Quillen's Lemma for affinoid enveloping algebras

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1 Introduction

We fix some notation.

- R is a discrete valuation ring, with maximal ideal πR,
- $k = R/\pi R$ is the residue field of R, and
- $K = Q(R)$ is the field of fractions of R.

It may be helpful to keep the examples $R = \mathbb{Z}_p$ and $R = \mathbb{C}[[t]]$ in mind.

Definition 1.1. 1. Let A be a K-algebra, with a $\mathbb{Z}$-filtration $F_\bullet A$. We say that A is a *sliced K-algebra* if

- $F_\bullet A$ is complete,
- $R \subseteq F_0 A$, and
- $F_n A = \pi^{-n} F_0 A$ for all $n \in \mathbb{Z}$.

We call the k-algebra $\operatorname{gr}_0 A := F_0 A / \pi F_0 A$ the *slice* of A.

2. Let B be a k-algebra, with an $\mathbb{N}$-filtration $F_\bullet B$. We say that B is an *almost commutative k-algebra* if

- $k \subseteq F_0 B$, and
- $\operatorname{gr} B$ is a finitely generated commutative k-algebra.

3. An *almost commutative affinoid K-algebra* is a sliced K-algebra with an almost commutative slice. We write $\operatorname{Gr}(A) := \operatorname{gr}(\operatorname{gr}_0 A)$.

Remarks 1.2.

- The filtration on a sliced K-algebra is completely determined by the "unit ball" $F_0 A$. When $R = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$, sliced K-algebras are examples of p-adic Banach algebras.
- The associated graded ring of a sliced K-algebra is always of the form $\operatorname{gr} A = (\operatorname{gr}_0 A)[s, s^{-1}]$ where s is the principal symbol of $\pi \in F_0 A$. It is therefore completely determined by the slice $\operatorname{gr}_0 A$.

- To define an almost commutative affinoid K-algebra, one needs extra data of an $\mathbb{N}$-filtration on the slice $\operatorname{gr}_0 A$ of a sliced K-algebra A.

Examples 1.3. 1. Let $\mathfrak{g}$ be an R-Lie algebra which is free of finite rank as an R-module. Let

$$F_0 A := \widehat{\mathcal{U}(\mathfrak{g})} := \varprojlim \frac{\mathcal{U}(\mathfrak{g})}{\pi^n \mathcal{U}(\mathfrak{g})}$$

be the π-adic completion of the R-enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. Then

$$A := \widehat{\mathcal{U}(\mathfrak{g})_K} := F_0 A \left[\frac{1}{\pi}\right]$$

becomes a sliced K-algebra with slice

$$\operatorname{gr}_0 A = \frac{\mathcal{U}(\mathfrak{g})}{\pi \mathcal{U}(\mathfrak{g})} \cong \mathcal{U}(\mathfrak{g}_k),$$

the enveloping algebra of the $\bmod \pi$ reduction $\mathfrak{g}_k := \mathfrak{g}/\pi\mathfrak{g}$ of $\mathfrak{g}$. Since this last enveloping algebra is well-known to be almost commutative, we see that $\widehat{\mathcal{U}(\mathfrak{g})_K}$ is an almost commutative affinoid K-algebra. We call it an *affinoid enveloping algebra.*

2. Let $A_m(R) = R[x_1, \ldots, x_m; \partial_1, \ldots, \partial_m]$ be the m^{th} Weyl algebra over R; thus

$$A_m(R) = \frac{R\langle x_1, \ldots, x_m, y_1, \ldots, y_m\rangle}{\langle [x_i, x_j] = 0, \quad [y_i, y_j] = 0, \quad [y_i, x_j] = \delta_{ij}\rangle}.$$

We form $\widehat{A_m(R)_K}$ in the same way; thus

$$\widehat{A_m(R)} = \varprojlim \frac{A_m(R)}{\pi^n A_m(R)}$$

is the π-adic completion of $A_m(R)$ and

$$\widehat{A_m(R)_K} = \widehat{A_m(R)} \left[\frac{1}{\pi}\right]$$

is again an almost commutative affinoid K-algebra. We call it the m^{th}-*affinoid Weyl algebra.*

