The Geometry of the Manifold of the Dirichlet-Neumann Map for a Square Network of Resistors

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Abstract

For a square network of resistors, with n nodes per side, the Dirichlet-Neumann map is a linear map which can be given in terms of the conductances of the resistors. Thus, the set of all such mappings forms a manifold that is embedded in the set of matrices. The geometry of this manifold when $n = 1$ will be examined.

1 Introduction

1.1 The $n \times n$ array

Consider a square array of resistors, with n boundary nodes on each side of the array. An example where $n = 5$ is shown in Figure 1. It can be seen that there are n^2 interior nodes, $4n$ boundary nodes, and $2n(n+1)$ resistors. Each resistor has a conductivity γ_i ($i = 1, 2, \ldots\ 2n(n+1)$). For convenience, the conductances of all of the resistors in a given network shall simply be referred to as the conductance γ of the network.

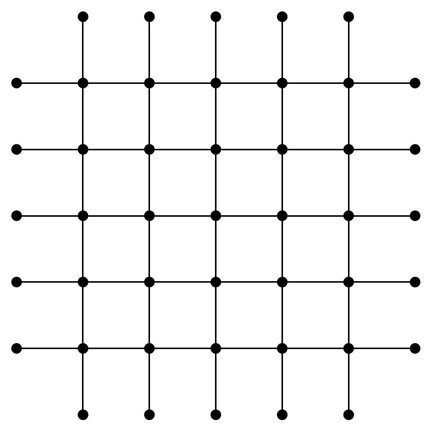


Figure 1

For a given network with conductance γ, there exists a map $\Lambda = \Lambda_\gamma$ from the set of voltages at the boundary nodes to the set of currents at the boundary nodes. This map is linear and is dependent on the conductivities of the resistors. As it is linear, Λ can be represented as matrix $\Lambda = [\lambda_{i,j}]$. Thus, we find the relation

$$\vec{v} = \Lambda \vec{b}$$

where $\vec{v}$ is the vector of boundary currents and $\vec{b}$ is the vector of boundary voltages. Note also, that this relation can give an interpetation to the $\lambda_{i,j}$: $\lambda_{i,j}$ is the current that results at the ith boundary node when a voltage of 1 is applied at boundary node j and zero at all other boundary nodes.

In Curtis and Morrow[1], the following properties of the matrix Λ_γ are given:

(R1) Let k be an integer with $1 \leq k \leq n$, and take $m = 4n - k + 1$. Then there is a unique set of numbers $\alpha_1, \alpha_2, \ldots \alpha_k$ such that for each i with $k < i < m$,

$$\lambda_{i,m} + \sum_{j=1}^{k} \lambda_{i,j}\alpha_j = 0$$

A similar relation holds for any node in any face, and columns from faces either clockwise or anticlockwise from that node.

(R2) Λ_γ is symmetric: $\lambda_{i,j} = \lambda_{j,i}$.

(R3) For each $i = 1, 2, \ldots 4n$,

$$\sum_{j=1}^{4n} \lambda_{i,j} = 0$$

Proofs of these properties are given in Curtis and Morrow[1]. Another property of any matrix Λ_γ is that $\lambda_{i,j} > 0$ for $i = j$ and $\lambda_{i,j} < 0$ when $i \neq j$.

1.2 The case n=1: background

We can now examine what this case looks like. A diagram of the network for this case can be seen in Figure 2.

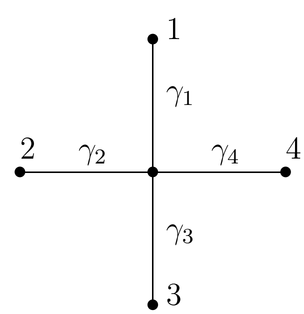


Figure 2

Λ will be a 4×4 matrix in this case, as shown in Figure 3. As there are only four γ_i's, the set of all Dirichlet-Neumann mappings for this case forms a four-dimensional manifold embedded within the set of 4×4 matrices, which has sixteen dimensions. However, because all such matrices Λ are symmetric, there are at most ten independent components.However, the rows of any Λ_γ add up to zero. Hence, from (R2) and (R3), we find that the set of matrices

$\Lambda = \Lambda_\gamma$ exists within a six dimensional subspace of the space of 4×4 matrices, corresponding to the fact that there are six matrix elements above the main diagonal. (See Figure 3.) This means that the four dimensional manifold M_Λ of matrices $\Lambda = \Lambda_\gamma$ can be examined in six dimensions. This is what will be examined in this paper.

$\Lambda =$

•	∗	∗	∗
•	•	∗	∗
•	•	•	∗
•	•	•	•

Figure 3

$*$ = elements that are independent by (R2) and (R3).

2 Parametrization of M_Λ

2.1 Parametization in terms of γ

Using Kirchoff's Law at the interior node, we can find the values of these six independent elements in terms of γ. Kirchoff's Law states that the sum of all currents going into that node must add up to zero, with outgoing currents being considered as negative. For purposes of this discussion, we will consider the interior node to be point 0 and the exterior nodes will be numbered as they are in Figure 2. Given this and Ohm's Law, Kirchoff's Law becomes

$$\gamma_1(u(1) - u(0)) + \gamma_2(u(2) - u(0)) + \gamma_3(u(3) - u(0)) + \gamma_4(u(4) - u(0)) = 0$$

where u(P) is the voltage at point P. This equation can be rewritten as

$$\sum_{i=1}^{4} \gamma_i u(i) = u(0) \sum_{i=1}^{4} \gamma_i \tag{1}$$

Thus, when we put a voltage at node k and zeroes everywhere else, we find

$$\lambda_{j,k} \quad = \quad \gamma_j(u(j) - u(0))$$

$$= \gamma_j[u(j) - \frac{\gamma_k}{\gamma_1+\gamma_2+\gamma_3+\gamma_4}]$$

The second term of this expression is obtained by solving eq.(1). Thus when $j \neq k$, which would correspond to node j having a voltage of zero,

$$\lambda_{j,k} = \frac{\gamma_j \gamma_k}{\sum_{i=1}^4 \gamma_i} \tag{2}$$

This is the parametrization of M_Λ in terms of the γ.

Some important things can be noted from this parametrization. First, it follows trivially from eqs.(2) that

$$\lambda_{1,2}\lambda_{3,4} = \lambda_{1,3}\lambda_{2,4} = \lambda_{1,4}\lambda_{2,3} \tag{3}$$

since each of these products equals $\frac{\gamma_1\gamma_2\gamma_3\gamma_4}{(\sum_{i=1}^4 \gamma_i)^2}$.

