1. *(3 points per part)* Suppose **a** and **b** are nonzero vectors in $\mathbf{R}^3$. Decide whether each of the following statements is **always** true, **sometimes** true, or **never** true. (Circle one.)

   If your answer is **always** or **never**, briefly explain why (one sentence is enough).

   If your answer is **sometimes**, give an example where it's true **and** an example where it's false.

   (a) $\mathbf{a} \cdot \mathbf{a} > 0$ **Always** **Sometimes** **Never**

   Remember, for full credit, you must include a short explanation (for **Always** *or* **Never***) or examples (for* **Sometimes***)!*

   $\vec{a} \cdot \vec{a} = |\vec{a}|^2$, and $|\vec{a}| \neq 0$ because $\vec{a}$ isn't $\vec{0}$

   (b) $\mathbf{a} \times \mathbf{b} = 2\mathbf{a}$ **Always** **Sometimes** **Never**

   Never: $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$, but $2\vec{a}$ can't be (if $\vec{a} \neq \vec{0}$)

   (c) $|\mathbf{a} \times \mathbf{b}| = \mathbf{a} \cdot \mathbf{b}$ **Always** **Sometimes** **Never**

   Yes: $\vec{a} = \langle 1, 1, 0 \rangle$, $\vec{b} = \langle 1, 0, 0 \rangle$ (whenever $\sin\theta = \cos\theta$)

   No whenever $\theta \neq \frac{\pi}{4}$, e.g. $\vec{a} = \langle 1, 0, 0 \rangle$ $\vec{b} = \langle 2, 0, 0 \rangle$

   (d) $\text{comp}_{\mathbf{a}}\mathbf{b} > |\mathbf{b}|$ **Always** **Sometimes** **Never**

   $\text{comp}_{\vec{a}}\vec{b} = |\vec{b}| \underbrace{\cos\theta}_{\text{never} > 1}$

   (e) $\text{proj}_{\mathbf{a}}\mathbf{b} = \mathbf{b}$ **Always** **Sometimes** **Never**

   Yes $\vec{a}$ & $\vec{b}$ parallel, eg $\vec{a} = \langle 1, 0, 0 \rangle$, $\vec{b} = \langle 2, 0, 0 \rangle$

   No: any other time, eg $\vec{a} = \langle 1, 0, 0 \rangle$ $\vec{b} = \langle 0, 1, 0 \rangle$

2. *(4 points per part)* Consider the vector function $\mathbf{r}(t) = \langle 3\cos(t) + 1, 4\cos(t) + 2, 5\sin(t) + 7\rangle$.

(a) The space curve for $\mathbf{r}(t)$ lies in a plane. Find the equation of that plane.

$4x = 12\cos t + 4$

$3y = 12\cos t + 6$ or, find 3 pts and use cross product

$\downarrow$

$\boxed{4x + 2 = 3y}$

(b) Find parametric equations for the line tangent to $\mathbf{r}(t)$ at $\underbrace{(1, 2, 2)}$.

$t = \frac{-\pi}{2}$

$\vec{r}\,'(t) = \langle -3\sin t, -4\cos t, 5\cos t\rangle$

$\vec{r}\,'(-\frac{\pi}{2}) = \langle 3, 4, 0\rangle$

$$\boxed{\begin{aligned} x &= 1 + 3t \\ y &= 2 + 4t \\ z &= 2 \end{aligned}}$$

(c) Find $\mathbf{T}(t)$, the unit tangent vector to $\mathbf{r}(t)$.

$|\vec{r}\,'(t)| = \sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t} = 5$

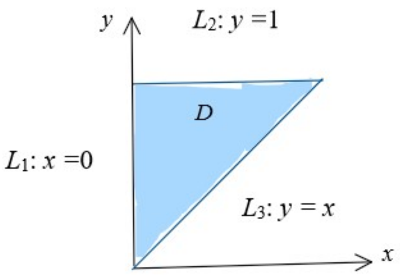
$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \boxed{\left\langle \frac{-3}{5}\sin t, \frac{-4}{5}\sin t, \cos t \right\rangle}$

3. *(7 points per part)* Consider the function $f(x, y) = xy - xy^3$.

   (a) Find all the critical points of f on $\mathbf{R}^2$ and classify each critical point.

   $f_x = y - y^3 = y(1 - y^2) = 0 \Rightarrow y = 0$ or $y = \pm 1$;
   $f_y = x - 3xy^2 = x(1 - 3y^2) = 0 \Rightarrow x = 0$ or $y = \pm\frac{1}{\sqrt{3}}$.
   Therefore the critical points are $(0, 0)$ and $(0, \pm 1)$.
   $f_{xx} = 0, f_{yy} = -6xy, f_{xy} = 1 - 3y^2$, Hessian $D = 0 - (1 - 3y^2)^2$.
   $D(0, 0) = -1 < 0; D(0, \pm 1) = -4 < 0$.
   $\boxed{\text{The critical points } (0,0) \text{ and } (0,\pm 1) \text{ are saddle points.}}$

   (b) Find the absolute maximum and minimum values of f on the triangular region bounded by the lines $y = x$, $y = 1$ and $x = 0$.



