat{d}([\hat{V}]) = [\overline{M}] \in G_0(H_{\mathbb{k}})$, where M is any $R\mathcal{B}$ -lattice in \hat{V} and $\overline{M} := M/\mathfrak{p}M$.

Again using the facts about groups, the set $Bch(H_{\mathbb{k}})$ of *virtual Brauer characters*, that is \mathbb{Z} -linear combinations of Brauer characters of $\mathbb{k}G$ -modules, is a ring under addition and multiplication of functions, and $Bch(H_{\mathbb{k}}) \cong G_0(H_{\mathbb{k}})$. Using that

$G_0(\mathbb{K}G) \cong ch(\mathbb{K}G)$, the ring of virtual characters of $H_{\mathbb{K}}$, it follows that the decomposition map \hat{d} induces a map

$$\hat{d}' : ch(H_{\mathbb{K}}) \rightarrow Bch(H_{\mathbb{k}}),$$

where \hat{d}' is the restriction map $\hat{\psi} \rightarrow \hat{\psi}|_{\mathcal{B}_{p'}}$.

Consequently Equation (3.5) implies that if $\hat{\chi}$ is the character for \hat{V} and $\hat{\chi}|_{\mathcal{B}_{p'}} = \sum_j \alpha_j \hat{\phi}_j$, where $\hat{\phi}_j$ is the Brauer character of the simple $H_{\mathbb{k}}$ -module \hat{W}_j , and $\hat{d}([\hat{V}]) = [\overline{M}] \in G_0(H_{\mathbb{k}})$, then

$$(4.11) \quad [\overline{M}] = \sum_j \alpha_j [W_j].$$

From now on we wish to distinguish the characters (over \mathbb{k} or \mathbb{K}) which arise from stabilizers of elements in different F -orbits of G . Assume that there are exactly s distinct orbits of F on G and that we fix $x_q \in \mathcal{O}_q$, the q^{th} orbit. Thus for a fixed $x = x_q \in G$ with stabilizer $F_x = F_{x_q}$, we will write $\chi_{i,x}$ for an irreducible character of $\mathbb{K}F_x$, and $\hat{\chi}_{i,x}$ for its induction up to $\mathbb{K}F$, which becomes an irreducible character of $H_{\mathbb{K}}$.

Similarly $\psi_{j,x}$ denotes an irreducible character of $\mathbb{k}F_x$, and $\hat{\psi}_{j,x}$ its induction up to $\mathbb{k}F$, which becomes an irreducible character of $H_{\mathbb{k}}$. Also $\phi_{j,x}$ denotes the Brauer character corresponding to $\psi_{j,x}$, and $\hat{\phi}_{j,x}$ the Brauer character corresponding to $\hat{\psi}_{j,x}$.

Lemma 4.12. *Let ϕ_x be a virtual Brauer character of $\mathbb{k}F_x$ and assume that $\phi_x = \sum_j z_{j,x} \phi_{j,x}$, where as above the $\phi_{j,x}$ are the Brauer characters of $\mathbb{k}F_x$.*

Then $\hat{\phi}_x = \sum_j z_{j,x} \hat{\phi}_{j,x}$.

The lemma follows from Definition 4.6 of a Brauer character $\hat{\phi}$ for $H_{\mathbb{k}}$ in terms of a Brauer character ϕ for $\mathbb{k}F_x$. Moreover Lemma 2.9 becomes

$$\hat{\chi}_{i,x}(p_y \# a) = \sum_{t \in T_x \text{ and } t^{-1}at \in F_x} \delta_{y \triangleleft t,x} \chi_{i,x}(t^{-1}at).$$

Applying Equation (3.5) to F_x , there are integers $d_{ij,x}$ such that

$$\chi_{i,x}|_{(F_x)_{p'}} = \sum_j d_{ij,x} \phi_{j,x},$$

where the $\phi_{j,x}$ are in $IBr(\mathbb{k}F_x)$.

