AN ALGORITHM FOR DETERMINING RESISTORS AND CAPACITORS IN A 2-LAYER NETWORK

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Abstract

In this paper it is shown that the resistors and capacitors of a 2-layer network can be determined by measuring alternating currents and voltages at boundary nodes.

Key terms:

Resistor. A resistor is the electrical device that limits current flow in accordance with *Ohm's Law*, $\Delta u = IR$. Here, Δu represents an electric potential difference, I the current, and R the resistance. Henceforth, resistors shall be referred to as *conductors*; numerically, a conductor's *conductance*, γ, is equivalent to R^{-1}. Ohm's Law can thus be rewritten as $I = \gamma\Delta u$. All conductances to be considered are assumed to be greater than zero.

Capacitor. A capacitor is the electrical device that condenses charge. Its *capacitance*, C, relates how much charge it can collect for some Δu across its terminals. All capacitances will be assumed to be greater than zero. These voltages will be considered to be of the form $Ae^{i\omega t}$, where A is some arbitrary amplitude, $i = \sqrt{-1}$, ω stands for frequency, and t represents time. The expression $i\omega C$ is unitwise equivalent to conductance; both $i\omega C$ and γ will be termed *admittances*, a.

Network. Any network to be considered in this work is a graph of an electrical construction with conductors and capacitors as edges. Each of these conductors and capacitors must be connected to at least one other element, but not necessarily to all others. Networks will be denoted by either Ω or T.

Node. In a network, the nodes are those points that represent a terminal of at least one element. A node may be shared by the terminals of several elements.

Boundary node. A boundary node is a node that is arbitrarily chosen to have a given net current, which may be nonzero. A node that is not a boundary node is called an *interior* node.

Kirkhoff's Current Law. Kirkhoff's Current Law states that an interior node of a network must always have a net current flow of zero. Boundary nodes, by definition, are exempt from Kirkhoff's Law.

2-Layer network. A 2-layer network is one that consists of 2 layers of conductors, with capacitors connecting the nodes between them. The conductors in the top layer lead to boundary nodes, while the lower layer conductors all meet at one interior node. Figure 1 shows a 2-layer network consisting of three boundary nodes (denoted by 1, 2, and 3), four interior nodes, (denoted by 4 through 7), two layers of three conductors each, and two capacitors:

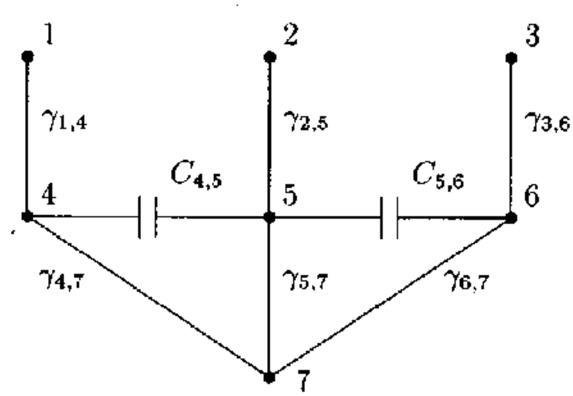

FIG. 1

This specific example of a 2-layer network will now be called the *3-prong, 2-layer network.* An *n-prong, 2-layer network*, with $n \geq 2$, consists of n

boundary nodes, $n+1$ interior nodes, $2n$ conductors, and $n-1$ capacitors. Figure 2 shows the 5-prong, 2-layer network:

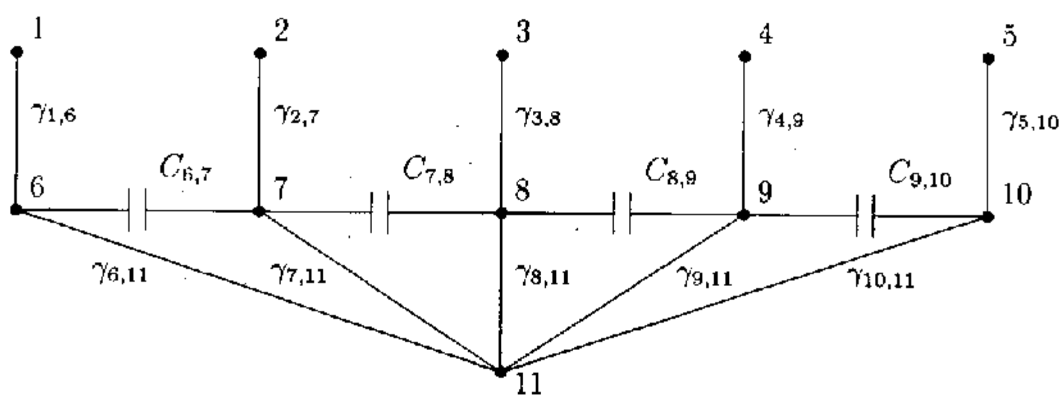

FIG. 2

Forward problem. Assume that a network is known. That is, all conductances and capacitances are known, as well as the connections between them. The forward problem uses given potentials at the network's boundary nodes to find potentials at its interior nodes which satisfy Kirkhoff's Law. By solving the forward problem, it is then possible to determine boundary currents.