By (a), f has no critical point in the interior of D.
On the boundary component $L_1 : x = 0, 0 \leq y \leq 1$, $f(x, y) = f(0, y) = 0$.
On the boundary component $L_2 : y = 1, 0 \leq x \leq 1$, $f(x, y) = f(x, 1) = x - x = 0$.
On the boundary component $L_3 : y = x$, $f(x, y) = f(x, x) = x^2 - x^4$, i.e., f can be expressed as a single variable function $g(x) = x^2 - x^4$ with domain $0 \leq x \leq 1$,
$g'(x) = 2x - 4x^3 = 2x(1 - 2x^2) = 0$ gives a critical point $x = \frac{1}{\sqrt{2}}$, $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = g(\frac{1}{\sqrt{2}}) = \frac{1}{4}$.
(At the end points of the domain $0 \leq x \leq 1$, $g(0) = g(1) = 0$.)
Therefore, the absolute maximum value is $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 1/4$;
the absolute minimum value is $f(0, y) = f(x, 1) = 0$.

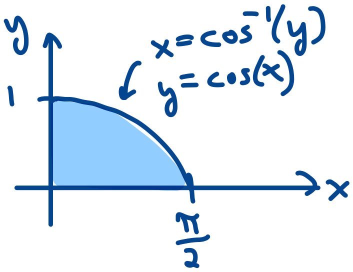
4. *(8 points)* Find $\dfrac{\partial z}{\partial x}$ if x, y, z are related by the implicit equation

$$x \sin z + e^{xy} = z.$$

$$\begin{aligned} 1 \sin z + x \cos z \frac{\partial z}{\partial x} + e^{xy} y &= \frac{\partial z}{\partial x} \\ (x \cos z - 1) \frac{\partial z}{\partial x} &= -1 \sin z - y e^{xy} \\ \implies \boxed{\frac{\partial z}{\partial x} = \frac{\sin z + y e^{xy}}{1 - x \cos z}} \end{aligned}$$

5. *(7 points per part)* Compute the following integrals.

(a) $\displaystyle\int_0^1 \int_0^{\cos^{-1}(y)} \sin(\sin(x))\, dx\, dy.$



$$= \int_0^{\frac{\pi}{2}} \int_0^{\cos x} \sin(\sin x)\, dy\, dx$$

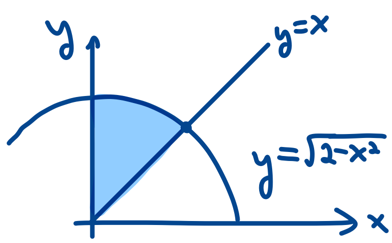
$$= \int_0^{\frac{\pi}{2}} \Big(y \sin(\sin x)\Big)\Big]_{y=0}^{y=\cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x \sin(\sin x)\, dx = \int_0^1 \sin(u)\, du$$

$u = \sin x$
$du = \cos x\, dx$

$$= -\cos(u)\Big]_0^1 = \boxed{1 - \cos(1)}$$

(b) $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} e^{x^2+y^2}\, dy\, dx.$



Polar!

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r e^{r^2}\, dr\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1}{2} e^{u}\right)\Big]_0^2 =$$

$u = r^2$
$du = 2r\, dr$

$$= \int_{\frac{\pi}{4}}^{\pi/2} \frac{1}{2}(e^2 - 1)\, d\theta = \boxed{\frac{\pi}{8}\left(e^2 - 1\right)}$$

6. *(12 points)* A lamina occupies the rectangle $\mathcal{R} = [0,4] \times [0,2]$. Find its center of mass if the density at each point is given by the function $\rho(x,y) = x + y^2$.

$$m = \int_0^4 \int_0^2 (x+y^2)\,dy\,dx = \int_0^4 \left(xy + \frac{1}{3}y^3\right)\Big]_{y=0}^{y=2} dx = \int_0^4 \left(2x + \frac{8}{3}\right) dx$$

$$= \left(x^2 + \frac{8}{3}x\right)\Big]_0^4 = 16 + \frac{32}{3} = \frac{80}{3}$$

$$M_y = \int_0^4 \int_0^2 (x^2 + xy^2)\,dy\,dx = \int_0^4 \left(x^2 y + \frac{1}{3}xy^3\right)\Big]_{y=0}^{y=2} dx = \int_0^4 \left(2x^2 + \frac{8}{3}x\right) dx$$

$$= \left(\frac{2}{3}x^3 + \frac{4}{3}x^2\right)\Big]_0^4 = \frac{128}{3} + \frac{64}{3} = 64$$

$$M_x = \int_0^4 \int_0^2 (xy + y^3)\,dy\,dx = \int_0^4 \left(\frac{1}{2}xy^2 + \frac{1}{4}y^4\right)\Big]_0^2 dx = \int_0^4 (2x+4)\,dx$$

$$= (x^2 + 4x)\Big]_0^4 = 32$$

$$(\bar{x}, \bar{y}) = \left(\frac{64}{\left(\frac{80}{3}\right)}, \frac{32}{\left(\frac{80}{3}\right)}\right) = \boxed{\left(\frac{12}{5}, \frac{6}{5}\right)}$$

