Calculating the Resistors in a Network

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The purpose of this paper is to report results of research conducted during the summer of 1988 on the inverse problem associated with the conductivity equation

$$\nabla(\gamma \cdot \nabla u) = 0.$$

The approximation to the continuous medium of varying conductivity is a network of resistors of varying conductivity. I will describe an algorithm for determining the resistors in a network from measurements of currents and potentials on the boundary. I will then give some results of its application on networks of known resistance.

1 Construction of Lambda

Consider a rectangular network like the one in Figure 1. For convenience, I consider a square network with n nodes on each side. Thus there are n^2 interior nodes, $4n$ boundary nodes, and $2n(n+1)$ resistors. Voltages applied at the $(4n)$ boundary nodes will induce currents to flow throughout the network so it seems reasonable to consider the currents flowing into the boundary nodes as a function of applied voltages. Call this mapping from voltages to currents Λ. By observing that Ohm's law, which determines the current flowing through each resistor, is linear, it is apparent that Λ is linear so a matrix representation for Λ is appropriate. Let $\Lambda_{i,j}$ be the current into boundary node i produced by applying 1 volt at boundary node j and grounding the other boundary nodes.

If the applied voltage is q_j at node j, then the current at node i, u_i, will be

$$u_i = \sum_{j=1}^{4n} \Lambda_{i,j} q_j$$

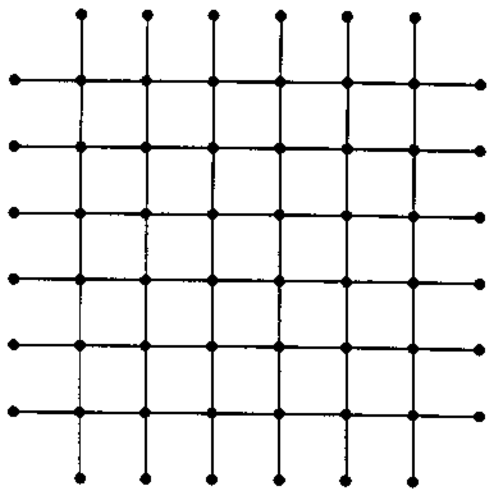

Figure 1

2 Curtis-Morrow Method

Figure 2 shows a diagonal along which the potential is to be zero.

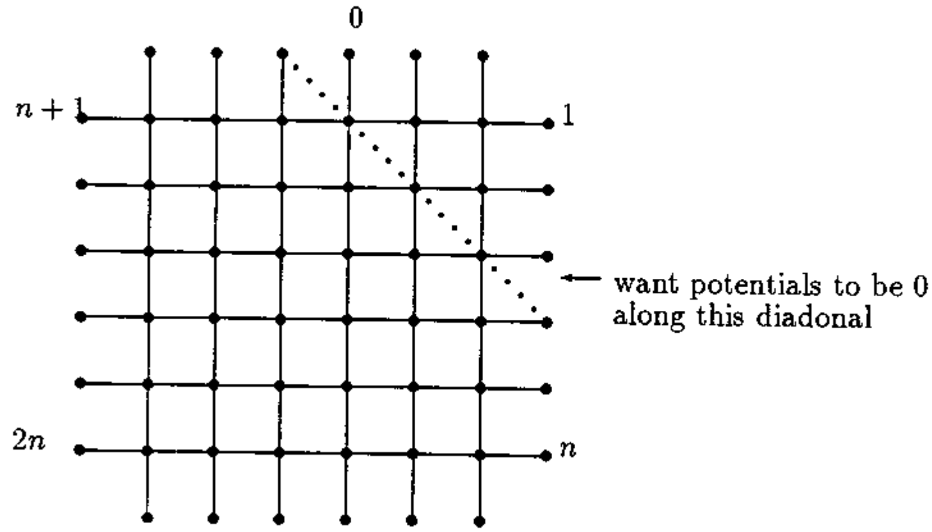

Figure 2: numbering of nodes

It was shown by Curtis and Morrow ([5]) that it is always possible to find boundary potentials that insure a potential of zero along any diagonal of a rectangular resistor network. To find the appropriate potentials the nodes with non-zero potentials are numbered. The potential at node 0 is 1, the potential at node i, for $i = 1 \dots n$ is α_i. The potential along the diagonal will be zero if and only if the currents on the left hand side are zero. Curtis and Morrow also proved that all of the currents on the left will be zero if at least n of them are known to be zero. Number n of the left side nodes as $n+1 \dots 2n$. Then the currents at these nodes are zero when

$$\sum_i \alpha_i \Lambda_{j,i} + \Lambda_{j,0} = 0$$

for each $j = n+1 \dots 2n$. These are n linearly independent equations in n unknowns so there is a unique solution.