Inverse problem. For an inverse problem, we are given a graph Ω with unknown values of admittances. The inverse problem uses currents and potentials at Ω's boundary nodes to determine its admittances. To "solve" an inverse problem means that all of a network's internal components can be recovered by measuring its boundary currents and potentials.

1. Preliminaries. While the inverse problem can be solved for many types of networks, there exists a relatively simple class of networks whose inverse problems do not yield unique solutions. The networks of this class have n connected spokes, n boundary nodes, $2mn$ conductors (m on each spoke, $m > 1$), and $n(m-1)+1$ interior nodes. Figure 3 shows an example similar to those in this class, ($n = 3$ and $m = 1$), which *can* be solved, and will be shortly:

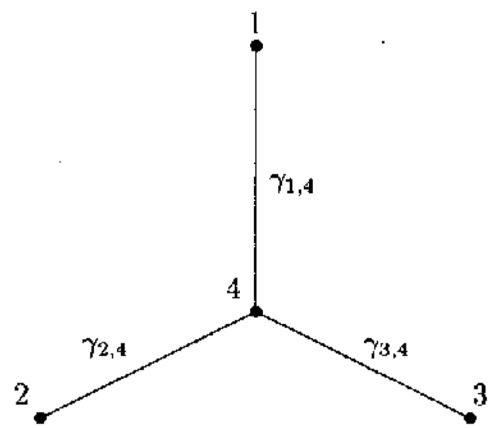

FIG. 3

In this network, to be referred to as T_1 (for *T*ripod 1), nodes 1, 2, and 3 are boundary nodes. For networks with a greater number of spokes, the boundary nodes will always be those at the end of each spoke (those that are common to only one conductor).

Figure 4 shows a network of this type with $n = 3$ and $m = 2$:

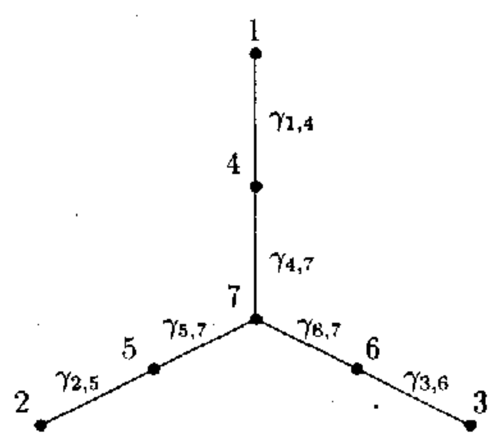

FIG. 4

The inverse problem of the network of Figure 4, T_2, can *not* be solved, because of the multiple conductors on each spoke. Any of these networks with m greater than 1 can not be solved. (It is important to note that all conductivities in these networks come from resistors. A tripod with $m > 1$ may have a solvable inverse problem if capacitors take the places of some resistors.)

The following network is of a different type for which the inverse problem can not be solved:

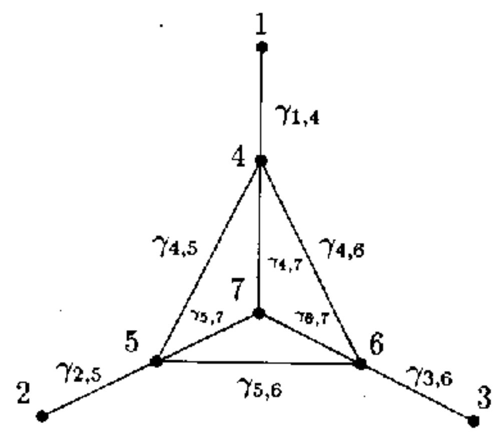

FIG. 5

If $\gamma_{4,5}$, $\gamma_{5,6}$, and $\gamma_{4,6}$ in this network were replaced with capacitors, it *would* have a solvable inverse problem. (Such a network would be very similar to the network in Figure 9 on page 10. The algorithm to be described is based on the fact that this substitution of capacitors for conductors allows such a network to have a solvable inverse problem.)

Two representative matrices exist for any network which play important roles in forward and inverse problems: the *Kirkhoff matrix*, K, and the *Lambda matrix*, Λ. The Kirkhoff matrix is square, with the number of rows (and columns) being equal to the total number of nodes in the network.

Construction of K is dependent on the indexing of the nodes. For a network with p boundary nodes and q interior nodes, the boundary nodes must be indexed from 1 to p, and the interior nodes indexed from $p+1$ to $p+q$. Each node will then correspond to one row and one column in K. K will consist of $(p+q)^2$ entries labeled $k_{i,j}$, with $i,j = 1$ to $(p+q)$.