Lifting the $\chi_{i,x}$ to $\hat{\chi}_{i,x}$ on $\mathcal{B}_{p'}$, we see that

$$\hat{\chi}_{i,x}|_{\mathcal{B}_{p'}} = \sum_j d_{ij,x} \hat{\phi}_{j,x}.$$

That is, the decomposition numbers for the $\hat{\chi}_{i,x}|_{\mathcal{B}_{p'}}$ with respect to the $\hat{\phi}_{j,x}$ are the same as the decomposition numbers for the $\chi_{i,x}|_{(F_x)_{p'}}$ with respect to the $\phi_{j,x}$ for the group F_x . Thus the decomposition matrix $\hat{D}_x = [d_{ij,x}]$ for the $\hat{\chi}_{i,x}|_{\mathcal{B}_{p'}}$ with respect to the $\hat{\phi}_{j,x}$ is the same as the decomposition matrix D_x for the $\chi_{i,x}|_{(F_x)_{p'}}$ with respect to the $\phi_{j,x}$.

The above discussion proves

Proposition 4.13. *As above, assume that there are exactly s distinct orbits \mathcal{O} of F on G and choose $x_q \in \mathcal{O}_q$, for $q = 1, \dots, s$. Then*

(1) $\hat{D}_{x_q} = D_{x_q}$

(2) *The decomposition matrix for the $\hat{\chi}_i|_{\mathcal{B}_{p'}}$ with respect to the $\hat{\phi}_j$ is the block matrix*

$$\hat{D} = \begin{bmatrix} \hat{D}_{x_1} & 0 & \cdots & 0 \\ 0 & \hat{D}_{x_2} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \hat{D}_{x_s} \end{bmatrix} \text{ where } \hat{D}_{x_q} \text{ is the decomposition matrix of } \hat{\chi}_{i,x_q}|_{\mathcal{B}_{p'}} \text{ with}$$

respect to $\hat{\phi}_{j,x_q}$.

As for groups, $\hat{C} = \hat{D}^t \hat{D}$ is called the *Cartan matrix*. We are now able to extend the theorem of Brauer we need (3.6).

Theorem 4.14. *Det(\hat{C}) is a power of p .*

Proof. First, \hat{C} is also a block matrix, with blocks $\hat{C}_{x_q} = (\hat{D}_{x_q})^t \hat{D}_{x_q}$. By Brauer's theorem applied to each group F_{x_q} , we know that $\text{Det}(\hat{C}_{x_q})$ is a power of p . Thus $\text{Det}(\hat{C})$ is a power of p . \square

5. INVARIANT FORMS: WITT KERNELS AND LATTICES

A first step in the direction of extending Thompson's theorem concerns the Witt kernel of a module with a bilinear form as in Theorem 2.10. We will show that, for an arbitrary field \mathbb{E} , the notion of Witt kernel of an $\mathbb{E}G$ -module extends to $H_{\mathbb{E}}$ -modules. One can then follow the argument in [Th].

Let V be a finitely-generated $H_{\mathbb{E}}$ -module which is equipped with a non-degenerate $H_{\mathbb{E}}$ -invariant bilinear form $\langle -, - \rangle : V \otimes_{\mathbb{E}} V \rightarrow \mathbb{E}$, which is either symmetric or skew symmetric. For example, if \mathbb{E} is algebraically closed, then any irreducible self-dual $H_{\mathbb{E}}$ -module has such a form by Theorem 2.10. For any submodule U of V ,

$$U^\perp = \{v \in V \mid \langle v, U \rangle = 0\}.$$

Since the form is H -invariant and S is the adjoint of the form, for all $u \in U^\perp$,

$$\langle h \cdot v, U \rangle = \langle v, S(h) \cdot U \rangle = \langle v, U \rangle = 0.$$

Thus U^\perp is also a submodule of V . Note also that $U^{\perp\perp} = U$ since V is finite-dimensional over \mathbb{E} . Let

$$\mathcal{M} = \mathcal{M}_V = \{V_0 \mid V_0 \text{ is an } H_{\mathbb{E}}\text{-submodule of } V \text{ and } \langle V_0, V_0 \rangle = 0\},$$

that is, $V_0 \subseteq V_0^\perp$.

Obviously, $\{0\} \in \mathcal{M}$, and \mathcal{M} is partially ordered by inclusion. If $V_0 \in \mathcal{M}$, then V_0^\perp/V_0 inherits a non-degenerate form given by

$$(v_0 + V_0, v'_0 + V_0)_{V_0^\perp/V_0} := \langle v_0, v'_0 \rangle, \quad v_0, v'_0 \in V_0.$$

If V_1 is a maximal element of \mathcal{M} , it is not difficult to see that V_1^\perp/V_1 is a completely reducible H_E -module, and the restriction of $(\ , \)_{V_1^\perp/V_1}$ to any H_E -submodule of V_1^\perp/V_1 is non-degenerate.

