Pointed topf actions on commutative domains

A study in NCIT:

Classic IT: \( G \) finite group

\( R = \mathbb{k}[x_1, \ldots, x_n] \) & study \( R^G, R \times G \)

\( H \) finite dual Hopf algebra

\( A \) quantumpolyalgebra

Basic Question: when does such a Hopf action exist?

Lots of choices for \( A \)

- Commutative
- Graded
- Domain
- (PI) (not PI)
- Associative (not associative)
- Edited (a quantization/ deformation of \( R \))
- Any algebra that looks like \( R \)

NCIT (talk)

\( H \) induced action of \( H \) on \( \mathbb{k} \)?

\( I^\alpha \) Hopf ideal of \( H \)

\( A^\alpha, A^\alpha \), etc.

\( \alpha \) results add new that a Hopf exists
Traditionally two choices for $H$

- Semisimple (as an $k$-alg)
- Pointed (all simple $H$-comodules are trivial)
  $\iff H^*$ is basic

This story is much more complicated

Theorem (Petrowsky, 2014)

If $H$ is a semisimple Hopf algebra that acts inner faithfully on a commutative domain, then $H$ is co-commutative (i.e., $H = kG$, $G$ finite group).

Program led by Andruskiewitsch & Schneider to classify all finite Hopf algebras $H$.

For $H$, $\Delta(x) \equiv x \otimes x + x \otimes y, y \otimes y$ (in $k(H)$)

Conj [AJS] Such $H$ are generated by simple-like $G$-graded primitive algebras.

Let $g \in H \implies \Delta(g) = g \otimes g$

$G(H)$

Theorem [Angiono, 2013] Conjecture is true for $H$ if $G(H)$ abelian.

We assume this.

All such $H$ are lifts of $B(V) \# kG$ ($\text{gr } H = B(V) \# kG$).

Nichols algebra is not a Hopf algebra but a braided Hopf algebra in $\mathcal{YD}$.

G-graded $G$-comod.
B(V) is hard to understand / present

\[ \langle x_i, -x_i \rangle \]

\[ A(x_i) = g_i \otimes x_i + x_i \otimes 1 \]

hard to understand

Restrict again to \( H \) of finite Cartan type in all variations of small quantum groups. \( U_q(g) \)

Here, \( g_i x_j = q_{ij} x_j g_i \) with \( A_{ij} = q_{ij} \) for \( (a_{ij}) \) a Cartan matrix of a Dynkin

\[ B_{ij} q_{ii} = q_{ij} \]

Clause form small of Dynkin affine Cartan

Example \( H = U_q(sl_2) \) given by \( k, \ e \)

\[ \begin{align*}
   c = & k, & e = & 1 \\
   \Delta(e) = & q^2 e + e q^{-2} & \Lambda(1) = & q \end{align*} \]

\[ \begin{align*}
   k e = & q^2 e f & k f = & q^{-2} f k & e f - f e = & \frac{k - k^{-1}}{q - q^{-1}} \\
   k^m = & 1 & e^m = & f^m = 0 \\
   q^{12} q^{11} = & q^{10} & q^{21} q^{22} = & q^{20} \\
   q_{ij} q_{ji} = & q_{ji} \]

\[ U_q(sl_2) \text{ is q-type } A_1 \times A_1 \]

Pointed Dynkin \( sl_2 \) and fundamental \( q \)-comatrix in \( \mathfrak{a} \)

Boilo down to actions on fields.

Lemma A commutative domain, \( Q \) as quotient field. If \( H \rightrightarrows \mathfrak{a} \) inner faithfully, then \( H \rightrightarrows Q \) more faithfully.
Definition: A Hopf algebra \( H/ \mathbb{K} \) is Galois-theoretical if \( H \) acts
faithfully and \( \mathbb{K} \)-linearly on a field containing \( \mathbb{K} \).

