Inverse Problem Results for Triangular and Hexagonal Networks

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One of the central research themes this last summer was to find accurate numerical techniques for solving the inverse problem $\nabla(\gamma\nabla u)=0$, i.e. to find the conductivity γ on some region given boundary potential and boundary current data.

We studied the finite difference method. To apply this method one must first solve the forward problem, i.e. determine the Dirichlet-Neumann mapping which maps boundary potentials to boundary currents. Since this mapping is a linear transformation we can represent it as a matrix denoted Λ. To determine Λ we must first, before establishing a boundary potential basis, specify the geometry of the network used in the numerical approximation. Most research has been done using a square network where each interior node has four neighboring nodes.

We felt a better approximation would result if one were to construct a network where each interior has six neighboring nodes. A choice had to be made as to the geometry of the boundary. We solved the forward problem for both a triangular and hexagonal boundary. However, we discoverd that for most (if not all) boundary configurations for a triangular network that the inverse problem cannot be solved in analogy with the square network. That is, instead of using only one boundary potential vector for each iteration (one-function approach) we must use two boundary potential vectors for each iteration (two-function approach). For large networks this can increase computation time significantly. The idea is to apply appropriate boundary potentials and then proceed to measure boundary currents and apply Kirchoff's law at interior nodes so as to generate a system of linear equations with conductances as unknowns. We have attempted a variety of schemes all of which result in an undetermined system using the one-function approach.

For example, consider the triangular network shown in figure 1. In analogy with the Curtis-Morrow method we begin by (1) applying a potential of 1 at the lower left corner (node 1) with zero potentials elsewhere on the boundary and (2) applying 0 currents at nodes 7 through 10. By measur~~e~~ing currents at nodes 2 and 15 we obtain conductances γ_1 and γ_2 respectively, i.e. $I_2 = \gamma_2$ and $I_{15} = \gamma_1$. Note that measuring currents here really means multiplying Λ by the boundary potential vector and reading off the corresponding entries in the resulting current vector.

For the next sequence of calculations refer to figure 2. We place 0 currents at nodes 7 through 10 and 0 potentials at all boundary nodes except nodes 15 and 2. By Cauchy-continuing the potentials (using Kirchoff's law) we obtain 0 potentials in the interior. And by fixing node 15 to a potential of 1 we are forced to accept a (yet to be measured) potential α_1 at node 2. α_1 could be physically measured but is better found using Λ and an a priori 0 boundary current, say entry 8 in the current vector:

with $\Lambda = \begin{pmatrix} \Lambda_{1,1} & \cdots & \Lambda_{1,n} \\ \Lambda_{n,1} & & \Lambda_{n,n} \end{pmatrix}$ potential $= \begin{pmatrix} 0 \\ \alpha_1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $I = \begin{pmatrix} i_1 \\ \vdots \\ i_7 \\ 0 \\ 0 \\ i_{..} \\ \vdots \\ i_{15} \end{pmatrix}$

have $\Lambda_{8,1} \cdot 0 + \Lambda_{8,2}\,\alpha_1 + \cdots + \Lambda_{8,n-1} \cdot 1 + \Lambda_{8,n} \cdot 0 = 0$

$\Rightarrow \quad \alpha_1 = -\Lambda_{8,n-1} / \Lambda_{8,2}$

Next applying Kirchoff's law at node 16 we encode information about γ_4 and γ_8, i.e.

(1) $\gamma_8(1-0) + \gamma_4(\alpha_1 - 0) = 0$

or $\gamma_8 + \alpha_1 \gamma_4 = 0$

Thus we obtain one equation in two unknowns. We try to alleviate this crisis by writing down current equations at nodes 2 and 15, i.e.

$I_{15} = \gamma_1(1-0) + \gamma_3(1-\alpha_1) + \gamma_7(1-0) + \gamma_8(1-0)$

$I_2 = \gamma_2(\alpha_1 - 0) + \gamma_3(\alpha_1 - 1) + \gamma_4(\alpha_1 - 0) + \gamma_5(\alpha_1 - 0)$

These last two equations have introduced three new unknowns, γ_3, γ_5, and γ_7, but we have not yet exhausted all of our information sources. We can still read off currents at nodes 3 and 14, i.e.

$I_3 = \gamma_5(0 - \alpha_1)$

$I_{14} = \gamma_7(0 - 1)$

Adding get

Figure 1.

