1. **[12 points]** Consider the implicitly defined surface $-x^2y - xz^2 + yz^4 = 16$.

Write an equation for the plane tangent to this surface at the point $(3, 4, 2)$.

$\frac{\partial}{\partial x}$

$$-2xy - z^2 - 2xz\frac{\partial z}{\partial x} + 4yz^3\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{2xy + z^2}{-2xz + 4yz^3} \xrightarrow{\substack{x=3\\ y=4\\ z=2}} \frac{\partial z}{\partial x} = \frac{28}{116} = \frac{7}{29}$$

$\frac{\partial}{\partial y}$

$$-x^2 - 2xz\frac{\partial z}{\partial y} + z^4 + 4yz^3\frac{\partial z}{\partial y} = 0$$

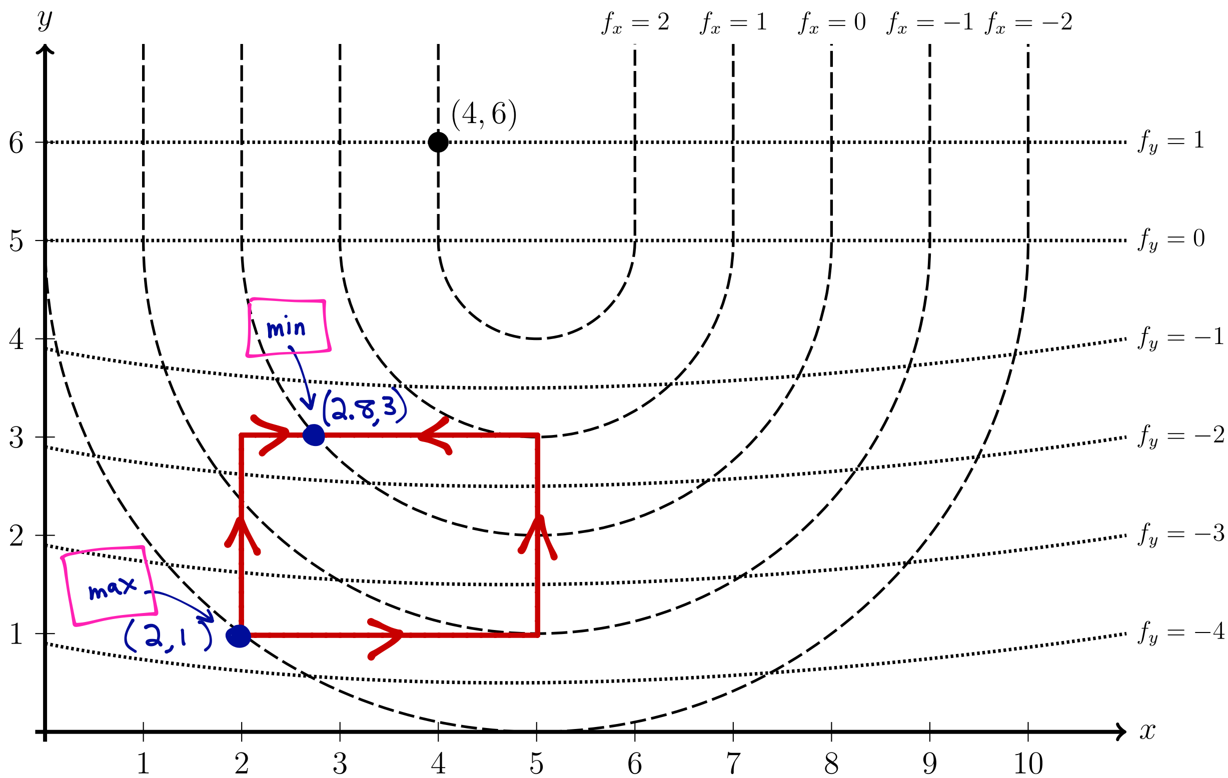
$$\frac{\partial z}{\partial y} = \frac{x^2 - z^4}{-2xz + 4yz^3} \xrightarrow{\substack{x=3\\ y=4\\ z=2}} \frac{\partial z}{\partial y} = \frac{-7}{116}$$

$$z = z_0 + \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0)$$

$$\boxed{z = 2 + \frac{7}{29}(x-3) - \frac{7}{116}(y-4)}$$

2. **[6 points per part]** Oh, nice, it's this graph again.

Below are the **level curves** of the **partial derivatives** of a function $f(x, y)$.



(a) The point $(4, 6)$ is marked on the graph above.

Suppose $f(4, 6) = 3$. Use linearization to approximate $f(4.2, 5.9)$.

$$L(x,y) = f(4,6) + f_x(4,6)(x-4) + f_y(4,6)(y-6) = 3 + 2(x-4) + 1(y-6)$$

$$f(4.2, 5.9) \approx L(4.2, 5.9) = 3 + 2(4.2-4) + 1(5.9-6) = \boxed{3.3}$$

(b) Consider the rectangle $R = \{(x, y) \,|\, 2 \le x \le 5, 1 \le y \le 3\}$.

Where are the absolute minimum and maximum of $f(x, y)$ on R?

Indicate each point on the graph, and explain your reasoning below.

$f_y < 0$ everywhere on R, so the maximum is somewhere on the bottom edge and the minimum is somewhere on the top edge. On the bottom, $f_x < 0$, so the max is at the bottom left. On the top, $f_x < 0$ and then $f_x = 0$ and finally $f_x > 0$. The minimum occurs when $f_x = 0$.