Proposition:
1. Any finite group algebra is GT.
2. If semisimple \( H \Rightarrow H \cong kH \).
3. GT preserved under taking:
   - Hopf subalgebra
   - Hopf dual
   - Crossed product

Theorem: If \( H \) is a finite pointed GT Hopf alg w/ \( H \)-module (field \( L \)). Then
1. \( L_H = L_{G(H)} \) (new primitive alg of \( H \) act by group \( L_{G(H)} \))
2. \( L_H \subseteq L \) is Galois alg Galois group \( G(H) \).

A generalization is for actions on commutative domains
- on Azumaya algebra.

<table>
<thead>
<tr>
<th>( H ) finite, a.m.</th>
<th>on commutative domain ( A )</th>
<th>on Azumaya algebra ( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H ) finite a.m.</td>
<td>( A^H = A^{Ho} )</td>
<td>( Z \cap A^H = Z \cap A^{Ho} ) (( Z^H = Z^G ))</td>
</tr>
<tr>
<td>( H ) finite a.m. &amp; pointed ( (Ho = 12e) ),</td>
<td>( A^H = A^G )</td>
<td>( Z^H = Z^G )</td>
</tr>
</tbody>
</table>
Theorem

1. The following are examples of finitely pointed GT Hopf algebras of finite Cartan type.

<table>
<thead>
<tr>
<th>GT Hopf algebra</th>
<th>finite Cartan type</th>
</tr>
</thead>
<tbody>
<tr>
<td>T(n) (Taft alg)</td>
<td>A_1</td>
</tr>
<tr>
<td>E(2n) (dim 2^{n+1})</td>
<td>A_1 x n</td>
</tr>
<tr>
<td>H_3(3,1) (book alg)</td>
<td>A_1 x A_1</td>
</tr>
<tr>
<td>A_1 (61-dim alg)</td>
<td>A_2</td>
</tr>
<tr>
<td>U_q(sl_2)</td>
<td>A_1 x A_1</td>
</tr>
<tr>
<td>U_q(su_2)</td>
<td>A_1 x A_1</td>
</tr>
<tr>
<td>Drinfeld ^2 \U_q(su_2)</td>
<td>A_{n-1} x A_{n-1}</td>
</tr>
<tr>
<td>\text{trace of } U_q(su_2)</td>
<td>\text{same type as } g</td>
</tr>
</tbody>
</table>

2. The following are non-examples of finitely pointed GT Hopf algebras of finite Cartan type.

<table>
<thead>
<tr>
<th>Non-GT Hopf alg</th>
<th>finite Cartan type</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{generalized Taft algebras}</td>
<td>A_1</td>
</tr>
<tr>
<td>T(n,m,n,a) \neq T(n)</td>
<td>A_1 x A_1</td>
</tr>
<tr>
<td>H_3(p,p) for p \neq 1</td>
<td>A_1 x A_1</td>
</tr>
<tr>
<td>\text{trace of } U_q(sl_2)</td>
<td>A_1 x A_1</td>
</tr>
</tbody>
</table>

Preview of Part II of paper: Therefore essentially it in some sense...
Some questions to consider (in the Haebara setting):

1. Quasiclassical analogue of $H \xrightarrow{\text{finite-dim. commutative}} A$

$G = \text{algebraic group w/ Poisson bracket (not necessarily zero)}.$

$X = \text{irreducible algebraic variety w/ zero Poisson bracket.}$

Determine which $G$ can act on $X$ and classify all such $G \times X$.

*This includes the problem of classifying 'Poisson homogeneous spaces' $X = G/\sigma$, ($G'$ is closed subgroup of $G$).

2. (Alexandru Chirvadin < postdoc at UW next year, UC Berkeley student)

Compact Quantum Groups (CQG)

Conjecture (Inos - Goswami - Joural): Smooth

If $H$ CQG acts faithfully on a connected compact manifold, then $H$ a classical compact group.