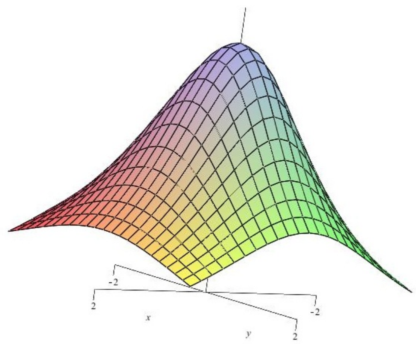
Math 126, Sections A and B, Winter 2011, Solutions to Midterm II

1. Answer the following.

   (a) (4 points) Below is a graph of the surface $z = f(x, y)$.



   Decide if the following partial derivatives are positive or negative.

   $f_x(0.2, 0.1) < 0$ $\quad$ $f_y(1, 2) < 0$ $\quad$ $f_{xy}(0.1, 0.1) > 0$ $\quad$ $f_{xx}(0.1, 0.1) < 0$

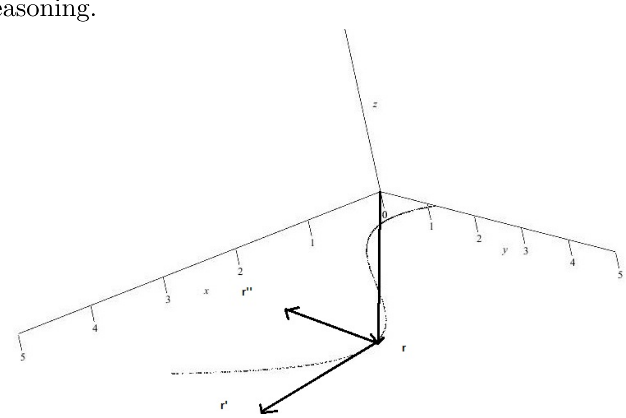
   (b) (3 points) Compute the curvature of $\mathbf{r} =< \sin t, \sin(2t), t >$ at the point when $t = \pi/3$.

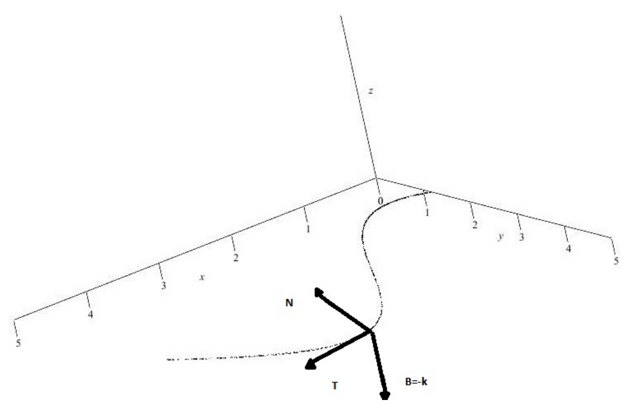
$$\mathbf{r}'(t) = \langle \cos t, 2\cos 2t, 1\rangle\,, \qquad \mathbf{r}'(\pi/3) = \langle 1/2, -1, 1\rangle$$

$$\mathbf{r}''(t) = \langle -\sin t, -4\sin 2t, 0\rangle\,, \qquad \mathbf{r}''(\pi/3) = \left\langle -\sqrt{3}/2, -2\sqrt{3}, 0\right\rangle$$

$$\kappa(\pi/3) = \frac{|\langle 1/2, -1, 1\rangle \times \left\langle -\sqrt{3}/2, -2\sqrt{3}, 0\right\rangle|}{|< 1/2, -1, 1 >|^3} = \frac{|\left\langle 2\sqrt{3}, -\frac{\sqrt{3}}{2}, -\frac{3\sqrt{3}}{2}\right\rangle|}{(3/2)^3} = \frac{4\sqrt{78}}{27}$$

   (c) (3 points) The vector function $\mathbf{r}(t)$ has the graph below. The curve is on the xy-plane. As t increase, it is traced in the direction of increasing x. The speed of the particle decreases as it gets further away from the origin. The point $(3.5, 5, 0)$ is on the curve and corresponds to the value $t = t_0$. Sketch the vectors $\mathbf{r}(t_0)$, $\mathbf{r}'(t_0)$, $\mathbf{r}''(t_0)$ on the first picture and the vectors $\mathbf{T}(t_0)$, $\mathbf{N}(t_0)$, $\mathbf{B}(t_0)$ on the second picture. Approximate the curvature from the graph explaining your reasoning.