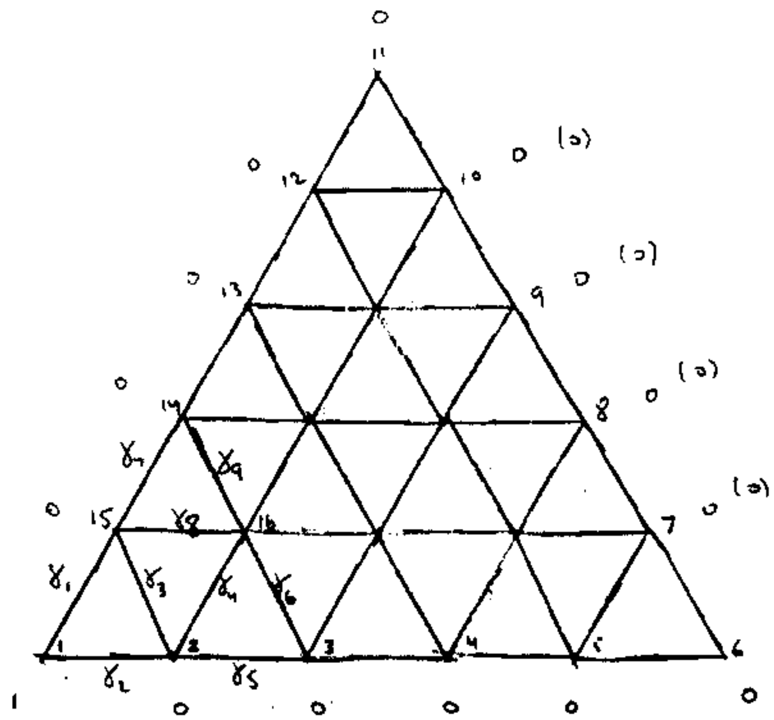
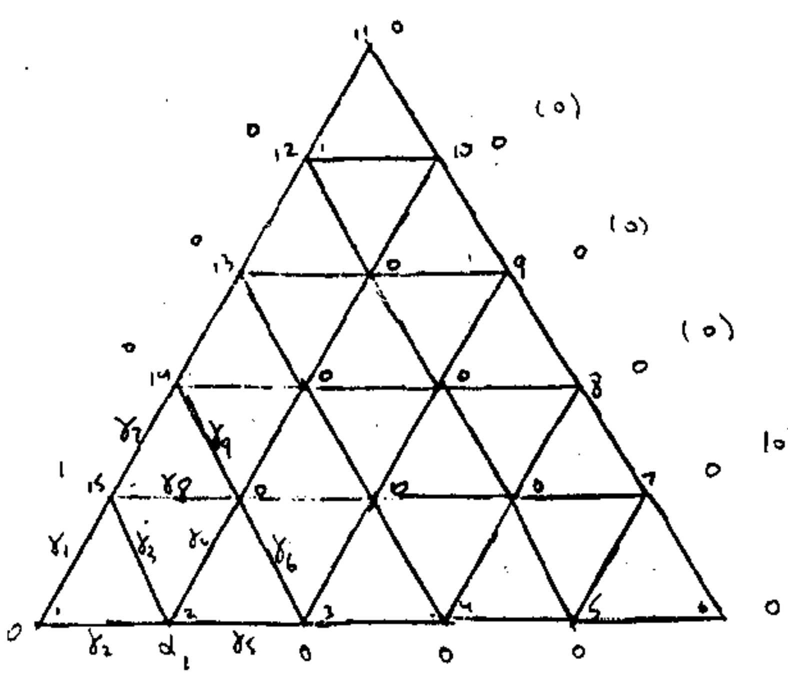


Figure 2



(2)

Furthermore, we have the current at node 1

(3) $I_1 = -(\gamma_1 + \gamma_2 \alpha_1)$

Thus adding (1), (2), and (3) get

$$I_1 + I_2 + I_3 + I_{14} + I_{15} = 0$$

i.e., the net current flowing in and out of the network is 0. Although this confirms our assumption of a steady-state network we have not solved for γ_4 and γ_8. With in our model no other physics can be employed to gain information about these conductances. Thus in essence we have shown one cannot circumvent the underdetermined nature of the triangular network using the one-function approach.