Thus, if we let each axis of $\Re^6$ correspond to one of the independent components, then there are three pairs of axes such that if we take one of the pairs, and interchange the two axes that make up that pair, then M_Λ remains unchanged. A proof of this will be given later. Those pairs are

- The axes corresponding to $\lambda_{1,2}$ and $\lambda_{3,4}$
- The axes corresponding to $\lambda_{1,3}$ and $\lambda_{2,4}$
- The axes corresponding to $\lambda_{1,4}$ and $\lambda_{2,3}$

These three pairs of axes shall be referred to as symmetry pairs.

2.2 Parametrization in terms of components

2.2.1 Parametrization in terms of four components

Here and herafter, the the following conventions wiil apply:

- The x^1 axis will correspond to $\lambda_{1,2}$.
- The x^2 axis will correspond to $\lambda_{1,3}$.
- The x^3 axis will correspond to $\lambda_{1,4}$.
- The x^4 axis will correspond to $\lambda_{2,3}$.

- The x^5 axis will correspond to $\lambda_{2,4}$.
- The x^6 axis will correspond to $\lambda_{3,4}$.

This is shown in Figure 4 below.

$$\Lambda = \begin{array}{|c|c|c|c|} \hline & x^1 & x^2 & x^3 \\ \hline & & x^4 & x^5 \\ \hline & & & x^6 \\ \hline & & & \\ \hline \end{array}$$

Figure 4

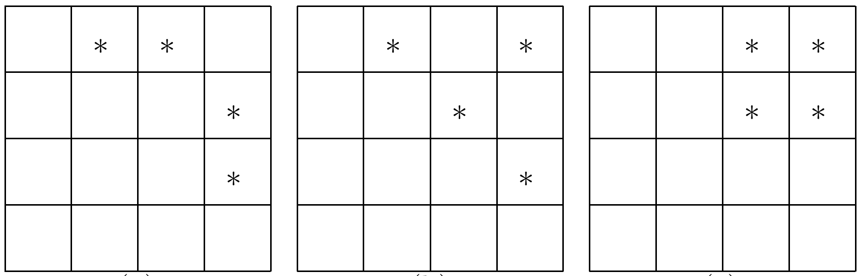
Note: It follows from this that the symmetry pairs are the pairs of axes such that their indices add up to seven.

We begin with the parametrization

$$\begin{aligned} x^1 &= u \\ x^2 &= v \\ x^3 &= w \\ x^4 &= vz/w \\ x^5 &= z \\ x^6 &= vz/u \end{aligned}$$

This corresponds to the parametrization that results from projecting the manifold onto the x^1,x^2,x^3, and x^5 axes. Equivalently, it is the parametrization that results when you are given the four elements u,v, w, and z, and solve for the other two using the relations given by (R1).

It should be noted that there are three such parametrizations in terms of four of the components, shown below, that do not form a well-defined coordinate system on M_Λ. Note that they involve two complete symmetry pairs, and don't include the other pair.



(a) (b) (c)

Figure 5

The fact that these projections aren't valid parametrizations can be seen if we look at Figure 5a. In this figure, from the parametrization, we are given x^1, x^2, x^5, and x^6. Thus, for this to be a valid parametrization, it remains to determine x^3 and x^4 in terms of those components just given. However, the only equation that we have is eq.(3), which gives us

$$x^3 x^4 = x^1 x^6 = k$$

where k is known. Thus, we have only one equation, but two variables and so the parametrization fails. A similar argument holds for Figs. 5b and 5c.

2.2.2 Parametrization in terms of three components

If we make the change of coordinates,

$$\begin{aligned} u &= u' \\ v &= v' \\ w &= w' \\ z &= k'/v' \end{aligned}$$

we get a parametrization that is a projection of M_Λ onto three axes, each of which comes from a different symmetry pair. The last coordinate, $k' = vz$, is the product of both of the coordinates in one of the symmetry pairs. By eq.(3), this product is the same for all of the symmetry pairs. Thus we can also say that $k' = (x^1 x^2 x^3 x^4 x^5 x^6)^{1/3}$. Thus, here we have a symmetry between the symmetry pairs where they can be permuted without changing the manifold. A proof of this will not given now, but it can be seen in a qualitative manner, since in this new coordinate system, the parametrization is

$$\begin{aligned} x^1 &= u \qquad & x^4 &= k/w \\ x^2 &= v \qquad & x^5 &= k/v \\ x^3 &= w \qquad & x^6 &= k/u \end{aligned}$$

where the primes that resulted from the change of coordinates have been removed.

2.2.3 The submanifold of constant k

Here we look at what results when we put on the restriction $k = 1$ in the above coordinate system. The resulting submanifold K has three dimensions and has the parametrization

$$\begin{aligned} x^1 &= u \qquad & x^4 &= 1/w \\ x^2 &= v \qquad & x^5 &= 1/v \\ x^3 &= w \qquad & x^6 &= 1/u \end{aligned}$$

The coordinate basis vectors, when embedded in $\Re^6$ that result from this are

$$\frac{\partial}{\partial u} = \vec{e_u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1/u^2 \end{bmatrix}, \quad \frac{\partial}{\partial v} = \vec{e_v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1/v^2 \\ 0 \end{bmatrix}, \quad \frac{\partial}{\partial w} = \vec{e_w} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1/w^2 \\ 0 \\ 0 \end{bmatrix}$$

¿From these, we can obtain an induced metric of

$$\tilde{g} = \begin{bmatrix} 1 + 1/u^4 & 0 & 0 \\ 0 & 1 + 1/v^4 & 0 \\ 0 & 0 & 1 + 1/w^4 \end{bmatrix}$$

Hence these are orthogonal coordinates.

Theorem 2.1 *Let K be the submanifold of M_Λ that is defined by*

$$x^1x^6 = x^2x^5 = x^3x^4 = 1$$

when embedded in $\Re^6$. Then K is flat.

Proof If we orthogonalize these coordinate basis vectors, we get a new set of basis vectors

$$\vec{e_u}' = \frac{u^2}{\sqrt{1+u^4}}\vec{e_u}, \quad \vec{e_v}' = \frac{v^2}{\sqrt{1+v^4}}\vec{e_v}, \quad \vec{e_w}' = \frac{w^2}{\sqrt{1+w^4}}\vec{e_w}$$

Upon inspection, we can see that the commutators of these three new basis vectors are all zero. For instance, if we look at $\vec{e_u}'$ and $\vec{e_v}'$, we find that

$$\begin{aligned}
[\vec{e_u}', \vec{e_v}'] &= \frac{u^2}{\sqrt{1+u^4}}\frac{\partial}{\partial u}\frac{v^2}{\sqrt{1+v^4}}\frac{\partial}{\partial v} - \frac{v^2}{\sqrt{1+v^4}}\frac{\partial}{\partial v}\frac{u^2}{\sqrt{1+u^4}}\frac{\partial}{\partial u} \\
&= \frac{u^2v^2}{\sqrt{(1+u^4)(1+v^4)}}\frac{\partial^2}{\partial u\partial v} - \frac{u^2v^2}{\sqrt{(1+v^4)(1+u^4)}}\frac{\partial^2}{\partial v\partial u} \\
&= 0
\end{aligned}$$

A similar argument applies to $\vec{e_w}'$ as well. Thus this new primed basis is also a coordinate basis, meaning that there is a parametrization for this submanifold which has a set of orthonormal vectors as its coordinate basis. Therefore, in this this parametrization, the induced metric will be the Euclidean metric. Hence, K is flat. QED.

