MATH 126 B/C **MIDTERM 2** **Autumn 2011**

Nov 22, 2011

NAME: SIGNATURE:

STUDENT ID #: TA SECTION:

Problem	Number of points	Points obtained
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

Instructions:

- Your exam consists of FIVE problems. Please check that you have all four of them.
- No books or notebooks allowed; you may use an A4 double-sided, handwritten sheet of notes *for personal use* (do not share).
- Place a box around your final answer to each question.
- No *graphing* calculators allowed (scientific calculators OK).
- **Answers with little or no justification may receive no credit.**
- **Answers obtained by guess-and-check work will receive little or no credit, even if correct.**
- Read problems *carefully.*
- Raise your hand if you have a question.
- If you need more space, use additional blank sheets which will be provided by your TA. It is your responsibility to have him/her staple the additional sheets to your exam before you turn it in.
- Please turn off cell phones. GOOD LUCK!

Problem 1. (10 pts.) Find the double integral of the function $f(x, y) = xy$ over the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

Solution. We get

$$\int\int f dA = \int_0^1 \int_0^x xydydx = \frac{1}{2}\int_0^1 x^3 dx = \frac{1}{8}.$$

Problem 2. (10 pts) Find equation of the tangent plane of the function $f(x, y) = \ln(x - 3y)$ at $(7, 2)$. Use it to approximate $f(6.9, 2.02)$.

Solution. Let $x_0 = 7$, $y_0 = 2$, and $z_0 = 0$. Also

$$f_x(7,2) = \frac{1}{x_0 - 3y_0} = 1, \quad f_y(7,2) = \frac{-3}{x_0 - 3y_0} = -3.$$

Thus the tangent plane is given by

$$z = (x - 7) - 3(y - 2) = x - 3y - 1.$$

Moreover

$$f(6.9, 2.02) \approx 0 + 1(6.9 - 7) - 3(2.02 - 2) = -.1 - .06 = -.16.$$

Problem 3. (10 pts) Answer the following questions.

(i) At what point on the curve $x = t^3, y = 3t, z = t^4$ is the normal plane parallel to the plane $6x + 6y - 8z = 1$?

Solution. Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. We want to find t such that $\mathbf{r}'(t)$ is a parallel to the normal vector of the given plane, i.e., $\langle 6, 6, -8\rangle$. Now $\mathbf{r}'(t) = \langle 3t^2, 3, 4t^3\rangle$. Thus, $t = -1$ is the only solution. Hence the point is $(-1, -3, 1)$.

(ii) Suppose the location of a particle at time t is given by the curve $\mathbf{r}(t) = \langle t, t^2, t\rangle$. At what point on its trajectory is the normal component of its acceleration maximum?

We know

$$a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}.$$

We see

$$\mathbf{r}'(t) = \langle 1, 2t, 1\rangle,\ \mathbf{r}''(t) = \langle 0, 2, 0\rangle,\ \mathbf{r}'(t) \times \mathbf{r}''(t) = \langle -2, 0, 2\rangle.$$

Thus

$$a_N = \frac{\sqrt{8}}{\sqrt{2 + 4t^2}}.$$

This is maximum at $t = 0$, i.e., at $(0, 0, 0)$.

Problem 4. (10 pts) Find the absolute maximum and minimum values of the function $f(x,y) = x+y$ in the domain $D = \big\{(x,y):\ 1 \le x^2+y^2 \le 4\big\}$.

Solution. First find critical points inside the domain by solving

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0.$$

There is no solution. So we move to the boundary. There is an outer and an inner circular boundary.

The inner boundary can be parametrized by $(\cos t, \sin t)$, $0 \le t < 2\pi$. The function is now

$$f(t) = \cos(t) + \sin(t), \quad f'(t) = -\sin(t) + \cos(t) = 0 \ \Rightarrow t = \frac{\pi}{4}, \text{ or, } \frac{5\pi}{4}.$$

We evaluate $f(\pi/4) = \sqrt{2}$, $f(5\pi/4) = -\sqrt{2}$.

The outer boundary can be parametrized by $(2\cos t, 2\sin t)$, $0 \le t < 2\pi$. Here

$$f(t) = 2\sin(t) + 2\cos(t), \quad f'(t) = 2\sin(t) - 2\cos(t) = 0.$$

The same solution gives us $f(\pi/4) = 2\sqrt{2}$, $f(5\pi/4) = -2\sqrt{2}$.

Thus the absolute maximum is $2\sqrt{2}$ and the absolute minimum is $-2\sqrt{2}$.

Problem 5. (10 pts) Use polar coordinates to find the volume of the solid that is inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.
Solution. The volume between the two bodies is

$$I = 2\left[\int\int_{x^2+y^2\le 16} \sqrt{16 - x^2 - y^2} dA - \int\int_{x^2+y^2\le 4} \sqrt{16 - x^2 - y^2}\right]$$

Using polar coordinates

$$\begin{aligned} I/2 &= \int_0^{2\pi}\int_0^4 r\sqrt{16 - r^2} dr d\theta - \int_0^{2\pi}\int_0^2 r\sqrt{16 - r^2} dr d\theta \\ &= 2\pi\left[-\frac{1}{3}(16 - r^2)^{3/2}\,|_2^4\right] = \frac{2\pi}{3}12^{3/2}. \end{aligned}$$

Thus $I = 4\pi 12^{3/2}/3$.