So instead we turned to the two-function approach for a hexagonal network with hexagonal boundary. Roughly speaking, the general scheme for solving this inverse problem is to first divide the hexagon into six wedges and then, starting at the boundary, penetrate to the center. Refer to figure 10 and 11. Label nodes as shown. Consider wedge I. To determine potentials in the interior it was previously thought that the currents at nodes 9 through 17 must all be set to 0. This introduced a more overdetermined system than is necessary. In fact we only have to set 0 currents at nodes 10 through 16 for all computations. This will improve the accuracy if using a least squares solver. At nodes 6 through 24 set the boundary potentials to 0 and at node 1 set the boundary potential to 1. This forces potentials α_1 through α_4 at nodes 2 through 5 and 0 potentials at all interior nodes. Apply Kirchoff's law at nodes 25 through 28, e.g.

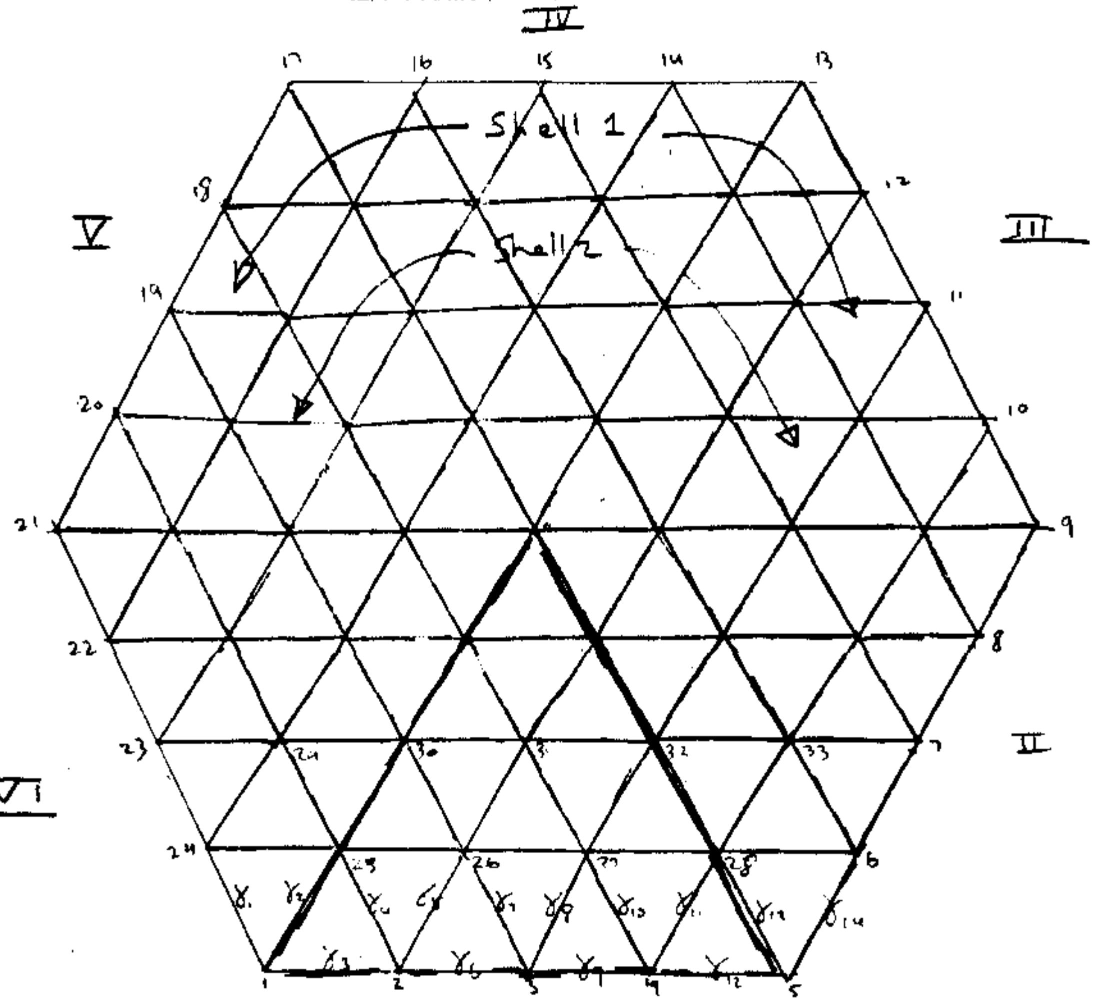
(10) $(1-0)\gamma_2 + (\alpha_1 - 0)\gamma_4 = 0$ (Two unknowns in one equation)

In accordance with the two-function approach set the potential at node 5 to 1 forcing potentials β_1 through β_4 at nodes 1 through 4. Again applying Kirchoff's law at the same nodes as before, e.g.