Note: This proof has the obvious extension to any k, where $k = x^1x^6 = x^2x^5 = x^3x^4$.

2.3 An orthogonal system of coordinates

Referring back to the parametrization given in §2.2.2, if we examine the x^1-x^6 symmetry pair, then we can easily see that the coordinate lines of constant k in that plane are hyperbolas with major axis at a 45 deg angle to both the x^1 and x^6 axes. The set of hyperbolas such that $(x^1)^2 - (x^6)^2 = u =$constant are a family of curves that is orthogonal to the hyperbolas of constant k. Thus, the coordinate $u = (x^1)^2 - (x^6)^2$ should be orthogonal to the coordinate

$k = 2x^1x^6$. Hence this can give us a parametrization of the manifold that is orthogonal if we solve the set of equations:

$$k = 2x^1x^6 = 2x^2x^5 = 2x^3x^4 \tag{4}$$
$$u = (x^1)^2 - (x^6)^2 \tag{5}$$
$$v = (x^2)^2 - (x^5)^2 \tag{6}$$
$$w = (x^3)^2 - (x^4)^2 \tag{7}$$

Note that eq.(4) is the same as eq.(3) except for a factor of two. This will be explained shortly. Upon solving these equations, we get the parametrization

$$x^1 = -\sqrt{\frac{u + \sqrt{k^2 + u^2}}{2}} \qquad x^4 = -\sqrt{\frac{-w + \sqrt{k^2 + w^2}}{2}}$$
$$x^2 = -\sqrt{\frac{v + \sqrt{k^2 + v^2}}{2}} \qquad x^5 = -\sqrt{\frac{-v + \sqrt{k^2 + v^2}}{2}}$$
$$x^3 = -\sqrt{\frac{w + \sqrt{k^2 + w^2}}{2}} \qquad x^6 = -\sqrt{\frac{-u + \sqrt{k^2 + u^2}}{2}}$$

It should be noted that the minus sign in front of all of the square roots is due to the fact that all matrices Λ_γ have negative off-diagonal components.

There are two reasons why there is a factor of two in eq.(4). One is because with the factor of two, the expression under the inner radical is $k^2 + u^2$ as opposed to $4k^2 + u^2$. Secondly, if we define three complex coordinates

$$z^1 = x^1 + ix^6 \quad z^2 = x^2 + ix^5 \quad z^3 = x^3 + ix^4$$

then the manifold has a very simple interpretation: it is the set of all complex triples (z^1, z^2, z^3) such that the imaginary component of the squares of all three numbers is the same. Then the k coordinate is the imaginary part of the squares of all three numbers and the u, v, and w coordinates are the real parts.

This parametrization gives rise to the induced metric

$$\tilde{g} = \frac{1}{4}\begin{bmatrix} U^{-1/2} + V^{-1/2} + W^{-1/2} & 0 & 0 & 0 \\ 0 & U^{-1/2} & 0 & 0 \\ 0 & 0 & V^{-1/2} & 0 \\ 0 & 0 & 0 & W^{-1/2} \end{bmatrix}$$

where $U = k^2 + u^2$, $V = k^2 + v^2$, and $W = k^2 + w^2$.

¿From this, we get the following geodesic equations

$$\begin{aligned}\frac{d^2k}{d\lambda^2} &= \frac{k}{2E}(U^{-3/2} + V^{-3/2} + W^{-3/2})\left(\frac{dk}{d\lambda}\right)^2 + \frac{u}{EU^{3/2}}\left(\frac{dk}{d\lambda}\right)\left(\frac{du}{d\lambda}\right) \\ &\quad + \frac{v}{EV^{3/2}}\left(\frac{dk}{d\lambda}\right)\left(\frac{dv}{d\lambda}\right) + \frac{w}{EW^{3/2}}\left(\frac{dk}{d\lambda}\right)\left(\frac{dw}{d\lambda}\right) \\ &\quad - \frac{k}{2EU^{3/2}}\left(\frac{du}{d\lambda}\right)^2 - \frac{v}{2EV^{3/2}}\left(\frac{dv}{d\lambda}\right)^2 - \frac{w}{2EW^{3/2}}\left(\frac{dw}{d\lambda}\right)^2\end{aligned}$$

$$\frac{d^2u}{d\lambda^2} = -\frac{u}{2U}\left(\frac{dk}{d\lambda}\right)^2 + \frac{k}{U}\left(\frac{dk}{d\lambda}\right)\left(\frac{du}{d\lambda}\right) + \frac{u}{2U}\left(\frac{du}{d\lambda}\right)^2$$

$$\frac{d^2v}{d\lambda^2} = -\frac{v}{2V}\left(\frac{dk}{d\lambda}\right)^2 + \frac{k}{V}\left(\frac{dk}{d\lambda}\right)\left(\frac{dv}{d\lambda}\right) + \frac{v}{2V}\left(\frac{dv}{d\lambda}\right)^2$$

$$\frac{d^2w}{d\lambda^2} = -\frac{w}{2W}\left(\frac{dk}{d\lambda}\right)^2 + \frac{k}{W}\left(\frac{dk}{d\lambda}\right)\left(\frac{dw}{d\lambda}\right) + \frac{w}{2W}\left(\frac{dw}{d\lambda}\right)^2$$

where here $E = 4g_{uu} = U^{-1/2} + V^{-1/2} + W^{-1/2}$.

¿From this, we can obtain the curvature tensor; a list of the curvature components can be found in appendix A.

3 The symmetries of M_Λ

In §2, it was mentioned that M_Λ has symmetries both within and between the symmetry pairs. More precisely, there are isometries of $\Re^6$ which permute the axes and preserve M_Λ. The main result of this section is that these isometries are the only isometries which preserve M_Λ, and moreover, their group structure is the same as that of the symmetry group of the cube.

We first define a "symmetry" of M_Λ, as a bijective map $G : \Re^6 \to \Re^6$, such that the range of $G|_{M_\Lambda}$ is M_Λ. That is, it's some function from $\Re^6$

to $\Re^6$, that when restricted to M_Λ, simply acts like a reparametrization of M_Λ. With this, we can eliminate a very large number of isometries form the symmetry group of M_Λ right away.