Definition 5.1. Let V_1 be a maximal element of \mathcal{M} . Then the *Witt kernel* of V is $V' := V_1^\perp/V_1$.

It is not clear from this definition that the Witt kernel is independent of the choice of the maximal element of \mathcal{M} . However, we have

Lemma 5.2. *If V_1, V_2 are maximal elements of \mathcal{M} , then there is an $H_{\mathbb{E}}$ -isomorphism*

$$\Phi : V_1^\perp/V_1 \rightarrow V_2^\perp/V_2,$$

such that

$$(v_1, v'_1)_{V_1^\perp/V_1} = (\Phi(v_1), \Phi(v'_1))_{V_2^\perp/V_2}, \text{ for all } v_1, v'_1 \in V_1^\perp/V_1.$$

The proof follows exactly the proof of [Th, Lemma 2.1] for group algebras.

We next extend the facts shown in [Th] about G -invariant forms on RG -lattices to the case of lattices for bismash products. Our proofs follow [Th] very closely.

Assume that $\mathbb{L}, \mathbb{K}, \pi, R$ and \mathbb{k} are as at the end of Section 2, with $\mathbb{k} = R/\pi R$.

Fix an irreducible $H_{\mathbb{L}}$ -module $V_{\mathbb{L}}$ whose indicator is non-zero. Since \mathbb{L} is a splitting field for $H_{\mathbb{Q}}$, so is \mathbb{K} , and thus

$$V := V_{\mathbb{L}} \otimes_{\mathbb{L}} \mathbb{K}$$

is an irreducible $H_{\mathbb{K}}$ -module. Moreover the bilinear form on $V_{\mathbb{L}}$ extends to a bilinear form on V , and thus there is a non-singular $H_{\mathbb{K}}$ -invariant bilinear form $\langle \ , \ \rangle$ on V , with values in \mathbb{K} , which is symmetric or skew-symmetric by 2.10.

Recall the basis \mathcal{B} of $H_{\mathbb{K}}$ from Section 2.

Definition 5.3. Let $V = V_{\mathbb{L}} \otimes_{\mathbb{L}} \mathbb{K}$ be as above. Then an *$R\mathcal{B}$ -lattice* in V is a finitely generated $R\mathcal{B}$ -submodule L of V such that $\mathbb{K}L = V$.

Let $\mathcal{L} = \mathcal{L}_V$ be the family of $R\mathcal{B}$ -sublattices of V . If $L \in \mathcal{L}$, then L^* denotes the *dual lattice* defined by

$$L^* = \{l \in V \mid \langle L, l \rangle \subseteq R\}.$$

Since R is Noetherian, L^* is also an $R\mathcal{B}$ -lattice by [CR, 4.24]. In particular L^* is also finitely-generated. We also let

$$\mathcal{L}_I = \mathcal{L}_{V,I} = \{R\mathcal{B}\text{-lattices } L \in \mathcal{L}_V \mid \langle L, L \rangle \subseteq R\}$$

denote the set of integral lattices. If L is any element of \mathcal{L} , there is an integer n such that $\pi^n L \in \mathcal{L}_I$. Obviously, \mathcal{L}_I is partially ordered by inclusion and if $L_1, L_2 \in \mathcal{L}_I$ with $L_1 \subseteq L_2$, then $L_2 \subseteq L_1^*$. Thus any chain of sublattices starting with L_1 is contained in L_1^* , which is a Noetherian $R\mathcal{B}$ -module, and so the chain must stop. Thus every element of \mathcal{L}_I is contained in a maximal element of \mathcal{L}_I .

In the following discussion, L denotes a *fixed* maximal element of \mathcal{L}_I .

Lemma 5.4. (1) $\pi L^* \subseteq L$.

(2) Let $M = L^*/L$. There is a non singular $R\mathcal{B}$ -invariant form $\langle \ , \ \rangle_M$ on M , with values in \mathbb{k} , defined as follows: if $m_1, m_2 \in M$, $m_i = x_i + L$ then $\langle m_1, m_2 \rangle_M :=$ image in \mathbb{k} of $\pi \langle x_1, x_2 \rangle$.