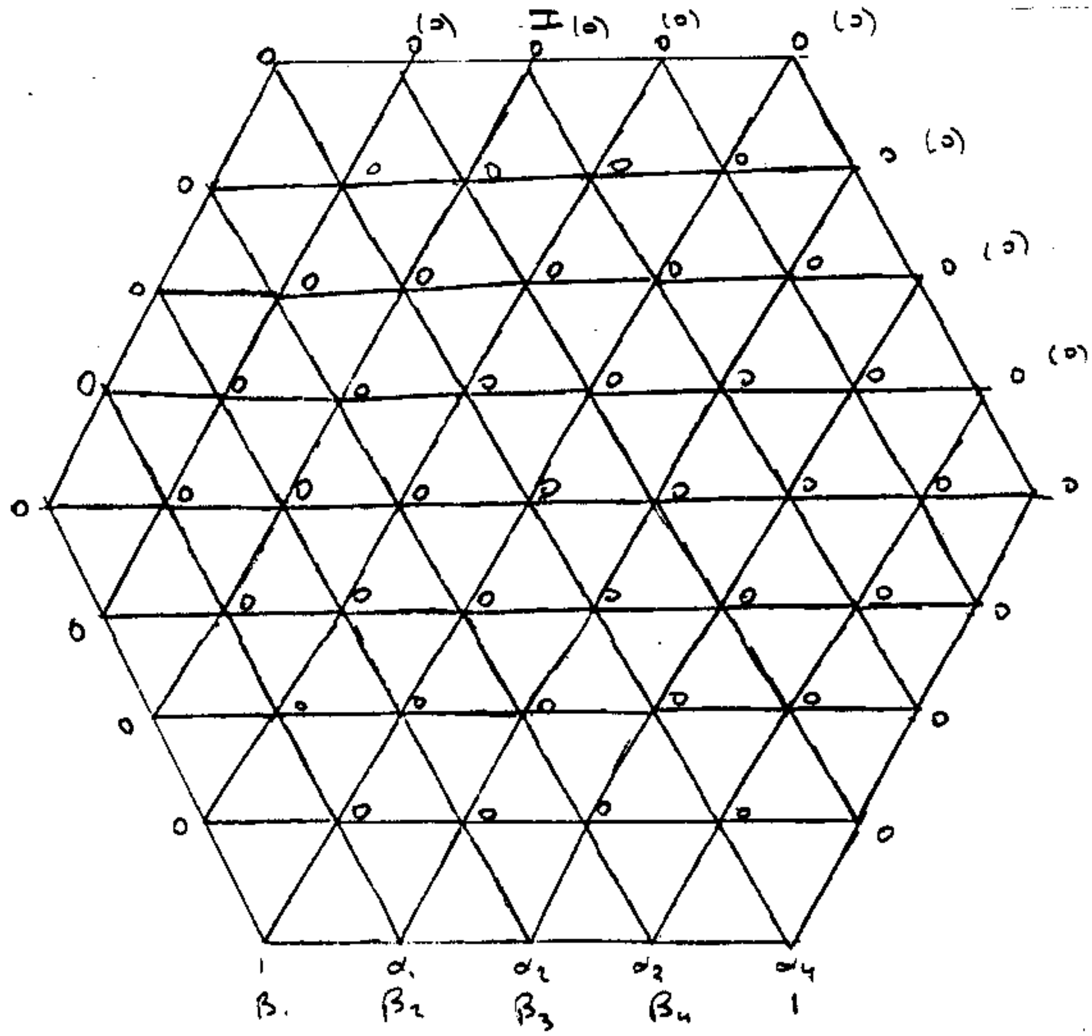
(11) $(\beta_1 - 0)\gamma_2 + (\beta_2 - 0)\gamma_4 = 0$

Solving (10) and (11) simultaneously we obtain γ_2 and γ_4. Next measure current at node 24 to find γ_1, i.e.