3.1 Translations

In this section we consider translations. There is only one theorem concerning translations:

Theorem 3.1 *No isometry G which is a translation will preserve M_Λ.*

Proof If the original parametrization of M_Λ was the projectional parametrization given in §2.2.2, then the new parametrization after we apply G to M_Λ will be

$$\begin{array}{ll} x^1 = u + a^1 & x^4 = k/w + a^4 \\ x^2 = v + a^2 & x^5 = k/v + a^5 \\ x^3 = w + a^3 & x^6 = k/u + a^6 \end{array} \tag{8}$$

where at least one of the $a^i \neq 0$. If this were only a reparametrization of M_Λ, then there should exist a change of coordinates

$$\begin{aligned} k &= k(k', u', v', w') \\ u &= u(k', u', v', w') \\ v &= v(k', u', v', w') \\ w &= w(k', u', v', w') \end{aligned}$$

such that in the new coordinate system, the parametrization of M_Λ is

$$\begin{array}{ll} x^1 = u' & x^4 = k'/w' \\ x^2 = v' & x^5 = k'/v' \\ x^3 = w' & x^6 = k'/u' \end{array} \tag{9}$$

¿From the parametrization in eqs.(8), $u' = u + a^1$ or $u = u' - a^1$.Thus, from eqs.(8),

$$x^6 = \frac{k}{u} + a^6 = \frac{k}{u' - a^1} + a^6 \tag{10}$$

In order for eq.(10) to be true, we must have

$$k = \frac{(k' - a^6 u')(u' - a^1)}{u'} \tag{11}$$

since in eqs.(9), $x^6 = k'/u'$. From the parametrizations for x^2, we find that $v = v' - a^2$. Substituting this and eq.(11) into the parametrization for x^5 in eqs.(8), we find

$$\begin{aligned} x^5 = \frac{k}{v} + a^5 &= \frac{(k' - a^6 u')(u' - a^1)}{u'(v' - a^2)} + a^5 \\ &\neq \frac{k'}{v'} \end{aligned}$$

Hence, we do not have a coordinate system of the form given in eqs.(9), so that G does not correspond to a reparametrization of M_Λ. Thus, M_Λ is not preserved under G. QED.

Note:in this proof, it is assumed that at least one of the a^i, $(i = 1, 2, 5, 6)$ are nonzero. If all of these are zero, then it is a trivial matter to modify the proof so as to include the nonzero a^i.

3.2 Orthogonal transformations

Here we are looking at isometries of $\Re^6$ which preserve the origin, that is, rotations and reflections.

Theorem 3.2 *Any orthogonal transformation* $A = [a^i_j]$ *of* $\Re^6$ *which does not take axes into other axes, will not preserve* M_Λ.

Proof Again, we let the initial parametrization be the one given in §2.2.2. Thus, after A acts on M_Λ, the parametrization will be

$$\begin{aligned} x^1 &= a^1_1 u + a^1_2 v + a^1_3 w + a^1_4 \frac{k}{w} + a^1_5 \frac{k}{v} + a^1_6 \frac{k}{u} \\ x^2 &= a^2_1 u + a^2_2 v + a^2_3 w + a^2_4 \frac{k}{w} + a^2_5 \frac{k}{v} + a^2_6 \frac{k}{u} \\ x^3 &= a^3_1 u + a^3_2 v + a^3_3 w + a^3_4 \frac{k}{w} + a^3_5 \frac{k}{v} + a^3_6 \frac{k}{u} \end{aligned}$$

$$\begin{aligned} x^4 &= a^4_1 u + a^4_2 v + a^4_3 w + a^4_4 \frac{k}{w} + a^4_5 \frac{k}{v} + a^4_6 \frac{k}{u} \\ x^5 &= a^5_1 u + a^5_2 v + a^5_3 w + a^5_4 \frac{k}{w} + a^5_5 \frac{k}{v} + a^5_6 \frac{k}{u} \\ x^6 &= a^6_1 u + a^6_2 v + a^6_3 w + a^6_4 \frac{k}{w} + a^6_5 \frac{k}{v} + a^6_6 \frac{k}{u} \end{aligned} \tag{12}$$

where $\sum_{i=1}^{6}(a^i_j)^2 = 1$ for all j, and at least one of the nonzero $a^i_j \neq \pm 1$. Note that this last condition will require that at least four of the nonzero $a^i_j \neq \pm 1$. Again, we are looking for a change of coordinates that will take us from the parametrization given in eqs.(12) to the parametrization

$$\begin{aligned} x^1 = u' \qquad & x^4 = k'/w' \\ x^2 = v' \qquad & x^5 = k'/v' \\ x^3 = w' \qquad & x^6 = k'/u' \end{aligned} \tag{13}$$

where the primed variables are the new coordinates. Comparing eqs.(13) with eqs.(12), we can see that

$$\begin{bmatrix} u' \\ v' \\ w' \\ k'/w' \\ k'/v' \\ k'/u' \end{bmatrix} = \begin{bmatrix} a^1_1 & \cdots & a^1_6 \\ \vdots & \ddots & \vdots \\ a^6_1 & \cdots & a^6_6 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ k/w \\ k/v \\ k/u \end{bmatrix}$$

As A is an orthogonal tranformation, we can easily solve this system of equations for the unprimed coordinates

$$\begin{bmatrix} u \\ v \\ w \\ k/w \\ k/v \\ k/u \end{bmatrix} = \begin{bmatrix} a^1_1 & \cdots & a^6_1 \\ \vdots & \ddots & \vdots \\ a^1_6 & \cdots & a^6_6 \end{bmatrix} \begin{bmatrix} u' \\ v' \\ w' \\ k'/w' \\ k'/v' \\ k'/u' \end{bmatrix} \tag{14}$$

Among others, we get the equations

$$u = a^1_1 u' + a^2_1 v' + a^3_1 w' + a^4_1 \frac{k'}{w'} + a^5_1 \frac{k'}{v'} + a^6_1 \frac{k'}{u'} \tag{15}$$

$$v = a_2^1 u' + a_2^2 v' + a_2^3 w' + a_2^4 \frac{k'}{w'} + a_2^5 \frac{k'}{v'} + a_2^6 \frac{k'}{u'} \quad (16)$$

$$w = a_3^1 u' + a_3^2 v' + a_3^3 w' + a_3^4 \frac{k'}{w'} + a_3^5 \frac{k'}{v'} + a_3^6 \frac{k'}{u'} \quad (17)$$