Proof. (1) Let h be the smallest integer ≥ 0 such that $\pi^h L^* \subseteq L$. If $h \leq 1$, then (1) holds. So suppose $h \geq 2$.

Let $L_1 = L + \pi^{h-1} L^*$. Then $L_1 \in \mathcal{L}$. Moreover, if $u_1, u_2 \in L_1$, say $u_i = l_i + \pi^{h-1} l_i^*$, $l_i \in L$, $l_i^* \in L^*$, then

$$\langle u_1, u_2 \rangle = \langle l_1, l_2 \rangle + \pi^{h-1} (\langle l_1, l_2^* \rangle + \langle l_1^*, l_2 \rangle) + \pi^{h-2} \langle \pi^h l_1^*, l_2^* \rangle \in R$$

by definition of L^* and of h . Thus, $L_1 \in \mathcal{L}_I$. Since $L \subseteq L_1$, this violates the maximality of L . So (1) holds.

(2) If $l_1^*, l_2^* \in L^*$, then $\pi l_1^* \in L$, so $\langle \pi l_1^*, l_2^* \rangle \in R$. Since $\langle L, L^* \rangle$ and $\langle L^*, L \rangle$ are contained in R , and since π is a generator for the maximal ideal of R , it follows that $\langle \cdot, \cdot \rangle_M$ is well defined. To see that this form is non singular, suppose $l^* \in L^*$ and $\langle l^*, L^* \rangle = 0$. Then $l^* \in L^{**} = L$, so $l^* + L = 0$ in M . This proves (2). \square

Lemma 5.5. $\{l \in L \mid \langle l, L \rangle \subseteq \pi R\} = \pi L^*$.

Proof. By Lemma 5.4(1), $\pi L^* \subseteq L$. By definition of L^* , $\pi L^* \subseteq \{l \in L \mid \langle l, L \rangle \subseteq \pi R\}$. Thus it suffices to show that if $l \in L$ and $\langle l, L \rangle \subseteq \pi R$, then $l \in \pi L^*$. This is clear, since $\langle 1/\pi l, L \rangle \subseteq R$, so that by the definition of L^* , we have $1/\pi l \in L^*$. \square

6. INDICATORS AND BRAUER CHARACTERS

In this section we combine our work in the previous sections to prove the analog of a theorem of Thompson.

Theorem 6.1. Thompson [Th] *Let k be an algebraically closed field of odd characteristic, let G be a finite group, and let W be an irreducible $\mathbb{k}G$ -module. If W has non-zero Frobenius-Schur indicator, then W is a composition factor (of odd multiplicity) in the reduction mod p of an irreducible $\mathbb{K}G$ -module with the same indicator as W .*

By reduction mod p , we mean to use the p -modular system $(\mathbb{K}, R, \mathbb{k})$ as described in Example 2.12, and then the induced decomposition map as in (4.11).

We first prove the analog of [Th, Lemma 3.3]. Recall the notation in Section 4:

$V_{\mathbb{L}}$ is a fixed irreducible $H_{\mathbb{L}}$ -module which is self-dual and thus $V = V_{\mathbb{L}} \otimes_{\mathbb{L}} \mathbb{K}$ is an irreducible self-dual $H_{\mathbb{K}}$ -module, with character χ . V has a non-degenerate $H_{\mathbb{K}}$ -invariant bilinear form $\langle \cdot, \cdot \rangle$ with values in \mathbb{K} , which is symmetric or skew-symmetric by Theorem 2.10.

As in Section 5, \mathcal{L} is the family of $R\mathcal{B}$ -sublattices of V and \mathcal{L}_I is the subset of integral lattices. Let L denote a *fixed* maximal element of \mathcal{L}_I with dual lattice L^* .

Consider the following $H_{\mathbb{k}}$ -modules: let $X = L^*/\pi L^*$, $Y = L/\pi L^*$, and $Z = L^*/L$. Note that Y is a submodule of X . Then there is an exact sequence of $H_{\mathbb{k}}$ -modules

$$0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0.$$

Using the non-degenerate form on V , it follows from the argument in Lemma 5.4(2) that both X and Z have a non-degenerate form; moreover Lemma 5.5 gives us a non-degenerate form on Y . These three forms are all of the same type, that is, either all are symmetric or all are skew symmetric, and the type is given by the indicator $\nu(\chi)$ of V .

