Invariants of AS-Regular Algebras: Complete Intersections
Preliminary Report

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Group Actions on $\mathbb{C}[x_1, \cdots, x_n]$

Let $G$ be a finite group of $n \times n$ matrices acting on $\mathbb{C}[x_1, \cdots, x_n]$

$$g = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$g \cdot x_j = \sum_{i=1}^{n} a_{ij} x_i$$

Extend to an automorphism of $\mathbb{C}[x_1, \cdots, x_n]$. 
When is $\mathbb{C}[x_1, x_2, \ldots, x_n]^G$:

- **A polynomial ring?**
  Shephard-Todd-Chevalley Theorem (1954): if and only if $G$ is generated by reflections (all eigenvalues except one are 1)

- **A Gorenstein ring?**
  Watanabe’s Theorem (1974): if $G \subseteq SL_n(\mathbb{C})$
  Stanley’s Theorem (1978): iff $H_{A^G}(t^{-1}) = \pm t^m H_{A^G}(t)$.

- **A complete intersection?**
  Groups classified by Nakajima (1984), Gordeev (1986)
Noncommutative Generalizations

Replace commutative polynomial ring with AS-regular algebra over \( \mathbb{C} \).

Let \( G \) be a finite group of graded automorphisms of \( A \).

Replace commutative Gorenstein ring with AS-Gorenstein algebra.

Replace reflection by quasi-reflection

\[
Tr_A(g, t) = \sum_{k=0}^{\infty} \text{trace}(g | A_k) t^k
\]

\[
= \frac{p(t)}{(1 - t)^{n-1} q(t)}
\]

for \( q(1) \neq 0 \) and \( n = \text{GKdim} A \).
Replace determinant by homological determinant

P. Jørgensen- J. Zhang:

When $A$ is AS-regular of dimension $n$, then when the trace is written as a Laurent series in $t^{-1}$

$$Tr_A(g, t) = (-1)^n(hdet g)^{-1}t^{-\ell} + \text{lower terms}$$
Conjectures: (Proven in some cases):

Shephard-Todd-Chevalley Theorem:
\( A^G \) is AS-regular if and only if \( G \) is generated by quasi-reflections.

Watanabe’s Theorem: \( A^G \) is AS-Gorenstein when all elements of \( G \) have homological determinant 1.

Stanley’s Theorem: \( A^G \) is AS-Gorenstein if and only if 
\[
(H_{A^G}(t^{-1}) = \pm t^m H_{A^G}(t)).
\]
A^G a complete intersection:

Theorem: (Kac and Watanabe – Gordeev) (1982). If \( \mathbb{C}[x_1, \ldots, x_n]^G \) is a complete intersection then \( G \) is generated by bi-reflections (all but two eigenvalues are 1).

For an AS-regular algebra \( A \) a graded automorphism \( g \) is a quasi-bi-reflection of \( A \) if

\[
\text{Tr}_A(g, t) = \sum_{k=0}^{\infty} \text{trace}(g|A_k)t^k
\]

\[
= \frac{p(t)}{(1 - t)^{n-2}q(t)},
\]

\( n = \text{GKdim} \ A \), and \( q(1) \neq 0 \).
Example:

$A^G$ a complete intersection

$A = \mathbb{C}_{-1}[x, y, z]$ is regular of dimension 3, and

$$g = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

acts on it. The eigenvalues of $g$ are $-1, i, -i$ so $g$ is not a bi-reflection of $A_1$. However,

$Tr_A(g, t) = 1/((1 + t)^2(1 - t)) = -1/t^3 + \text{lower degree terms}$

and $g$ is a quasi-bi-reflection with $h\text{det } g = 1$.

$$A^g \cong \frac{k[X, Y, Z, W]}{\langle W^2 - (X^2 + 4Y^2)Z \rangle},$$

a commutative complete intersection.
Commutative Complete Intersections

Let $A$ be a connected graded noetherian commutative algebra. Then the following are equivalent.

1. $A$ is isomorphic to $k[x_1, x_2, \ldots, x_n]/(d_1, \ldots, d_m)$ for a homogeneous regular sequence.

2. The Ext-algebra $\text{Ext}^*_A(k, k)$ is noetherian.

3. The Ext-algebra $\text{Ext}^*_A(k, k)$ has finite GK-dimension.
Noncommutative Complete Intersections

Let $A$ be a connected graded noetherian algebra.

1. We say $A$ is a \textit{classical complete intersection ring} if there is a connected graded noetherian AS regular algebra $R$ and a regular sequence of homogeneous elements $\{d_1, \cdots, d_n\}$ of positive degree such that $A$ is isomorphic to $R/(d_1, \cdots, d_n)$.

2. We say $A$ is a \textit{complete intersection ring of type NP} if the Ext-algebra $\Ext^*_A (k, k)$ is noetherian.

3. We say $A$ is a \textit{complete intersection ring of type GK} if the Ext-algebra $\Ext^*_A (k, k)$ has finite Gelfand-Kirillov dimension.