gure 10



gure 11



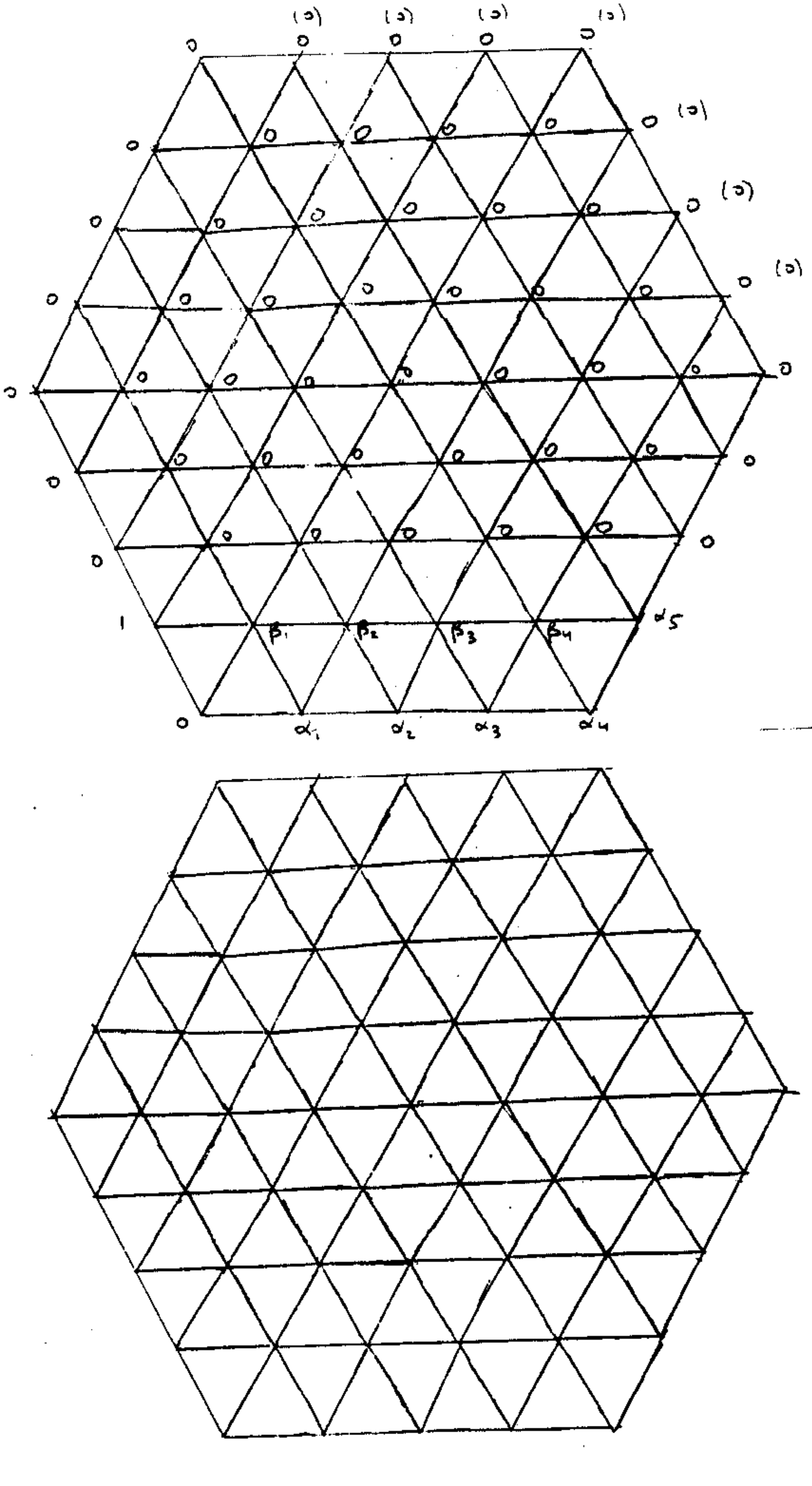
$$I_{24} = (0-1)\gamma_1$$

Also $I_1 = (1-0)\gamma_1 + (1-0)\gamma_2 + (1-\alpha_1)\gamma_3$

This determines γ_3. Measuring the current at node 2 will yield γ_6 since γ_2, γ_3, γ_4, and γ_5 are already known. Proceeding in a similar manner we obtain the remaining conductances in shell 1. For shell 2 we alter the boundary potential as shown in figure 12 and employ similar techniques only now we have to contend with nonzero interior potentials β_1 through β_4 at nodes 25 through 28.

Algorithms had been developed to run this scheme but computer account problems prevented the completion of a computer implementation.

Figure 12


(0) (0) (0) (0)
(0)
(0)
(0)
1
β_1 β_2 β_3 β_4 α_5
0
α_1 α_2 α_3 α_4