Proposition 6.2. *Let $V, L, X, Y,$ and Z be as above. Suppose that P is an irreducible $H_{\mathbb{k}}$ -module with Brauer character $\hat{\phi}$, such that*

- (1) $\hat{\phi}^* = \hat{\phi}$;
- (2) $d(\hat{\chi}|_{\mathcal{B}_{p'}}, \hat{\phi})$ is odd.

Let Y', Z' be the Witt kernels of Y, Z respectively. Then the multiplicity of P in $Y' \oplus Z'$ is odd. Consequently P has the same type as V .

Proof. We let $M = L^*$; then $X = L^*/\pi L^* = M/\mathfrak{p}M$. By Proposition 4.9, the restriction $\chi|_{R\mathcal{B}_{p'}}$ is the Brauer character of the $H_{\mathbb{k}}$ -module $\overline{M} := M/\mathfrak{p}M$.

By hypothesis, the multiplicity of $\hat{\phi}$ in $\hat{\chi}|_{\mathcal{B}_{p'}}$ is odd, and thus using the decomposition map, the multiplicity of P in $X = \overline{M}$ is odd. Thus the multiplicity of P in $Y \oplus Z$ is odd. Since $\hat{\phi} \circ S = \hat{\phi}$, it follows from the definition of Brauer characters that also $\hat{\psi} \circ S = \hat{\psi}$ on P , and so $P \cong P^*$ as $H_{\mathbb{k}}$ -modules.

As in Section 5, let Y_1 be an $H_{\mathbb{k}}$ -submodule of Y which is maximal subject to $\langle Y_1, Y_1 \rangle_Y = 0$. Then the multiplicity of P in Y_1 equals the multiplicity of P in Y/Y_1^\perp (since $Y_1^* \cong Y/Y_1^\perp$ and $P^* = P$), and so the parity of the multiplicity of P in Y equals the parity of the multiplicity of P in the Witt kernel $Y' = Y_1^\perp/Y_1$.

The same argument applies to Z , and thus the multiplicity of P in $Y' \oplus Z'$ is odd.

For the second part, by Section 5 we know that Y' is completely reducible, and thus if P appears in Y' , the non-degenerate bilinear form on Y' restricts to a non-degenerate form on P . By uniqueness, this form must agree with the given form on P , and thus P and Y' , and so P and Y , have the same type.

Similarly, if P appears in Z' , then P and V have the same type. But since P appears an odd number of times in $Y' \oplus Z'$, it must appear in either Y' or Z' . \square

Theorem 6.3. *Let \hat{P} be a self-dual simple $H_{\mathbb{k}}$ -module, and let $\hat{\phi}$ be its Brauer character. Then there is an irreducible \mathbb{K} -character $\hat{\chi}$ of $H_{\mathbb{K}}$ such that*

- (1) $\hat{\chi}^* = \hat{\chi}$, and
- (2) $d(\hat{\chi}|_{\mathcal{B}_{p'}}, \hat{\phi})$ is odd.

Moreover if $\hat{\chi}$ is any irreducible \mathbb{K} -character of $H_{\mathbb{K}}$ satisfying (1) and (2), then $\nu(\hat{\chi}) = \nu(\hat{P})$.

To prove the theorem, it will suffice to show that $\hat{\chi}$ exists, since the equality $\nu(\hat{\chi}) = \nu(\hat{P})$ follows from Proposition 6.2.

We follow the outline of Thompson's argument, although we must look at the RF_x -blocks separately. We know that for some $x = x_q$, \hat{P} is induced from a simple $\mathbb{k}F_x$ -module P . Let B_x be the block of RF_x containing the Brauer character ϕ of P , let $\{\chi_1, \dots, \chi_m\}$ be all of the irreducible \mathbb{K} -characters in B_x , and let $\{\phi_1, \dots, \phi_n\}$ be all of the irreducible Brauer characters in B_x .

Let D_x be the decomposition matrix of the χ_i with respect to the ϕ_j , and $C_x = D_x^t D_x$ the Cartan matrix. From Brauer's theorem 3.6, $\text{Det}(C_x)$ is a power of p and so is odd since p is odd.