   The vectors $\mathbf{r}'(t_0)$ and $\mathbf{T}(t_0)$ have to be tangent to the curve, point in the direction of increasing x, and $\mathbf{T}(t_0)$ must have length 1. The vector $\mathbf{r}(t_0)$ is the position vector from the origin to the point. The acceleration vector $\mathbf{r}''(t_0)$ should point inside and back (the speed is decreasing), roughly point towards a point on the x axis to the right of $(3.5, 0, 0)$. The normal should be perpendicular to the Tangent vector and point towards the x axis. The Binormal vector is $-\mathbf{k}$. You can estimate the curvature by drawing a circle and estimating its radius. The curvature should be the reciprocal of the radius.

2. Given the implicit function
$$2x^2 + 4yz + 3z^3 - 5xz + x - 13 = 0$$

(a) Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial x}$.
Differentiation both sides with respect to x

$$4x + 4yz_x + 9z^2 z_x - 5(z + xz_x) + 1 = 0$$

so

$$z_x = \frac{-4x + 5z - 1}{4y + 9z^2 - 5x}$$

and differentiating with respect to y

$$4(z + yz_y) + 9z^2 z_y - 5xz_y = 0$$

so

$$z_y = \frac{-4z}{4y + 9z^2 - 5x}$$

(b) Check that the point $(1, -2, -1)$ is on the surface $2x^2 + 4yz + 3z^3 - 5xz + x - 13 = 0$ and find the equation of its tangent plane at that point.
At the point $(1, -2, -1)$ the values for the partial derivatives are

$$z_x = \frac{-4 + 5(-1) - 1}{4(-2) + 9(-1)^2 - 5} \frac{-10}{-4} = \frac{5}{2}$$

and

$$z_y = \frac{-4(-1)}{4(-2) + 9 - 5} = -1$$

so the tangent plane has equation

$$z = -1 + \frac{5}{2}(x - 1) - (y + 2)$$

(c) Use linearization to approximate the value of z when $x = 1.1$ and $y = -1.95$.

$$z \approx -1 + \frac{5}{2}(x - 1) - (y + 2) = -1 + \frac{5}{2}(1.1 - 1) - (-1.95 + 2) = -0.8$$

3. Find and classify the critical points of

$$f(x,y) = 2x^3 + y^3 - 3x^2 - 12x - 3y.$$

The critical points are given by

$$f_x(x,y) = 6x^2 - 6x - 12 = 6(x-2)(x+1) = 0$$

and

$$f_y(x,y) = 3y^2 - 3 = 3(y-1)(y+1) = 0$$

The second order partial derivatives are

$$f_{xx}(x,y) = 12x - 6, \qquad f_{yy}(x,y) = 6y, \qquad f_{yy}(x,y) = 0$$

(x,y)	f_{xx}	f_{yy}	f_{xy}	D	type
$(2,1)$	18	6	0	108	min
$(2,-1)$	18	-6	0	-108	saddle
$(-1,1)$	-18	6	0	-108	saddle
$(-1,-1)$	-18	-6	0	108	Max

4. Find the volume of the solid under the hyperboloid $z = xy$ and above the triangle in the xy-plane with vertices $(0,0)$, $(1,3)$ and $(3,1)$.

   Whether you integrate x first or y first, you need to split the integrals into two:

$$\int_0^1 \int_{\frac{1}{3}x}^{3x} xy \;\; dydx + \int_1^3 \int_{\frac{1}{3}x}^{-x+4} xy \;\; dydx = \frac{22}{3}$$

or

$$\int_0^1 \int_{\frac{y}{3}}^{3y} xy \;\; dxdy + \int_1^3 \int_{\frac{y}{3}}^{-y+4} xy \;\; dxdy = \frac{22}{3}$$