Lemma 1.4. *Let A be an almost commutative affinoid K-algebra. Then*

1. *A is Noetherian.*

2. *Every quotient of A is again almost commutative affinoid.*

3. *If $\operatorname{Gr}(A)$ is a domain/has finite global dimension/is Auslander-regular/ $\cdots$, then A also has the corresponding property.*

2 Motivation

- Affinoid enveloping algebras $\widehat{\mathcal{U}(\mathfrak{g})_K}$ arise as particular microlocalisations of Iwasawa algebras.
- Affinoid Weyl algebras $\widehat{A_m(R)_K}$ arise in Berthelot's theory of *arithmetic differential operators.*
- The slice of the affinoid Weyl algebra is just the Weyl algebra $A_m(k)$ over the residue field k of R. In particular, if $k = \overline{\mathbb{F}_p}$ is the algebraic closure of $\mathbb{F}_p$ then we can view this slice as the algebra of crystalline differential operators on the affine m-space $\mathbb{A}_k^m$.
- Crystalline differential operators were used by Bezrukavnikov, Mirkovic and Rumynin to study representations of Lie algebras in prime characteristic.
- Soibelman also constructed examples of almost commutative affinoid algebras by π-adically completing quantized function algebras and quantized enveloping algebras.

However, the main motivation comes from *rigid analytic geometry.*

3 Rigid analytic geometry

Let $\mathfrak{g} = Rx$, the one-dimensional Lie algebra over R. Then it is easy to see that

$$K\langle x\rangle := \widehat{\mathcal{U}(\mathfrak{g})_K} = \left\{ \sum_{a=0}^{\infty} \lambda_a x^a \in K[[x]] : \lambda_a \to 0 \quad \text{as} \quad a \to \infty \right\}$$

is an algebra. It is known as the *Tate algebra.* Note that every $f \in K\langle x\rangle$ can be evaluated at any point ξ in the unit ball

$$o_{\overline{K}} := \{\xi \in \overline{K} : |\xi| \leqslant 1\}.$$

Fact 3.1. Let $G_K := \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K. Evaluation at $\xi \in o_{\overline{K}}$ induces a bijection between the G_K-orbits on $o_{\overline{K}}$, and the maximal ideals in $K\langle x\rangle$:

$$o_{\overline{K}}/G_K \xrightarrow{\cong} \operatorname{MaxSpec} K\langle x\rangle.$$

Here are the classical definitions in the commutative theory.

Definition 3.2.

1. An *affinoid algebra* is a quotient of some *Tate algebra*

   $$K\langle x_1, \dots, x_n\rangle := \widehat{\mathcal{U}(\mathfrak{a})_K}$$

   for some abelian R-Lie algebra $\mathfrak{a} = Rx_1 \oplus \cdots \oplus Rx_n$.
2. A *rigid analytic variety* is $\mathrm{MaxSpec}(A)$ for some affinoid K-algebra A.

Examples 3.3. 1. The set of G_K-orbits on the n-dimensional polydisc $o_{\overline{K}}^n$ arises as the maximal ideal spectrum of the n^{th}-Tate algebra $K\langle x_1, \dots, x_n\rangle$.

2. The descending chain of Tate algebras

$$K\langle x\rangle \supset K\langle \pi x\rangle \supset K\langle \pi^2 x\rangle \supset \cdots$$

corresponds to a cover of the full π-adic line $\overline{K}$ by closed balls of ever-increasing radius:

$$o_{\overline{K}} \subset \frac{1}{\pi} o_{\overline{K}} \subset \frac{1}{\pi^2} o_{\overline{K}} \subset \cdots$$

3. The "unit circle" $\{\xi \in \overline{K} : |\xi| = 1\}/G_K$ is a rigid analytic variety, since

$$\{\xi \in \overline{K} : |\xi| = 1\}/G_K = \operatorname{MaxSpec} K\langle x, x^{-1}\rangle = \operatorname{MaxSpec} \frac{K\langle x, y\rangle}{\langle xy - 1\rangle}.$$

It is *open* in the usual π-adic topology by the strict triangle inequality $|a + b| \leqslant \max |a|, |b|$ which is always valid in the p-adic world.