Lifting this set-up to $H_{\mathbb{K}}$, \hat{B}_x is the block of $R\mathcal{B}$ containing the Brauer character $\hat{\phi}$ of \hat{P} , $\{\hat{\chi}_1, \dots, \hat{\chi}_m\}$ are all the irreducible \mathbb{K} characters in \hat{B}_x , and $\{\hat{\phi}_1, \dots, \hat{\phi}_n\}$ are all of the irreducible Brauer characters in \hat{B}_x .

Choose notation so that $n = 2n_1 + n_2$, where $\{\hat{\phi}_1, \hat{\phi}_2\}, \{\hat{\phi}_3, \hat{\phi}_4\}, \dots, \{\hat{\phi}_{2n_1-1}, \hat{\phi}_{2n_1}\}$, are pairs of non self-dual characters, that is, $(\hat{\phi}_{2i-1})^* = \hat{\phi}_{2i}$, and $\hat{\phi}_{2n_1+1}, \dots, \hat{\phi}_n$ are self-dual. By hypothesis, $n_2 \neq 0$ since $\hat{\phi}$ is one of the $\hat{\phi}_i$.

Write C_x in block form as $C_x = \begin{bmatrix} C_0 & C_2 \\ C_2^t & C_1 \end{bmatrix}$, where C_0 is $2n_1 \times 2n_1$ and C_1 is $n_2 \times n_2$.

The Theorem will now follow from the next two lemmas:

Lemma 6.4. *Det(C_1) is odd.*

Proof. For $i = 1, 2, \dots, n$, let P_i be the projective indecomposable $\mathbb{k}F_x$ -module whose socle has Brauer character ϕ_i , and let Φ_i be the Brauer character of P_i . Then $c_{ij} = (\Phi_i, \Phi_j)$.

Let $\sigma = (1, 2)(3, 4) \cdots (2n_1 - 1, 2n_1) \in \mathcal{S}_n$; also let σ denote the corresponding permutation matrix. Let $\tilde{\mathcal{S}}_n$ be the set of all permutations in \mathcal{S}_n which do not fix $\{2n_1 + 1, \dots, n\}$. Since $\tilde{\mathcal{S}}_n$ is the complement in \mathcal{S}_n of the centralizer of σ , it follows that $\sigma^{-1}\tilde{\mathcal{S}}_n\sigma = \tilde{\mathcal{S}}_n$, and σ has no fixed points on $\tilde{\mathcal{S}}_n$.

Looking at the matrix C_x , it follows that $\sigma^{-1}C_x\sigma = C_x$ since $(\hat{\phi}_i)^* = \hat{\phi}_{i+1}$ for $i = 1, 3, \dots, 2n_1 - 1$ and $(\hat{\phi}_i)^* = \hat{\phi}_i$ for $i = 2n_1 + 1, \dots, n$. Then

$$\text{Det}(C_x) = \text{Det}(C_0)\text{Det}(C_1) + \sum_{\tau \in \tilde{\mathcal{S}}_n} \text{sgn}(\tau)c_{1\tau(1)}c_{2\tau(2)} \cdots c_{n\tau(n)}.$$

Moreover $c_{ij} = c_{\sigma(i)\sigma(j)}$, again since $\sigma^{-1}C_x\sigma = C_x$. Choose $\tau \in \tilde{\mathcal{S}}_n$ and set $\tau' = \sigma\tau\sigma$. Then $\tau' \neq \tau$, and it follows that

$$\prod_{i=1}^n c_{i\tau(i)} = \prod_{i=1}^n c_{i\tau'(i)}.$$

Thus $\text{Det}(C_x) \equiv \text{Det}(C_0)\text{Det}(C_1) \pmod{2}$. This proves the Lemma. \square

Lemma 6.5. *For each $j = 2n_1 + 1, \dots, n$, there exists $i \in \{1, 2, \dots, m\}$ such that $\hat{\chi}_i^* = \hat{\chi}_i$ and the decomposition number $d_{ij} = d(\hat{\chi}_i|_{\mathcal{B}_{p'}}, \hat{\phi}_j)$ is odd.*

Proof. Let $m = 2m_1 + m_2$, where the notation is chosen so that $\{\hat{\chi}_1, \hat{\chi}_2\}, \{\hat{\chi}_3, \hat{\chi}_4\}, \dots, \{\hat{\chi}_{2m_1-1}, \hat{\chi}_{2m_1}\}$, are pairs of non self-dual characters, that is, $(\hat{\chi}_{2i-1})^* = \hat{\chi}_{2i}$, and $\hat{\chi}_{2m_1+1}, \dots, \hat{\chi}_m$ are self-dual.