If we substitute these into the parametrization of x^6 in eqs.(12), then we get

$$\begin{aligned}
x^6 &= a_1^6 u + a_2^6 v + a_3^6 w + a_4^6 \frac{k}{w} + a_5^6 \frac{k}{v} + a_6^6 \frac{k}{u} \\
&= a_1^6 (a_1^1 u' + a_1^2 v' + a_1^3 w' + a_1^4 \frac{k'}{w'} + a_1^5 \frac{k'}{v'} + a_1^6 \frac{k'}{u'}) \\
&\quad + a_2^6 (a_2^1 u' + a_2^2 v' + a_2^3 w' + a_2^4 \frac{k'}{w'} + a_2^5 \frac{k'}{v} + a_2^6 \frac{k'}{u'}) \\
&\quad + a_3^6 (a_3^1 u' + a_3^2 v' + a_3^3 w' + a_3^4 \frac{k'}{w'} + a_3^5 \frac{k'}{v} + a_3^6 \frac{k'}{u'}) \\
&\quad + k \left(\frac{a_4^6}{a_3^1 u' + a_3^2 v' + a_3^3 w' + a_3^4 \frac{k'}{w'} + a_3^5 \frac{k'}{v'} + a_3^6 \frac{k'}{u'}} \right. \\
&\quad + \frac{a_5^6}{a_2^1 u' + a_2^2 v' + a_2^3 w' + a_2^4 \frac{k'}{w'} + a_2^5 \frac{k'}{v'} + a_2^6 \frac{k'}{u'}} \\
&\quad \left. + \frac{a_6^6}{a_1^1 u' + a_1^2 v' + a_1^3 w' + a_1^4 \frac{k'}{w'} + a_1^5 \frac{k'}{v'} + a_1^6 \frac{k'}{u'}} \right)
\end{aligned}$$

And since, in the primed coordinates, $x^6 = k'/u'$, we have

$$k = \frac{B}{C} \quad (18)$$

where

$$\begin{aligned}
B &= k - u' [a_1^6 (a_1^1 u' + a_1^2 v' + a_1^3 w' + a_1^4 \frac{k'}{w'} + a_1^5 \frac{k'}{v'} + a_1^6 \frac{k'}{u'}) \\
&\quad + a_2^6 (a_2^1 u' + a_2^2 v' + a_2^3 w' + a_2^4 \frac{k'}{w'} + a_2^5 \frac{k'}{v} + a_2^6 \frac{k'}{u'}) \\
&\quad + a_3^6 (a_3^1 u' + a_3^2 v' + a_3^3 w' + a_3^4 \frac{k'}{w'} + a_3^5 \frac{k'}{v} + a_3^6 \frac{k'}{u'})] \\
C &= u' \left(\frac{a_4^6}{a_3^1 u' + a_3^2 v' + a_3^3 w' + a_3^4 \frac{k'}{w'} + a_3^5 \frac{k'}{v'} + a_3^6 \frac{k'}{u'}} \right. \\
&\quad + \frac{a_5^6}{a_2^1 u' + a_2^2 v' + a_2^3 w' + a_2^4 \frac{k'}{w'} + a_2^5 \frac{k'}{v'} + a_2^6 \frac{k'}{u'}}
\end{aligned}$$

$$+ \frac{a_6^6}{a_1^1 u' + a_1^2 v' + a_1^3 w' + a_1^4 \frac{k'}{w'} + a_1^5 \frac{k'}{v'} + a_1^6 \frac{k'}{u'}} \Bigg)$$

If we substitute eqs.(15-17) and eq.(18) into the parametrization of x^5 in eqs.(12), we get

$$\begin{aligned}
x^5 &= a_1^5 u + a_2^5 v + a_3^5 w + a_4^5 \frac{k}{w} + a_5^5 \frac{k}{v} + a_6^5 \frac{k}{u} \\
&= a_1^5 (a_1^1 u' + a_1^2 v' + a_1^3 w' + a_1^4 \frac{k'}{w'} + a_1^5 \frac{k'}{v'} + a_1^6 \frac{k'}{u'}) \\
&\quad + a_2^5 (a_2^1 u' + a_2^2 v' + a_2^3 w' + a_2^4 \frac{k'}{w'} + a_2^5 \frac{k'}{v} + a_2^6 \frac{k'}{u'}) \\
&\quad + a_3^5 (a_3^1 u' + a_3^2 v' + a_3^3 w' + a_3^4 \frac{k'}{w'} + a_3^5 \frac{k'}{v} + a_3^6 \frac{k'}{u'}) \\
&\quad + \frac{B}{C} \Bigg(\frac{a_4^6}{a_3^1 u' + a_3^2 v' + a_3^3 w' + a_3^4 \frac{k'}{w'} + a_3^5 \frac{k'}{v'} + a_3^6 \frac{k'}{u'}} \\
&\quad + \frac{a_5^6}{a_2^1 u' + a_2^2 v' + a_2^3 w' + a_2^4 \frac{k'}{w'} + a_2^5 \frac{k'}{v'} + a_2^6 \frac{k'}{u'}} \\
&\quad + \frac{a_6^6}{a_1^1 u' + a_1^2 v' + a_1^3 w' + a_1^4 \frac{k'}{w'} + a_1^5 \frac{k'}{v'} + a_1^6 \frac{k'}{u'}} \Bigg) \\
&\neq \frac{k'}{v'}
\end{aligned}$$

as long as some of the nonzero a_j^i are not equal to ± 1. Thus, an isometry of $\Re^6$ which doesn't take axes into other axes does not preserve M_Λ as it doesn't correspond to a change of coordinates on M_Λ. QED.

A corollary to this is that the symmetry group for M_Λ must be finite.

We can now look at orthogonal transformations $[a_j^i]$ where all of the nonzero elements of the transformation matrix are ± 1. Note that in such a matrix each row and each column has exactly one nonzero element. Our next lemma concerns the signs of all such elements.

Lemma 3.3 *All of the nonzero elements of an orthogonal transformation* $[a_j^i]$ *which preserves* M_Λ *are equal to one.*

Proof We know from the previous lemma that all of the nonzero elements of an orthogonal transformation $A = [a_j^i]$ which preserves M_Λ are equal to

± 1. Suppose that at least one of those is equal to -1. We also know from §1.1 that the parametrizations of all of the x^i must always have a negative value. But, if one of the nonzero elements of A were -1, then that would correspond to one of the x^i having a positive value in its parametrization after A had been applied. Thus, the manifold would only be preserved if all of the nonzero elements of A had a value of one. QED.

The implication of this lemma is that the only orthogonal transformations in $\Re^6$ which preserve M_Λ are those which permute the axes of $\Re^6$. Thus we have the following theorem.

Theorem 3.4 *An orthogonal transformation A preserves M_Λ if and only if for every symmetry pair, both of the axes in that pair are mapped to the same pair under A.*

Definition 1 *The axes x^1,x^2, and x^3 shall be called corresponding axes within the symmetry pairs. The axes x^4,x^5,and x^6 are also corresponding axes. The axes x^1,x^2,x^3 shall be said to not correspond to the axes x^4, x^5, and x^6.*

Note that in each set of three corresponding axes, there is one axis from each of the three symmetry pairs. Also, from here on, the word transformation will refer to orthogonal transformations.