7. *(5 points per part)* For all parts, consider $f(x) = \ln(x+2)$ based at $b = 1$. (**NOT based at zero!**)

(a) Find the third Taylor polynomial, $T_3(x)$, for $f(x)$ based at $b = 1$.

$$f(1) = \ln 3,$$

$$f'(x) = \frac{1}{x+2},\ f'(1) = \frac{1}{3},$$

$$f''(x) = \frac{-1}{(x+2)^2},\ f''(1) = -\frac{1}{9},$$

$$f'''(x) = \frac{2}{(x+2)^3},\ f'''(1) = \frac{2}{27}.$$

$$T_3(x) = \ln 3 + \frac{1}{3}(x-1) - \frac{1/9}{2!}(x-1)^2 + \frac{2/27}{3!}(x-1)^3$$

$$\boxed{= \ln 3 + \frac{1}{3}(x-1) - \frac{1}{18}(x-1)^2 + \frac{1}{81}(x-1)^3.}$$

(b) Use Taylor's inequality to find an upper bound (as sharp as possible) for the error $|f(x) - T_2(x)|$ on the interval $[-0.5, 2.5]$, where $T_2(x)$ is the second Taylor polynomial of $f(x)$ centered at $b = 1$.

The interval $[-0.5, 2.5]$ is centered at $b = 1$ with radius $r = 1.5$, on this interval,

$$|f'''(x)| = \left|\frac{2}{(x+2)^3}\right| \le \frac{2}{(-0.5+2)^3} = \frac{2}{1.5^3}$$ (max occurs at the left end point $x = -0.5$).

By Taylor's inequality,

$$|f(x) - T_2(x)| \le \frac{\max\limits_{x\in[-0.5,2.5]} |f'''(x)|}{3!}|x-1|^3 \le \frac{2/(1.5)^3}{3!}r^3 = \frac{2/(1.5)^3}{3!}1.5^3 = \boxed{\frac{1}{3}.}$$

(c) Find the smallest value of n such that Taylor's inequality guarantees that the error $|f(x) - T_n(x)| < 0.02$ for all x in the interval $[-0.5, 2.5]$, where $T_n(x)$ is the n^{th} Taylor polynomial of $f(x)$ centered at $b = 1$.

From the pattern of f', f'', f''', $f^{(k)}(x) = \dfrac{(-1)^{k+1}(k-1)!}{(x+2)^k} \Rightarrow \left|f^{(n+1)}(x)\right| = \left|\dfrac{n!}{(x+2)^{n+1}}\right|.$

On the interval $[-0.5, 2.5]$,

$$|f^{n+1}(x)| = \left|\frac{(-1)^{n+1}n!}{(x+2)^{n+1}}\right| \le \left|\frac{n!}{(-0.5+2)^{n+1}}\right| = \left|\frac{n!}{1.5^{n+1}}\right|$$ (max occurs at $x = -0.5$).

By Taylor's inequality,

$$|f(x) - T_n(x)| \le \frac{\max\limits_{x\in[-0.5,2.5]} |f^{(n+1)}(x)|}{(n+1)!}r^{n+1} = \frac{n!/(1.5)^{n+1}}{(n+1)!}1.5^{n+1} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

solve $\dfrac{1}{n+1} < 0.02 \Longrightarrow n+1 > 50 \Longrightarrow n > 49 \Longrightarrow \boxed{n \ge 50}$

8. Consider the function $f(x) = x\sin(x^2)$.

(a) *(6 points)* Find the Taylor series of $f(x) = x\sin(x^2)$ based at $b = 0$. Use the sigma sum notation $\sum_{k=\ldots}^{\infty}$ to express the Taylor series.

Apply the Taylor series $\sin x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$ (for all $x \in (-\infty, \infty)$),

$$x\sin(x^2) = x\sum_{k=0}^{\infty}(-1)^k \frac{(x^2)^{2k+1}}{(2k+1)!} = \boxed{\sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+3}}{(2k+1)!}}.$$

(b) *(4 points)* Use the series found in (a) to find $f^{(507)}(0)$ (i.e., the 507$^{\text{th}}$ order derivative of f at 0.)

In the Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$, $\frac{f^{(507)}(0)}{507!}$ is the coefficient of x^{507}.

In the Taylor series found in (a), x^{507} occurs when $4k + 3 = 507 \Rightarrow k = 126$.

Therefore the coefficient $\frac{f^{(507)}(0)}{507!} = \frac{(-1)^{126}}{((2)(126)+1)!} \Rightarrow \boxed{f^{(507)}(0) = \frac{507!}{253!}}$