3 Finding Potentials and Currents

The boundary potentials determined by the Curtis-Morrow method together with the currents they produce can be used to calculate for the nth diagonal:

(1) the potential at each node

(2) the total current flowing from the north and east into each node if the value of each resistor above the nth diagonal has been determined.

Consider Figure 3, which shows the first, (k-1)th, and kth diagonals..

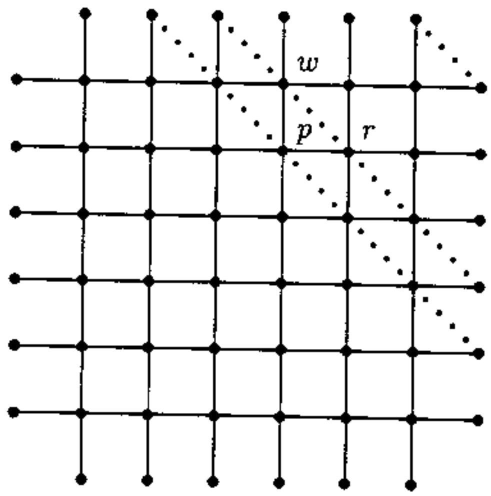

Figure 3

Suppose that we know (1) and (2) for the (k-1)th diagonal. I will show that I can use this to find (1) and (2) for the kth diagonal. Then the nth diagonal will follow from the 1st diagonal. Choose a node on the kth diagonal not a boundary node and call it p. Let the node to the north be q. q is on the (k-1)th diagonal. Let the resistance between p and q be $1/\gamma(pq)$ and let the current from q to p be I(qp). If I(qp) is known then the potential Q(p) is by Ohm's law

$$(Q(q) - Q(p)) \cdot \gamma(pq) = I(qp)$$

Let the node to the east of p be r. r is on the (k-1)th diagonal. If r is not a boundary node then I(pr) is given by

$$(Q(p) - Q(r)) \cdot \gamma(pr) = I(pr)$$

Now I know that the total current into node p from the north and east is

$$I(qp) - I(pr)$$

and that the current out of node r to the south is the total current into r from the north and east plus I(pr). This process can be continued until node r is a boundary node in which case I(pr) is a boundary current. This process must have started at a node p for which the current into it from the north is know, that is, at the 2nd node on the kth diagonal. The 1st and last nodes on the kth diagonal are determined by boundary potentials and currents. Thus the kth diagonal is determined by the (k-1)th diagonal. The first diagonal is determined by boundary data.

4 Calculating a band of resistors

Now that (1) and (2) are known for the nth diagonal the resistors between the nth and (n+1)th diagonal, which I will call the nth resistor band, can be calculated. Choose a node p on the (n+1)th diagonal and let q be the node to the north of p. q is on the nth diagonal. Suppose that we know the current I(qp) flowing from q to p. Then by Ohm's law the value of the resistor between q and p, 1/gamma(qp), is

$$Q(q) \cdot \gamma(qp) = I(qp)$$

since the potential of p is zero. I(qp) is the current flowing into node p from the north. This current must flow out of p to the east since the nodes to the west and south are also at zero potential. Let r be the node to the east of p. r is on the nth diagonal so the resistance between p and r, $1/\gamma(pr)$ is given by Ohm's law:

$$-Q(r) \cdot \gamma(pr) = I(qr)$$

If r is not a boundary node then I know that the current flowing to the south from node r is the total current flowing into it from the north and east plus I(qp). Thus this process can be continued until r is a boundary node, in which case the current into p and the potential at p are boundary data. This process must have started at a node p for which the current into it from the north is known, namely the 2nd node on the (n+1)th diagonal. The value of each resistor in the nth band can be determined in this way.

5 Improving the calculations

To determine each resistor value in a square network we could calculate the first band of resistors at a corner and use this to find the 2nd band and so on until half of the square is known, then do the same at the diagonally opposite corner. This is not the best method because a resistor that is far away from the present corner may be closer to an adjacent corner and could be calculated more accurately from there. To solve this problem I calculate the 1st band of resistors at each of the four corners, then the 2nd band, and so on. Each resistor will be calculated twice and the more accurate value is the one on the lower band (in the case that the bands are the same level either value will do). This is the value of the resistor that should be used in subsequent calculations, the other value being discarded.

Another observation that helped to improve my method of calculating resistors is that for any given resistor band the first half is calculated more accurately than the second half. So each band must be calculated twice, once in each direction, the reverse direction being equivalent to reflecting the network over the main diagonal perpendicular to the band and calculating in the forward direction. Then the more accurate value of each resistor can be saved, the other discarded.