Definition 2 *The u-pair is the x^1-x^6 symmetry pair. The v-pair is the x^2-x^5 symmetry pair. The w-pair is the x^3-x^4 symmetry pair.*

Proof We will use the parametrization given in §2.2.2. Suppose that a given transformation A sends both axes in each symmetry pair to the same pair. First, we suppose that each axis gets sent to a corresponding axis and let the m-pair be sent to the n-pair. Then the coordinate change

$$k = k' \qquad n = m' \tag{19}$$

for each m $(m = u, v, w)$ will recover the parametrization given in eqs.(9). Now suppose that the m-pair gets mapped to the n-pair, but the axes in the m-pair don't get mapped to their corresponding axes in the n-pair. Then first, we make the coordinate change

$$k = k' \qquad n = k'/n'$$

This will restore the two axes to their original sets of corresponding axes. We do this for each such pair of axes which didn't get mapped to their corresponding axes under A. We then make the coordinate change in eqs.(19) to get the original parametrization. This completes the proof in one direction.

Example Take the isometry

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

After applying A to M_Λ, we have the parametrization

$$\begin{aligned} x^1 &= v & \qquad x^4 &= u \\ x^2 &= w & \qquad x^5 &= k/w \\ x^3 &= k/u & \qquad x^6 &= k/v \end{aligned}$$

First note that the u-pair is mapped to the w-pair under A, but that the two axes in the u-pair didn't get mapped to their corresponding axes in the w-pair. So first, we make the change of coordinates

$$\begin{aligned} k &= k' & \qquad u &= k'/u' \\ v &= v' & \qquad w &= w' \end{aligned}$$

This gives us the parametrization

$$\begin{aligned} x^1 &= v & \qquad x^4 &= k/u \\ x^2 &= w & \qquad x^5 &= k/w \\ x^3 &= u & \qquad x^6 &= k/v \end{aligned}$$

Here we also note that A maps the v-pair to the u-pair and the w-pair to the v-pair. Thus, we now make the coordinate change

$$\begin{aligned} k &= k' & \qquad u &= w' \\ v &= u' & \qquad w &= v' \end{aligned}$$

This yields the parametrization

$$\begin{aligned} x^1 &= u' \qquad & x^4 &= k'/w' \\ x^2 &= v' \qquad & x^5 &= k'/v' \\ x^3 &= w' \qquad & x^6 &= k'/u' \end{aligned}$$

which is the parametrization that we are after.

We must now prove the theorem going in the opposite direction. We now assume that we have an orthogonal transformation A which preserves M_Λ. This means that the parametrization after we apply A to M_Λ will still satisfy (R1). Hence, from eq.(3), the parametrization will still satisfy the equation

$$k = x^1x^6 = x^2x^5 = x^3x^4 \tag{20}$$

after we apply A. Suppose that the axes which get mapped to the l-pair come from the m-pair and the n-pair and that the axis x^i was in the l-pair before A was applied and had the parametrization $x^i = l$. Depending on which two axes are mapped to the l-pair, there are four possibilities for the product x^ix^{7-i} after we apply A. They are

(i) $x^ix^{7-i} = mn$

(ii) $x^ix^{7-i} = \frac{km}{n}$

(iii) $x^ix^{7-i} = \frac{kn}{m}$

(iv) $x^ix^{7-i} = \frac{k^2}{mn}$

If $m \neq n$, then none of these are equal to k and eq.(20) is not satisfied. Thus, A does not preserve M_Λ, which is a contradiction. If $m = n$ for all l, then only two of these possibilities exist, namely cases (ii) and (iii). In both of these cases $x^ix^{7-i} = k$ and hence M_Λ is preserved. Hence, if A preserves M_Λ, then it takes both axes of a given symmetry pair to the same symmetry pair. QED.

3.3 The structure of the symmetry group of M_Λ

3.3.1 Preliminaries: the symmetry group of the cube

The goal of this section is to show that the set of isometries G_Λ which preserve M_Λ has the same group structure as that of the cube. But, before we can do that we must discuss the symmetry group of the cube itself.

First, we note that within the symmetry group G_C of the cube, there exists for each face f of the cube a subgroup G_f of G_C which preserves the face f. In Sullivan[3], it is shown that G_f is the same as D_f, the dihedral group for the face f, which in this case happens to be equal to D_4. Since $o(G_C)/o(D_4)$ =number of faces of the cube, which equals 6, and since $o(D_4) = 8$, we have the result that $o(G_C) = 48$.

There is another way to look at the symmetry group of the cube, which is as follows: orient the cube so that its center is at the origin and each of the three coordinate axes pass through the centers of its faces. We have three symmetries when we look at the reflections of the cube through the x-y, the x-z, and the y-z planes. These transformations are equivalent to the mappings $z \to -z$, $y \to -y$, and $x \to -x$, respectively. These transformations are commutative with each other and they each have an order of two. Thus if we take the group H which consists of all possible combinations of these transformations, then $o(H) = 8$.

We have another symmetry by looking at the transformations which permute the axes but don't reverse their directions. These transformations form a group K which is the same as the symmetric group S_3. If we look at all possible compositions of transformations from K with transformations from H, then we find that there are 48 possible. Hence, as $o(G_C) = 48$, we have the result that

$$G_C = KH \tag{21}$$

3.3.2 The isomorphism between G_Λ and G_C

Theorem 3.5 *The groups G_Λ and G_C are isomrphic to each other.*

Proof When we look at M_Λ, we find that there are two types of symmetries: those transformations which interchange the the two axes within a symmetry pair and those transformations which permute the symmetry pairs among each other. The transformations which interchange the two axes within a symmetry pair map both axes of a given symmetry pair to that same symmetry pair, except that they don't send the axes to their corresponding axes. Each of these transformations commute with each other, so that if we take the group F which consists of all compositions of these transformations, then we find that it has order eight. If we compare H to F, then it is fairly easy to see that they are isomorphic to each other; just take the mapping which maps an interchange of axes in $\Re^6$ to a reversal of direction of an axis in $\Re^3$. Next, we look at the set of transformations which permute the symmetry pairs among each other and for each symmetry pair, each axis gets taken to a corresponding axis in some symmetry pair. This group L is also isomorphic to S_3. Since the elements of L preserve the ordering of the axes within the symmetry pairs, we can write L as

$$L = G_\Lambda / F$$

the set of cosets of L. Hence we can write

$$G_\Lambda = LF$$

Since L is isomorphic to S_3 and hence to K, and F is isomorphic to H, it follows that G_Λ is isomorphic to G_C from eq.(21). QED.

A Curvature components

Here is a table of the nonzero curvature components in the coordinate system given in §2.3.