Or, see the inequalities drawn on the border above.

3. **[12 points]** Consider the function $f(x, y) = x^2 + 2xy + y^3 - 4y^2 - 45y + 10$.

Find all critical points of f. Classify them as local maxima, local minima, or saddlepoints.

$$f_x(x,y) = 2x + 2y = 0 \rightarrow x = -y$$

$$f_y(x,y) = 2x + 3y^2 - 8y - 45 = 0$$

$$-2y + 3y^2 - 8y - 45 = 0$$

$$3y^2 - 10y - 45 = 0$$

$$y = \frac{10 \pm \sqrt{100 - 4(3)(-45)}}{6} \approx -2.55 \text{ or } 5.88$$

or $\frac{5 \pm 4\sqrt{10}}{3}$

Critical points: $(2.55, -2.55)$ & $(-5.88, 5.88)$

To classify:

$$f_{xx}(x,y) = 2$$

$$f_{yy}(x,y) = 6y - 8$$

$$f_{xy}(x,y) = 2$$

$$D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

$$= 2(6y - 8) - 2^2$$

$D(2.55, -2.55) < 0$,

so $\boxed{(2.55, -2.55) \text{ is a saddlepoint.}}$

$\left(\frac{-5 + 4\sqrt{10}}{3}, \frac{5 - 4\sqrt{10}}{3}\right)$

$D(-5.88, 5.88) > 0$ and $f_{xx}(x,y) > 0$,

so $\boxed{(-5.88, 5.88) \text{ is a local min.}}$

$\left(\frac{-5 - 4\sqrt{10}}{3}, \frac{5 + 4\sqrt{10}}{3}\right)$

4. **[7 points per part]** Evaluate each integral.

(a) $\displaystyle\int_1^3 \int_2^4 y e^{xy}\, dy\, dx$

$$= \int_2^4 \int_1^3 y e^{xy}\, dx\, dy = \int_2^4 \int_y^{3y} e^u\, du\, dy = \int_2^4 \left(e^u\right)\Big]_y^{3y} dy$$

$u = xy$

$du = y\, dx$

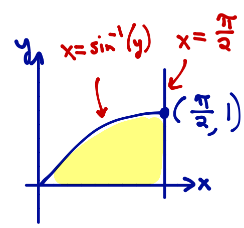
$$= \int_2^4 \left(e^{3y} - e^y\right) dy$$

$$= \left(\frac{1}{3}e^{3y} - e^y\right)\Big]_2^4$$

$$= \boxed{\left(\frac{1}{3}e^{12} - e^4\right) - \left(\frac{1}{3}e^6 - e^2\right)}$$

(b) $\displaystyle\int_0^1 \int_{\sin^{-1}(y)}^{\frac{\pi}{2}} \sqrt{1+\cos(x)}\, dx\, dy$

let's draw it:



$$= \int_0^{\frac{\pi}{2}} \int_0^{\sin(x)} \sqrt{1+\cos(x)}\, dy\, dx$$

$$= \int_0^{\frac{\pi}{2}} \left(y\sqrt{1+\cos(x)}\right)\Big]_0^{\sin(x)} dx$$

$$= \int_0^{\frac{\pi}{2}} \sin(x)\sqrt{1+\cos(x)}\, dx = \int_2^1 -\sqrt{u}\, du = \left(\frac{-2}{3}u^{3/2}\right)\Big]_2^1 = \boxed{\frac{-2}{3}\left(1 - 2\sqrt{2}\right)}$$

$u = 1 + \cos(x)$

$du = -\sin(x)dx$

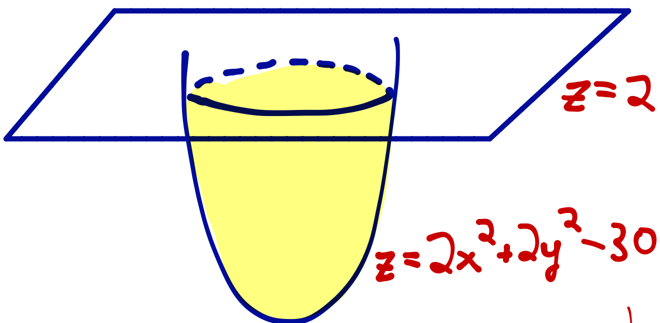
5. **[10 points]** Find the volume of the solid bounded by the paraboloid $z = 2x^2 + 2y^2 - 30$ and the plane $z = 2$.

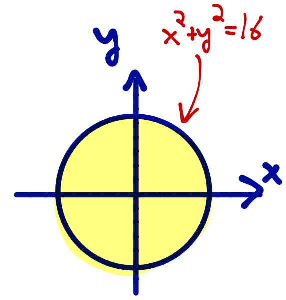
Intersection: $2 = 2x^2 + 2y^2 - 30$

$\downarrow$

$x^2 + y^2 = 16$

Side view:





polar!

$$\text{Volume} = \iint_D \left(2 - \left(2x^2 + 2y^2 - 30\right)\right) dA = \int_0^{2\pi} \int_0^4 (32 - 2r^2)\, r\, dr\, d\theta$$

circle of radius 4, centered @ (0,0)

$$= \int_0^{2\pi} \left(16r^2 - \frac{1}{2}r^4\right)\Bigg]_0^4 d\theta$$

$$= \int_0^{2\pi} (256 - 128)\, d\theta = 128\theta\Big]_0^{2\pi}$$

$$= \boxed{256\pi}$$