6 Results

I have described a complete algorithm incorporating the Curtis-Morrow method to determine the resistors in a network. Now I want to show some results.

The original network of resistors:

```
 1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
 1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
 1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
 1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   3.0 1   1   1   1   1
 1   1   1   1   3.0 3.0 1   1   1   1
1   1   1   1   3.0 3.0 3.0 1   1   1   1
 1   1   1   3.0 3.0 3.0 3.0 1   1   1
1   1   1   1   3.0 3.0 3.0 1   1   1   1
 1   1   1   1   3.0 3.0 1   1   1   1
1   1   1   1   1   3.0 1   1   1   1   1
 1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
 1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
 1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
 1   1   1   1   1   1   1   1   1   1
```

```
With Lambda accurate to 14 digits, the maximum relative error was: 3.33786e-06
With Lambda accurate to 13 digits, the maximum relative error was: 5.50032e-04
With Lambda accurate to 12 digits, the maximum relative error was: 8.50002e-03
With Lambda accurate to 11 digits, the maximum relative error was: 0.130117
With Lambda accurate to 10 digits, the maximum relative error was: 0.372457
With Lambda accurate to 9 digits, the maximum relative error was: 0.628177
```

The original network of resistors:

```
 1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
 1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
 1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
 1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   10  1   1   1   1   1
 1   1   1   1   10  10  1   1   1   1
1   1   1   1   10  10  10  1   1   1   1
 1   1   1   10  10  10  10  1   1   1
1   1   1   1   10  10  10  1   1   1   1
 1   1   1   1   10  10  1   1   1   1
1   1   1   1   1   10  1   1   1   1   1
 1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
 1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
 1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
 1   1   1   1   1   1   1   1   1   1
```

```
With Lambda accurate to 14 digits, the maximum relative error was: 8.09670e-05
With Lambda accurate to 13 digits, the maximum relative error was: 6.11973e-04
With Lambda accurate to 12 digits, the maximum relative error was: 4.96197e-03
With Lambda accurate to 11 digits, the maximum relative error was: 0.211887
```

The original network of resistors:

```
  1    1    1    1    1    1    1    1    1    1
1    1    1    1    1    1    1    1    1    1    1
  1    1    1    1    1    1    1    1    1    1
1    1    1    1    1    1    1    1    1    1    1
  1    1    1    1    1    1    1    1    1    1
1    1    1    1    1    1    1    1    1    1    1
  1    1    1    1    1    1    1    1    1    1
1    1    1    1    1    100  1    1    1    1    1
  1    1    1    1    100  100  1    1    1    1
1    1    1    1    100  100  100  1    1    1    1
  1    1    1    100  100  100  100  1    1    1
1    1    1    1    100  100  100  1    1    1    1
  1    1    1    1    100  100  1    1    1    1
1    1    1    1    1    100  1    1    1    1    1
  1    1    1    1    1    1    1    1    1    1
1    1    1    1    1    1    1    1    1    1    1
  1    1    1    1    1    1    1    1    1    1
1    1    1    1    1    1    1    1    1    1    1
  1    1    1    1    1    1    1    1    1    1
1    1    1    1    1    1    1    1    1    1    1
  1    1    1    1    1    1    1    1    1    1
```

With Lambda accurate to 14 digits, the maximum relative error was: 3.44002e-03
With Lambda accurate to 13 digits, the maximum relative error was: 1.21000e-02
With Lambda accurate to 12 digits, the maximum relative error was: 3.39200e-02

The original network of resistors:

```
  1     1     1     1     1     1     1     1     1     1
1     1     1     1     1     1     1     1     1     1     1
  1     1     1     1     1     1     1     1     1     1
1     1     1     1     1     1     1     1     1     1     1
  1     1     1     1     1     1     1     1     1     1
1     1     1     1     1     1     1     1     1     1     1
  1     1     1     1     1     1     1     1     1     1
1     1     1     1     1     100   1     1     1     1     1
  1     1     1     1     100   100   1     1     1     1
1     1     1     1     100   0.01  100   1     1     1     1
  1     1     1     100   0.01  0.01  100   1     1     1
1     1     1     1     100   0.01  100   1     1     1     1
  1     1     1     1     100   100   1     1     1     1
1     1     1     1     1     100   1     1     1     1     1
  1     1     1     1     1     1     1     1     1     1
1     1     1     1     1     1     1     1     1     1     1
  1     1     1     1     1     1     1     1     1     1
1     1     1     1     1     1     1     1     1     1     1
  1     1     1     1     1     1     1     1     1     1
1     1     1     1     1     1     1     1     1     1     1
  1     1     1     1     1     1     1     1     1     1
```