A.1 Riemann curvature tensor $R^{\alpha}{}_{\beta\gamma\delta}$

Note:

$$U = k^2 + u^2, \quad V = k^2 + v^2, \quad W = k^2 + w^2$$

$$E = g_{11} = \frac{1}{4}(U^{-1/2} + V^{-1/2} + W^{-1/2})$$

$$R^{k}{}_{uku} = \frac{1}{8EU^{3/2}}\left[\frac{1}{8E}(U^{-1/2} + k^2(V^{-3/2} + W^{-3/2})) - \frac{1}{2}\right]$$

$$R^{k}{}_{ukv} = \frac{uv}{64E^2U^{3/2}V^{3/2}}$$

$$R^{k}{}_{ukw} = \frac{uw}{64E^2U^{3/2}W^{3/2}}$$

$$R^{k}{}_{uuk} = -\frac{1}{8EU^{3/2}}\left[\frac{1}{8E}(U^{-1/2} + k^2(V^{-3/2} + W^{-3/2})) - \frac{1}{2}\right]$$

$$R^{k}{}_{uuv} = -\frac{kv}{64E^2U^{3/2}V^{3/2}}$$

$$R^{k}{}_{uuw} = -\frac{kw}{64E^2U^{3/2}W^{3/2}}$$

$$R^{k}{}_{uvk} = -\frac{uv}{64E^2U^{3/2}V^{3/2}}$$

$$R^k{}_{uvu} = \frac{kv}{64E^2U^{3/2}V^{3/2}}$$

$$R^k{}_{uwk} = -\frac{uw}{64E^2U^{3/2}W^{3/2}}$$

$$R^k{}_{uwu} = \frac{kw}{64E^2U^{3/2}W^{3/2}}$$

$$R^k{}_{vku} = \frac{uv}{64E^2U^{3/2}V^{3/2}}$$

$$R^k{}_{vkv} = \frac{1}{8EV^{3/2}}\left[\frac{1}{8E}(V^{-1/2} + k^2(U^{-3/2} + W^{-3/2})) - \frac{1}{2}\right]$$

$$R^k{}_{vkw} = \frac{vw}{64E^2V^{3/2}W^{3/2}}$$

$$R^k{}_{vuk} = -\frac{uv}{64E^2U^{3/2}V^{3/2}}$$

$$R^k{}_{vuv} = \frac{ku}{64E^2U^{3/2}V^{3/2}}$$

$$R^k{}_{vvk} = -\frac{1}{8EV^{3/2}}\left[\frac{1}{8E}(V^{-1/2} + k^2(U^{-3/2} + W^{-3/2})) - \frac{1}{2}\right]$$

$$R^k{}_{vvu} = -\frac{ku}{64E^2U^{3/2}V^{3/2}}$$

$$R^k{}_{vvw} = -\frac{kw}{64E^2V^{3/2}W^{3/2}}$$

$$R^k{}_{vwk} = -\frac{vw}{64E^2V^{3/2}W^{3/2}}$$

$$R^k{}_{vwv} = \frac{kw}{64E^2V^{3/2}W^{3/2}}$$

$$R^k{}_{wku} = \frac{uw}{64E^2U^{3/2}W^{3/2}}$$

$$R^k{}_{wkv} = \frac{vw}{64E^2v^{3/2}W^{3/2}}$$

$$R^k{}_{wkw} = \frac{1}{8EW^{3/2}}\left[\frac{1}{8E}(W^{-1/2} + k^2(U^{-3/2} + V^{-3/2})) - \frac{1}{2}\right]$$

$$R^k{}_{wuk} = -\frac{uw}{64E^2U^{3/2}W^{3/2}}$$

$$R^k{}_{wuw} = \frac{ku}{62E^2U^{3/2}W^{3/2}}$$

$$R^k{}_{wvk} = -\frac{vw}{64E^2V^{3/2}W^{3/2}}$$

$$R^k{}_{wvw} = \frac{kv}{64E^2V^{3/2}W^{3/2}}$$

$$R^k{}_{wwk} = -\frac{1}{8EW^{3/2}}\left[\frac{1}{8E}(W^{-1/2} + k^2(U^{3/2} + V^{-3/2})) - \frac{1}{2}\right]$$

$$R^k{}_{wwu} = -\frac{ku}{64E^2U^{3/2}W^{3/2}}$$

$$R^k{}_{wwv} = -\frac{kv}{64E^2V^{3/2}W^{3/2}}$$

$$R^u{}_{kku} = -\frac{1}{2U}\left[\frac{1}{8E}(U^{-1/2} + k^2(V^{-3/2} + W^{-3/2})) - \frac{1}{2}\right]$$

$$R^u{}_{kkv} = -\frac{uv}{16EUV^{3/2}}$$

$$R^u{}_{kkw} = -\frac{uw}{16EUW^{3/2}}$$

$$R^u{}_{kuk} = \frac{1}{2U}\left[\frac{1}{8E}(U^{-1/2} + k^2(V^{-3/2} + W^{-3/2})) - \frac{1}{2}\right]$$

$$R^u{}_{kuv} = \frac{kv}{16EUV^{3/2}}$$

$$R^u{}_{kuw} = \frac{kw}{16EUW^{3/2}}$$

$$R^u{}_{kvk} = \frac{uv}{16EUV^{3/2}}$$

$$R^u{}_{kvu} = -\frac{kv}{16EUV^{3/2}}$$

$$R^u{}_{kwk} = \frac{uw}{16EUW^{3/2}}$$

$$R^u{}_{kwu} = -\frac{kw}{16EUW^{3/2}}$$

$$R^u{}_{vku} = -\frac{kv}{16EUV^{3/2}}$$

$$R^u{}_{vkv} = \frac{ku}{16EUV^{3/2}}$$

$$R^u{}_{vuk} = \frac{kv}{16EUV^{3/2}}$$

$$R^{u}{}_{vuv} = -\frac{k^2}{16EUV^{3/2}}$$

$$R^{u}{}_{vvk} = -\frac{ku}{16EUV^{3/2}}$$

$$R^{u}{}_{vvu} = \frac{k^2}{15EUV^{3/2}}$$

$$R^{u}{}_{wku} = -\frac{kw}{16EUW^{3/2}}$$

$$R^{u}{}_{wkw} = \frac{ku}{16EUW^{3/2}}$$

$$R^{u}{}_{wuk} = \frac{kw}{16EUW^{3/2}}$$

$$R^{u}{}_{wuw} = -\frac{k^2}{16EUW^{3/2}}$$

$$R^{u}{}_{wwk} = -\frac{ku}{16EUW^{3/2}}$$

$$R^{u}{}_{wwu} = \frac{k^2}{16EUW^{3/2}}$$

$$R^{v}{}_{kku} = -\frac{uv}{16EU^{3/2}V}$$

$$R^{v}{}_{kkv} = -\frac{1}{2V}\left[\frac{1}{8E}(V^{-1/2} + k^2(U^{-3/2} + W^{-3/2})) - \frac{1}{2}\right]$$

$$R^v{}_{kkw} = -\frac{vw}{16EVW^{3/2}}$$

$$R^v{}_{kuk} = \frac{uv}{16EU^{3/2}V}$$

$$R^v{}_{kuv} = -\frac{ku}{16EU^{3/2}V}$$

$$R^v{}_{kvk} = \frac{1}{2V}\left[\frac{1}{8E}(V^{-1/2} + k^2(U^{-3/2} + W^{-3/2})) - \frac{1}{2}\right]$$