4. We say $A$ is a \textit{weak complete intersection ring} if the Ext-algebra $\Ext^*_A (k, k)$ has subexponential growth.
Noncommutative case:

classical complete intersection ring $\Rightarrow$
complete intersection ring of type GK

complete intersection ring of type NP (GK) $\Rightarrow$
weak complete intersection ring

complete intersection ring of type GK $\not\Rightarrow$
complete intersection ring of type NP

Example: $A = k\langle x, y \rangle/(x^2, xy, y^2)$ is a Koszul algebra with Ext-algebra $E := k\langle x, y \rangle/(yx)$; GKdim $E = 2$ but $E$ is not noetherian.
Examples of noncommutative complete intersections of type NP (GK) include noetherian Koszul algebras that have Ext-algebras that are Noetherian (finite GK) for example

\[ A = \frac{\mathbb{C}^{-1}[x, y]}{\langle x^2 - y^2 \rangle} \text{ with } \text{Ext}_A^*(k, k) = A! = \frac{\mathbb{C}[x, y]}{\langle x^2 + y^2 \rangle} \]

or

\[ A = \frac{\mathbb{C}[x, y]}{\langle x^2, y^2 \rangle} \text{ with } \text{Ext}_A^*(k, k) = A! = \frac{\mathbb{C}[x, y]}{\langle xy \rangle}; \]

in second case

\[ A \cong \frac{B}{\langle x^2, y^2 \rangle} \]

where \( B \) is the AS-regular algebra generated by \( x, y \) with \( yx^2 = x^2y \) and \( y^2x = xy^2 \).
Let $A$ be a connected graded Noetherian ring. We say $A$ is cyclotomic Gorenstein if the following conditions hold:

(i) $A$ is AS-Gorenstein;

(ii) $H_A(t)$, the Hilbert series of $A$, is a rational function $p(t)/q(t)$ for some relatively prime polynomials $p(t), q(t) \in \mathbb{Z}[t]$ where all roots of $p(t)$ are roots of unity.

Suppose that $A$ is isomorphic to $R^G$ for some Auslander regular algebra $R$ and a finite group $G \subseteq \text{Aut}(R)$. If $\text{Ext}_A^*(k, k)$ has subexponential growth, then $A$ is cyclotomic Gorenstein.

Hence if $A$ not cyclotomic Gorenstein, then $A$ is not a complete intersection of any type.
Veronese Subrings

For a graded algebra $A$ the $r$th Veronese $A^{(r)}$ is the subring generated by all monomials of degree $r$.

If $A$ is AS-Gorenstein of dimension $d$, then $A^{(r)}$ is AS-Gorenstein if and only if $r$ divides $\ell$ where $\text{Ext}_A^d(k, A) = k(\ell)$ (Jørgensen-Zhang).

Let $g = \text{diag}(\lambda, \cdots, \lambda)$ for $\lambda$ a primitive $r$th root of unity; $G = \langle g \rangle$ acts on $A$ with $A^{(r)} = A^G$.

If the Hilbert series of $A$ is $(1 - t)^{-d}$ then

$$\text{Tr}_A(g^i, t) = \frac{1}{(1 - \lambda^i t)^d}.$$ 

For $d \geq 3$ the group $G = \langle g \rangle$ contains no quasi-bi-reflections, so $A^G = A^{(r)}$ should not be a complete intersection.
Theorem:
Let $A$ be noetherian connected graded algebra.

1. Suppose the Hilbert series of $A$ is $(1 - t)^{-d}$. If $r \geq 3$ or $d \geq 3$, then $H_{A^{(r)}}(t)$ is not cyclotomic. Consequently, $A^{(r)}$ is not a complete intersection of any type.

2. Suppose $A$ is a quantum polynomial ring of dimension 2 (and $H_A(t) = (1 - t)^{-2}$). If $r = 2$, then $H_{A^{(r)}}(t)$ is cyclotomic and $A^{(r)}$ is a classical complete intersection.
Permutation Actions on

$$A = \mathbb{C}_{-1}[x_1, \cdots, x_n]$$

If $g$ is a 2-cycle then

$$Tr_A(g) = \frac{1}{(1 + t^2)(1 - t)^{n-2}}$$

$$= (-1)^n \frac{1}{t^n} + \text{lower terms}$$

so $\text{hdet } g = 1$, and all $A^G$ are AS-Gorenstein. Further a
permutation matrix $g$ is a quasi-bi-reflection if and only if it is
a 2-cycle or a 3-cycle.

Both $A^{S_n}$ and $A^{A_n}$ are classical complete intersections.
Example:

\[ g = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix} \]

Then \( A(g) \) has Hilbert series

\[
\frac{1 - 2t + 4t^2 - 2t^3 + t^4}{(1 + t^2)^2(1 - t)^4}
\]

whose numerator is not a product of cyclotomic polynomials, so \( A(g) \) is not any of our types of complete intersection.
Examples in Dimension 3:

Consider AS-Gorenstein fixed rings of AS-regular algebras of dimension 3 (e.g. 3-dimensional Sklyanin, down-up algebras).

Thus far all our examples are either classical complete intersections or not cyclotomic (so none of our types of complete intersection).