Remarks 3.4. 1. Because the p-adic topology is totally disconnected, the naive definition of analytic functions as those that locally have a power series expansion does not lead to a satisfying theory. Instead, John Tate defined in 1962 a very weak topology on p-adic spaces such as $\overline{K}/G_K$ (actually, a Grothendieck topology) — and also a sheaf $\mathcal{O}$ of *rigid analytic functions* on these p-adic spaces, by using affinoid algebras. For example, $\mathcal{O}(o_{\overline{K}}/G_K) = K\langle x\rangle$ and more generally, $\mathcal{O}(\operatorname{MaxSpec}(A)) = A$ for an affinoid algebra A.

2. An excellent introduction to this subject can be found in an expository paper by Peter Schneider, available here:

`http://www.math.uni-muenster.de/u/pschnei/publ/pap/rigid.ps`

3. Now return to our original setting, where $\mathfrak{g}_R$ was an arbitrary R-Lie algebra, free of rank n as an R-module. If $x_1, \dots, x_n$ is a basis for $\mathfrak{g}$ over R, then as a K-vector space we can still view the affinoid enveloping algebra $\widehat{\mathcal{U}(\mathfrak{g})}_K$ as an algebra of certain power series, this time in the *non-commuting* variables $x_1, \dots, x_n$:

$$\widehat{\mathcal{U}(\mathfrak{g})}_K = \left\{ \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha x^\alpha : \lambda_\alpha \to 0 \quad \text{as} \quad |\alpha| \to \infty \right\}.$$

It is tempting to therefore think of $\widehat{\mathcal{U}(\mathfrak{g})}_K$ as a *rigid analytic quantization* of the polydisc $o_{\overline{K}}^n/G_K$.

4. More generally, the descending chain of affinoid enveloping algebras

$$\widehat{\mathcal{U}(\mathfrak{g})_K} \supset \widehat{\mathcal{U}(\pi\mathfrak{g})_K} \supset \widehat{\mathcal{U}(\pi^2\mathfrak{g})_K} \supset \cdots$$

should be viewed as a rigid analytic quantization of the closed balls

$$o_K^n \subset \frac{1}{\pi} o_K^n \subset \frac{1}{\pi^2} o_K^n \subset \cdots$$

of course, up to the action of the Galois group G_K.

5. We are currently trying to "quantize" rigid analytic *symplectic* spaces, such as cotangent bundles of smooth rigid analytic varieties. To do this, we plan to use the affinoid Weyl algebra and its various deformations.

4 Simple modules

Recall "Quillen's Lemma", which is really a Theorem!

Theorem 4.1 (Quillen, 1969)**.** *Let M be a simple module over an almost commutative k-algebra A and let $\varphi : M \to M$ be an A-linear endomorphism. Then φ is algebraic over k.*

Corollary 4.2. *Every simple $\mathcal{U}(\mathfrak{g}_k)$-module has a central character.*

It is natural to try to prove a direct analogue of Quillen's Lemma in the affinoid world. Thus we make the following

Conjecture 4.3. *Let M be a simple module over an almost commutative affinoid K-algebra A and let $\varphi : M \to M$ be an A-linear endomorphism. Then φ is algebraic over K.*

Here is our main result, which has already been improved during the Workshop! Thanks are due to Michel Van den Bergh and Lance Small for several very helpful remarks.

Theorem 4.4. *Suppose that $\mathrm{Gr}(A)$ is Gorenstein. Then the conjecture holds.*

To explain some of the ideas involved in the proof of this result, let us begin by recalling Quillen's original argument.

Proof of Theorem 4.1. Give M some good filtration over $A[\varphi]$, and view $\operatorname{gr} M$ as a finitely generated $(\operatorname{gr} A)[\varphi]$-module. By the Generic Flatness Lemma, we may find some non-zero element $f \in k[\varphi]$ such that $(\operatorname{gr} M)_f = \operatorname{gr}(M_f)$ is free as a module over $k[\varphi]_f$. It follows that M_f is also free over $k[\varphi]_f$. But $k[\varphi]_f$ acts invertibly on M by Schur's Lemma so $k[\varphi]_f \cong k[\varphi, t]/\langle tf - 1\rangle$ has to be a field. By the Nullstellensatz, this can only happen if φ is algebraic over k. ☐

Definition 4.5. Let A, M be as in Conjecture 4.2.