$$R^v{}_{kvu} = \frac{ku}{16EU^{3/2}V}$$

$$R^v{}_{kvw} = \frac{kw}{16EVW^{3/2}}$$

$$R^v{}_{kwk} = \frac{vw}{16EVW^{3/2}}$$

$$R^v{}_{kwv} = -\frac{kw}{16EVW^{3/2}}$$

$$R^v{}_{uku} = \frac{kv}{16EU^{3/2}V}$$

$$R^v{}_{ukv} = -\frac{ku}{16EU^{3/2}V}$$

$$R^v{}_{uuk} = -\frac{kv}{16EU^{3/2}V}$$

$$R^v{}_{uuv} = \frac{k^2}{16EU^{3/2}V}$$

$$R^v{}_{uvk} = \frac{ku}{16EU^{3/2}V}$$

$$R^v{}_{uvu} = -\frac{k^2}{16EU^{3/2}V}$$

$$R^v{}_{wkv} = -\frac{kw}{16EVW^{3/2}}$$

$$R^v{}_{wkw} = \frac{kv}{16EVW^{3/2}}$$

$$R^v{}_{wvk} = \frac{kw}{16EVW^{3/2}}$$

$$R^v{}_{wvw} = -\frac{k^2}{16EVW^{3/2}}$$

$$R^v{}_{wwk} = -\frac{kv}{16EVW^{3/2}}$$

$$R^v{}_{wwv} = \frac{k^2}{16EVW^{3/2}}$$

$$R^w{}_{kku} = -\frac{uw}{16EU^{3/2}W}$$

$$R^w{}_{kkv} = -\frac{vw}{16EV^{3/2}W}$$

$$R^w{}_{kkw} = -\frac{1}{2W}\left[\frac{1}{8E}(W^{-1/2} + k^2(U^{-3/2} + V^{-3/2})) - \frac{1}{2}\right]$$

$$R^w{}_{kuk} = \frac{uw}{16EU^{3/2}W}$$

$$R^{w}{}_{kuw} = -\frac{ku}{16EU^{3/2}W}$$

$$R^{w}{}_{kvk} = \frac{vw}{16EV^{3/2}W}$$

$$R^{w}{}_{kvw} = -\frac{kv}{16EV^{3/2}W}$$

$$R^{w}{}_{kwk} = \frac{1}{2W}\left[\frac{1}{8E}(W^{-1/2} + k^2(U^{-3/2} + V^{-3/2})) - \frac{1}{2}\right]$$

$$R^{w}{}_{kwu} = \frac{ku}{16EU^{3/2}W}$$

$$R^{w}{}_{kwv} = \frac{kv}{16EV^{3/2}W}$$

$$R^{w}{}_{uku} = \frac{kw}{16EU^{3/2}W}$$

$$R^{w}{}_{ukw} = -\frac{ku}{16EU^{3/2}W}$$

$$R^{w}{}_{uuk} = -\frac{kw}{16EU^{3/2}W}$$

$$R^{w}{}_{uuw} = \frac{k^2}{16EU^{3/2}W}$$

$$R^{w}{}_{uwk} = \frac{ku}{16EU^{3/2}W}$$

$$R^{w}{}_{uwu} = -\frac{k^2}{16EU^{3/2}W}$$

$$R^{w}{}_{vkv} = \frac{kw}{16EV^{3/2}W}$$

$$R^{w}{}_{vkw} = -\frac{kv}{16EV^{3/2}W}$$

$$R^{w}{}_{vvk} = -\frac{kw}{16EV^{3/2}W}$$

$$R^{w}{}_{vvw} = \frac{k^2}{16EV^{3/2}W}$$

$$R^{w}{}_{vwk} = \frac{kv}{16EV^{3/2}W}$$

$$R^{w}{}_{vwv} = -\frac{k^2}{16EV^{3/2}W}$$

A.2 The Ricci curvature tensor $R_{\mu\nu}$

Here is a list of all sixteen components of the Ricci tensor.

$$R_{kk} = \frac{1}{16E}(U^{-3/2} + V^{-3/2} + W^{-3/2}) + \frac{k^2}{16E}[U^{-1}(V^{-3/2} + W^{-3/2})$$
$$+ V^{-1}(U^{-3/2} + W^{-3/2}) + W^{-1}(U^{-3/2} + V^{-3/2})] - \frac{1}{4}(U^{-1} + V^{-1} + W^{-1})$$

$$R_{ku} = R_{uk} = \frac{ku}{16EU^{3/2}}(V^{-1} + W^{-1})$$

$$R_{kv} = R_{vk} = \frac{kv}{16EV^{3/2}}(U^{-1} + W^{-1})$$

$$R_{kw} = R_{wk} = \frac{kw}{16EW^{3/2}}(U^{-1} + V^{-1})$$

$$R_{uu} = \frac{1}{64E^2U^{3/2}}[U^{-1/2} + k^2(V^{-3/2} + W^{-3/2})] - \frac{1}{16EU^{3/2}}[1 + k^2(V^{-1} + W^{-1})]$$

$$R_{uv} = R_{vu} = \frac{uv}{64E^2U^{3/2}V^{3/2}}$$

$$R_{uw} = R_{wu} = \frac{uw}{64E^2U^{3/2}W^{3/2}}$$

$$R_{vv} = \frac{1}{64E^2V^{3/2}}[V^{-1/2} + k^2(U^{-3/2} + W^{-3/2})] - \frac{1}{16EV^{3/2}}[1 + k^2(U^{-1} + W^{-1})]$$

$$R_{vw} = R_{wv} = \frac{vw}{64E^2V^{3/2}W^{3/2}}$$

$$R_{ww} = \frac{1}{64E^2V^{3/2}}[W^{-1/2} + k^2(U^{-3/2} + V^{-3/2})] - \frac{1}{16EV^{3/2}}[1 + k^2(U^{-1} + V^{-1})]$$

A.3 The Riemann curvature invariant R

The Riemann curvature invariant is

$$R = R^\mu_\mu = g^{\mu\nu}R_{\mu\nu} = \frac{1}{8E^2}(U^{-3/2} + V^{-3/2} + W^{-3/2}) - \frac{1}{2E}(U^{-1} + V^{-1} + W^{-1})$$

$$+ \frac{k^2}{8E^2}[U^{-1}(V^{-3/2} + W^{-3/2}) + V^{-1}(U^{-3/2} + W^{-3/2}) + W^{-1}(U^{-3/2} + V^{-3/2})]$$

$$- \frac{k^2}{2E}(U^{-1}V^{-1} + U^{-1}W^{-1} + V^{-1}W^{-1})$$

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