With Lambda accurate to 14 digits, the maximum relative error was: 0.137800

The original network of resistors:

```
   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1 0.1   1   1   1   1   1
   1   1   1   1 0.1 0.1   1   1   1   1
1   1   1   1 0.1  10 0.1   1   1   1   1
   1   1   1 0.1  10  10 0.1   1   1   1
1   1   1   1 0.1  10 0.1   1   1   1   1
   1   1   1   1 0.1 0.1   1   1   1   1
1   1   1   1   1 0.1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1
```

With Lambda accurate to 14 digits, the maximum relative error was: 2.88300e-02
With Lambda accurate to 13 digits, the maximum relative error was: 5.05700e-02
With Lambda accurate to 11 digits the maximum relative error was: 0.947370

The original network of resistors:

```
   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1  10   1   1   1   1   1
   1   1   1   1  10  10   1   1   1   1
1   1   1   1  10 0.1  10   1   1   1   1
   1   1   1  10 0.1 0.1  10   1   1   1
1   1   1   1  10 0.1  10   1   1   1   1
   1   1   1   1  10  10   1   1   1   1
1   1   1   1   1  10   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1
```

With Lambda accurate to 14 digits, the maximum relative error was: 3.00026e-04
With Lambda accurate to 13 digits, the maximum relative error was: 9.50024e-04
With Lambda accurate to 12 digits, the maximum relative error was: 6.15000e-02
With Lambda accurate to 11 digits, the maximum relative error was: 0.133520

Original network of resistors:

```
    1    1    1    1    1    1    1    1    1    1
 1    1    1    1    1    1    1    1    1    1    1
    1    1    1    1    1   10   10   10   10    1
 1    1    1    1    1    1   10   10   10    1    1
    1    1    1    1    1   10   10   10   10    1
 1    1    1    1    1    1   10   10   10    1    1
    1    1    1    1    1   10   10   10   10    1
 1    1    1    1    1    1   10   10   10    1    1
    1    1    1    1    1   10   10   10   10    1
 1    1    1    1    1    1    1    1    1    1    1
    1    1    1    1    1    1    1    1    1    1
 1    1    1    1    1    1    1    1    1    1    1
    1  0.1  0.1  0.1  0.1    1    1    1    1    1
 1    1  0.1  0.1  0.1    1    1    1    1    1    1
    1  0.1  0.1  0.1  0.1    1    1    1    1    1
 1    1  0.1  0.1  0.1    1    1    1    1    1    1
    1  0.1  0.1  0.1  0.1    1    1    1    1    1
 1    1  0.1  0.1  0.1    1    1    1    1    1    1
    1  0.1  0.1  0.1  0.1    1    1    1    1    1
 1    1    1    1    1    1    1    1    1    1    1
    1    1    1    1    1    1    1    1    1    1
```

With Lambda accurate to 14 digits, the maximum relative error was: 1.52999e-03
With Lambda accurate to 13 digits, the maximum relative error was: 8.60000e-03
With Lambda accurate to 12 digits, the maximum relative error was: 0.248300

Original network of resistors:

```
    1    1    1    1    1    1    1    1    1    1
 1    1    1    1    1    1    1    1    1    1    1
    1    1    5    1    1   10   10   10   10    1
 1    1    5    5    1    1   10   10   10    1    1
    1    5    5    5    1   10   10   10   10    1
 1    5    5    5    5    1   10   10   10    1    1
    1    5    5    5    1   10   10   10   10    1
 1    1    5    5    1    1   10   10   10    1    1
    1    1    5    1    1   10   10   10   10    1
 1    1    1    1    1    1    1    1    1    1    1
    1    1    1    1    1    1    1    1    1    1
 1    1    1    1    1    1    1    1    1    1    1
    1  0.1  0.1  0.1  0.1    1    1    3    1    1
 1    1  0.1  0.1  0.1    1    1    3    3    1    1
    1  0.1  0.1  0.1  0.1    1    3    3    3    1
 1    1  0.1  0.1  0.1    1    3    3    3    3    1
    1  0.1  0.1  0.1  0.1    1    3    3    3    1
 1    1  0.1  0.1  0.1    1    1    3    3    1    1
    1  0.1  0.1  0.1  0.1    1    1    3    1    1
 1    1    1    1    1    1    1    1    1    1    1
    1    1    1    1    1    1    1    1    1    1
```

With Lambda accurate to 14 digits, the maximum relative error was: 8.74001e-03
With Lambda accurate to 13 digits, the maximum relative error was: 1.63700e-02
With Lambda accurate to 12 digits, the maximum relative error was: 0.461000