1. An F_0A-*lattice* in M is a finitely generated F_0A-submodule N of M such that $M = NK$.

2. For any F_0A-lattice N in M, let

$$B_N := \{b \in K(\varphi) : bN \subseteq N\}$$

be its normalizer in $K(\varphi)$.

Remarks 4.6.
- B_N is always an order in the commutative field $K(\varphi)$, and $B_N/\pi B_N$ acts faithfully on $N/\pi N$.

- *If* we're lucky and $B_N/\pi B_N$ happens to be a field, then we can run Quillen's proof "on the slice" to deduce that $B_N/\pi B_N$ is algebraic over k.

- Because $\operatorname{gr}_0 A$ is strongly Noetherian, it follows that $B_N/\pi B_N$ is finite dimensional over k.

- This is enough to deduce that $K(\varphi)$ is finite dimensional over K.

We won't be so lucky in general. So we hope to improve matters by changing the lattice N.

Definition 4.7. We say that N is a *regular* φ*-lattice* if B_N is a discrete valuation ring.

This is a slightly weaker condition than what we had considered before, but in fact it suffices: the same proof shows that if M has a regular φ-lattice, then φ must be algebraic over K.

5 Finding a regular φ-lattice

Here is our strategy:

- Microlocalise M to make $\operatorname{gr}_0 M$ finite length.

- Use this condition to find a microlocal regular φ-lattice.

- Use dimension theory over Auslander-Gorenstein rings to lift this microlocal regular φ-lattice back to M.

Let us assume for simplicity that $\operatorname{gr}_0 A$ is commutative. After microlocalising, we may assume that Q is a sliced K-algebra with commutative Noetherian semilocal slice and that V is a simple Q-module with slice $\operatorname{gr}_0 V$ of finite length. By the functoriality of microlocalisaton, V is still a $K(\varphi)$-module.

- Fix some F_0Q-lattice L_0 in V.

- Let $\mathcal{L} = \{F_0Q\text{-lattices } L \subseteq V : L \subseteq L_0 \text{ but } L \nsubseteq \pi L_0\}$. We search for a microlocal regular φ-lattice in $\mathcal{L}$.

- Let $\mathcal{P} = \{B_L : L \in \mathcal{L}\}$, defined in a similar way.

Here are the main steps of our proof:

Theorem 5.1. *$\mathcal{P}$ has a maximal element.*

Theorem 5.2. *If $B \in \mathcal{P}$ is maximal, then it must be a discrete valuation ring.*

So the corresponding F_0Q-lattice in V is a regular φ-lattice.

Sketch proof of Theorem 5.1. Use Zorn's Lemma. Let $(B_\alpha)_{\alpha\in\mathcal{A}}$ be a chain in $\mathcal{P}$ and for each $\alpha \in \mathcal{A}$ let $L_\alpha \in \mathcal{L}$ be the largest $B_\alpha - F_0Q$ subbimodule of L_0. Then

$$\alpha \leqslant \beta \quad \Rightarrow \quad B_\alpha \subseteq B_\beta \quad \Rightarrow \quad B_\alpha L_\beta \subseteq L_\beta \quad \Rightarrow \quad L_\beta \subseteq L_\alpha,$$

so $(L_\alpha)_{\alpha\in\mathcal{A}}$ is a descending chain.

- Since $\operatorname{gr}_0 V$ has finite length and F_0Q is π-adically complete,

  $$L_\infty := \bigcap_{\alpha\in\mathcal{A}} L_\alpha$$

  is again a lattice in $\mathcal{L}$.

Therefore $B_\infty := B_{L_\infty} \in \mathcal{P}$. We claim that B_∞ is an upper bound for the chain (B_α) in $\mathcal{P}$. To see this, let $\alpha \in \mathcal{A}$; then

$$\beta \geqslant \alpha \quad \Rightarrow \quad B_\alpha L_\infty \subseteq B_\alpha L_\beta \subseteq B_\beta L_\beta \subseteq L_\beta$$

so

$$B_\alpha L_\infty \subseteq \bigcap_{\beta\geqslant\alpha} L_\beta = L_\infty \quad \text{for all} \quad \alpha \in \mathcal{A},$$

whence $\bigcup_{\alpha\in\mathcal{A}} B_\alpha \subseteq B_\infty$. $\square$

More details can be found in Section 8 of our preprint, available here:

`http://arxiv.org/abs/1102.2606`