Suppose $d_{ij} \equiv 0 \pmod{2}$, for all $i = 2m_1 + 1, \dots, m$. Then for each $k \in \{1, 2, \dots, n\}$, we have

$$c_{jk} = \sum_{i=1}^m d_{ij}d_{ik} \equiv \sum_{i=1}^{2m_1} d_{ij}d_{ik}.$$

On the other hand, $\phi_j^* = \phi_j$ and if $k \in \{2n_1 + 1, \dots, n\}$ then $\phi_k^* = \phi_k$ and so

$$d_{ij} = d_{i+1,k}, \quad d_{ik} = d_{i+1,k}, \quad i = 1, 3, \dots, 2m_1 - 1,$$

hence $c_{jk} \equiv 0 \pmod{2}$, for all such k . This means that some row of C_1 consists of even entries. This violates the previous lemma. \square

As in [GM, Theorem 4.4], we have the following consequence:

Corollary 6.6. *Consider the bismash products as above.*

- (1) *If all irreducible $H_{\mathbb{C}}$ -modules have indicator +1, the same is true for all irreducible $H_{\mathbb{k}}$ -modules.*
- (2) *If all irreducible $H_{\mathbb{C}}$ -modules have indicator 0 or 1, the same is true for all irreducible $H_{\mathbb{k}}$ -modules.*
- (3) *If all irreducible $H_{\mathbb{C}}$ -modules are self dual, the same is true for all irreducible $H_{\mathbb{k}}$ -modules.*

Proof. (3) This follows by Lemma 4.8.

Now consider (2). By Theorem 6.3 there are no irreducible kG -modules V with $\nu(V) = -1$. So (2) follows immediately.

Now (1) follows by (2) and (3). □

7. APPLICATIONS TO THE SYMMETRIC GROUP

In this section we apply the results of Section 6 to bismash products constructed from some specific groups.

Let \mathcal{S}_n be the symmetric group of degree n , consider $\mathcal{S}_{n-1} \subset \mathcal{S}_n$ by letting any $\sigma \in \mathcal{S}_{n-1}$ fix n , and let $C_n = \langle z \rangle$, the cyclic subgroup of \mathcal{S}_n generated by the n -cycle $z = (1, 2, \dots, n)$. Then $\mathcal{S}_n = \mathcal{S}_{n-1}C_n = C_n\mathcal{S}_{n-1}$ shows that $Q = \mathcal{S}_n$ is factorizable. Thus we may construct the bismash product $H_{n,\mathbb{E}} := \mathbb{E}^{C_n} \# \mathbb{E}^{\mathcal{S}_{n-1}}$. It was shown in [JM] that if \mathbb{E} is algebraically closed of characteristic 0, then H_n is totally orthogonal; that is, every irreducible module has indicator +1.

Corollary 7.1. *Let \mathbb{k} be algebraically closed of characteristic $p > 0$ and let $H_{n,\mathbb{k}} := \mathbb{k}^{C_n} \# \mathbb{k}^{\mathcal{S}_{n-1}}$. Then $H_{n,\mathbb{k}}$ is totally orthogonal.*

Proof. Apply Corollary 6.6 to the characteristic 0 result of [JM] mentioned above. □

Remark 7.2. In [GM] it is proved that $D(G)$ is totally orthogonal for any finite real reflection group G over any algebraically closed field. Corollary 6.6 shows that this result in characteristic $p > 0$ follows from the case of characteristic 0, which is somewhat easier to prove. When $G = \mathcal{S}_n$, the characteristic 0 case was shown in [KMM].

We close with a question.

Question 7.3. It would be interesting to know if our results could be extended to bicrossed products. However to extend our proof one would need a theory of Brauer characters for twisted group algebras (that is, for projective representations) which includes a version of Brauer's theorem on the Cartan matrix.

Acknowledgement: The authors would like to thank I. M. Isaacs, S. Witherspoon, and especially R. M. Guralnick, for helpful comments.

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