MATH 126 – Autumn 2007

Final Exam Hints, Answers, and Partial Solutions

1. (a) ANSWER: $T_2(x) = 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2$

   (b) ANSWER: $\sqrt{3.7} = f(0.7) \approx T_2(0.7) = 1.92359375...$

   (c) Taylor's inequality states that the error in the approximation is bounded by $\frac{M}{3!}|x-1|^3$, where $M$ is an upper bound of $|f'''(x)|$ on the interval $[0.7, 1.3]$. We can take $M$ to be $\frac{3}{8(3.7)^{5/2}}$. Then the error is less than or equal to $\frac{3}{(3!)(8)(3.7)^{5/2}}|x-1|^3$, which is less than or equal to $\frac{3}{(3!)(8)(3.7)^{5/2}}(0.3)^3$ on the interval $[0.7, 1.3]$.

   ANSWER: error $\leq 0.000064083$

2. (a) HINT: Integrate term-by-term the Taylor series for $\frac{1}{1-x}$ to obtain the Taylor series for $-\ln(1-x)$. Multiply by $-1$ and substitute $-3x^2$ for $x$ to obtain the Taylor series for $f(x) = \ln(1+3x^2)$.

   ANSWER: $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}}{n+1} x^{2(n+1)}$ or, equivalently, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^n}{n} x^{2n}$

   (b) ANSWER: $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

3. ANSWER: $-x + 7y + 5z = 30$

4. (a) HINT: Find the parametric equations of the line $\ell$ through $P$ and $Q$. Then $S$ is the point $(x, y, z)$ on $\ell$ such that $\vec{SR} \cdot \vec{PQ} = 0$.

   ANSWER: $\left(-\frac{1}{3}, \frac{4}{3}, \frac{4}{3}\right)$

   (b) ANSWER: $\sqrt{2}$

   (c) ANSWER: $\frac{3\sqrt{2}}{2}$

5. (a) ANSWER: $\sqrt{2}$

   (b) ANSWER: $2\sqrt{2}\pi$

   (c) ANSWER: $\kappa(t) = \frac{1}{\sqrt{2}}$

6. (a) ANSWER: One tangent line is: $x = t$, $y = -t$. The other is: $x = t$, $y = t$.

   (b) ANSWER: By symmetry, the area enclosed by the curve is 4 times the area in the first quadrant bounded by the curve and the $x$-axis. This curve is traversed once from right to left as $t$ increases from 0 to $\frac{\pi}{2}$. Using the formula from Stewart, Section 10.2 (page 662 of the 5th edition):

$$\text{area enclosed by the curve} = 4\int_{\pi/2}^{0} -\sin^2 t \cos t \, dt = \frac{4}{3}.$$

7. HINT: Use the fact that $\vec{a}(t) = \langle \sqrt{t}, t^2, t-1 \rangle$, $\vec{v}(0) = \vec{0}$, and $\vec{r}(0) = \langle 1, 2, 0 \rangle$ to find $\vec{r}(t)$. Then find the value of $t$ at which the $z$-component of $\vec{r}(t)$ is equal to 0.

   ANSWER: $t = 3$

8. (a) ANSWER: The level curve consists of the two lines $y = \pm\sqrt{\frac{2}{3}}x$.

   (b) ANSWER: $z = \frac{4}{3}(x-2) - (y-1) + 3$

   (c) ANSWER: $f(1.9, 1.2) \approx 2.66666.....$

9. ANSWER: The point (-2,3) is a saddle point, and the point (2,3) is a local minimum.

10. HINT: $f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\, dA$.

    ANSWER: $f_{ave} = \frac{1}{3}\left(e^{-1} - e^{-4}\right)$