In all the cases where $A^G$ is a complete intersection, $G$ is generated by quasi-bi-reflections.
Down-up algebra examples

Let $A$ be generated by $x, y$ with relations

$$y^2 x = xy^2 \text{ and } yx^2 = x^2 y.$$ 

Represent the automorphism $g(x) = ax + cy$ and $g(y) = bx + dy$ by the $2 \times 2$ matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$ 

Any invertible matrix induces a graded automorphism of $A$. The homological determinant of a graded automorphism $g$ with eigenvalues $\lambda_1$ and $\lambda_2$ is $(\lambda_1 \lambda_2)^2$. $A^G$ is AS-Gorenstein if and only if the $hdet(g) = (\lambda_1 \lambda_2)^2 = 1$ for all $g \in G$. 
quasi-bi-reflections

The trace of a graded automorphism \( g \) of \( A \) with eigenvalues \( \lambda_1 \) and \( \lambda_2 \) is

\[
Tr_A(g, t) = \frac{1}{(1 - \lambda_1 t)(1 - \lambda_2 t)(1 - \lambda_1 \lambda_2 t^2)}.
\]

Assuming \( (\lambda_1 \lambda_2)^2 = 1 \) for all \( g \in G \), quasi-bi-reflections are:

**Classical Reflections:** One eigenvalue of \( g \) is 1 and the other eigenvalue is a root of unity; since \( (\lambda_1 \lambda_2)^2 = 1 \) the other eigenvalue must be \(-1\).

**In \( SL_2(\mathbb{C}) \):** The eigenvalues of \( g \) are \( \lambda \) and \( \lambda^{-1} \) for \( \lambda \neq 1 \) (which forces the (homological) determinant to be 1).
Abelian Groups of Graded Automorphisms of $A$

Example:

$$G = \langle g_1, g_2 \rangle$$ for $g_1 = \text{diag } [\epsilon_n, \epsilon_n^{-1}]$ and $g_2 = \text{diag } [1, -1]$.

The group $G = \langle g_1, g_2 \rangle$ is a quasi-bi-reflection group of order $2n$ and $G \cong \mathbb{Z}_n \times \mathbb{Z}_2$.

When $n$ is even, $A^G$ is a classical complete intersection, and when $n$ is odd $A^G$ is not cyclotomic Gorenstein (so no kind of complete intersection).
For $n=2$ $G$ is a classical reflection group – the Klein-4 group. $A^G = k\langle x^2, y^2, (yx)^2, (xy)^2 \rangle$, the commutative hypersurface:

$$k[X, Y, Z, W] \frac{ZW - X^2Y^2}{\langle ZW - X^2Y^2 \rangle}.$$

For $n \geq 4$ $G$ is a quasi-bi-reflection group. $A^G = k\langle x^n, y^n, (xy)^2, (yx)^2, x^2y^2 \rangle$, the commutative complete intersection:

$$A^G \cong \frac{k[X, Y, Z, W, V]}{(XY - V^{n/2}, ZW - V^2)}.$$
n odd

\[ G = \langle g \rangle \] is generated by \( g = \text{diag} [\epsilon_n, -\epsilon_n^{-1}] \).

The numerator of the Hilbert series for \( A^G \) is

\[ = 1 + t^4 + 2t^{n+2} - 2t^{2n+2} - t^{3n} - t^{3n+4} \]

\[ = (1 - t^n)(1 + t^4 + t^n + 2t^{n+2} + t^{n+4} + t^{2n} + t^{2n+4}), \]

which we showed is NOT a product of cyclotomic polynomials for \( n > 1 \).
Dihedral Groups $G = \langle g_1, g_2 \rangle$

$g_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $g_2 = \begin{bmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{bmatrix}$

$n$ even:

$$A^G = \frac{k[X, Y, Z, W]}{\langle W^2 - X Y W - 4 Z^{n+2} + Y^2 Z + X^2 Z^n \rangle}$$

$n$ odd:

$$A^G = \frac{k[X, Y, Z][W; \sigma, \delta]}{\langle W^2 - Y^2 Z \rangle}.$$. 
Sklyanin Example

\[ ax^2 + yz + zy = 0 \]
\[ ay^2 + zx + xz = 0 \]
\[ az^2 + xy + yx = 0 \]

with \( a^3 \neq 1 \) and

\[ g = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

for \( \omega \) a primitive cubed root of unity. \( \det g = 1 \) and \( Tr_A(g, t) = 1/(1 - t^3) \), so \( g \) is a quasi-bi-reflection.

\[ A^g \cong \frac{\mathbb{C}_{-1}[x, x^3 - y^3][xy; \sigma, \delta][x^3; \sigma', \delta']}{\langle f \rangle}. \]
Questions:

If $A$ is a classical complete intersection, is $\text{Ext}_A^*(k, k)$ Noetherian?

Are there algebras $A$ with $\text{Ext}_A^*(k, k)$ Noetherian and finite GKdim that are not classical complete intersections?

What does $\text{Ext}_A^*(k, k)$ finite GKdim say about $A$?

Is there a version of the Kac-Watanabe-Gordeev Theorem in our context?

Classify the groups that give “complete intersections”.