Now, for example, consider some node n in T_1, corresponding to row n in K. The n,j entry of K will always have a value of zero, unless one of two conditions is true: either node n is directly connected to node j by a conductor, or $j = n$. If the first condition is true, $k_{n,j} = -\gamma_{n,j}$ where $\gamma_{n,j}$ is the conductance between nodes n and j. If $j = n$, $k_{n,n}$ is equal to the sum of all conductors touching node n.

A result of this construction is that the sum of all entries in any row or column of K will necessarily be zero. That is, $\Sigma_i \gamma_{i,j} = 0$ and $\Sigma_j \gamma_{i,j} = 0$. K is also symmetric, as $k_{m,n} = -\gamma_{m,n} = -\gamma_{n,m} = k_{n,m}$ for any nodes m and n in the network.

Figure 6 shows the Kirkhoff matrix for T_1:

$$K = \begin{bmatrix} \gamma_{1,4} & 0 & 0 & -\gamma_{1,4} \\ 0 & \gamma_{2,4} & 0 & -\gamma_{2,4} \\ 0 & 0 & \gamma_{3,4} & -\gamma_{3,4} \\ -\gamma_{1,4} & -\gamma_{2,4} & -\gamma_{3,4} & \sigma_4 \end{bmatrix}$$

FIG. 6

(With $\sigma_4 = \gamma_{1,4} + \gamma_{2,4} + \gamma_{3,4}$.)

The Kirkhoff matrix is now broken into four blocks, as shown by Figure 7:

$$K = \left[\begin{array}{c|c} K' & B^T \\ \hline B & A \end{array}\right]$$

FIG. 7

The break represented by the vertical line occurs between columns p and $p+1$, while the horizontal line occurs between rows p and $p+1$. In other words, the breaks separate boundary nodes and interior nodes. The part of K now called K' represents connections between boundary nodes, while that called A shows those between interior nodes. B and B^T show connections between one boundary node and one interior node. Returning to the example network of T_1, the components are:

$$K' = \begin{bmatrix} \gamma_{1,4} & 0 & 0 \\ 0 & \gamma_{2,4} & 0 \\ 0 & 0 & \gamma_{3,4} \\ & & \end{bmatrix}$$

$$B^T = \begin{bmatrix} -\gamma_{1,4} \\ -\gamma_{2,4} \\ -\gamma_{3,4} \end{bmatrix}$$

$$B = \begin{bmatrix} -\gamma_{1,4}, & -\gamma_{2,4}, & -\gamma_{3,4} \end{bmatrix}$$

$$A = [\sigma_4]$$

The other key matrix, Λ, is now calculated from K by:

$$\Lambda = K' - B^T A^{-1} B. \tag{1}$$

For T_1,

$$\Lambda = \begin{bmatrix} \gamma_{1,4} - \gamma_{1,4}^2/\sigma_4 & -\gamma_{1,4}\gamma_{2,4}/\sigma_4 & -\gamma_{1,4}\gamma_{3,4}/\sigma_4 \\ -\gamma_{1,4}\gamma_{2,4}/\sigma_4 & \gamma_{2,4} - \gamma_{2,4}^2/\sigma_4 & -\gamma_{2,4}\gamma_{3,4}/\sigma_4 \\ -\gamma_{1,4}\gamma_{3,4}/\sigma_4 & -\gamma_{2,4}\gamma_{3,4}/\sigma_4 & \gamma_{3,4} - \gamma_{3,4}^2/\sigma_4 \end{bmatrix}$$

FIG. 8

Λ shares two important characteristics of K. Λ is symmetric, and all of its rows and columns sum to zero. These two facts mean that only the off-diagonal entries of Λ are of interest. Furthermore, Λ has the same number of rows and columns as the network has boundary nodes, and the i, j entry of Λ represents the current flowing into boundary node i due to a potential of 1 applied at boundary node j.

The preceding follows the steps of the forward problem for T_1. Since we now know the form of Λ, the inverse problem–the problem of getting from boundary currents to conductances–is readily solved. We have, for T_1:

$$\lambda_{1,2} = -\gamma_{1,4}\gamma_{2,4}/\sigma_4 \quad (2)$$
$$\lambda_{1,3} = -\gamma_{1,4}\gamma_{3,4}/\sigma_4 \quad (3)$$
$$\lambda_{2,3} = -\gamma_{2,4}\gamma_{3,4}/\sigma_4 \quad (4)$$

At this point, the ratio $\gamma_{1,4}/\gamma_{2,4}$ can be found by dividing $\lambda_{1,3}$ by $\lambda_{2,3}$; $\gamma_{1,4}/\gamma_{3,4}$ and $\gamma_{2,4}/\gamma_{3,4}$ are obtained similarly. To proceed, choose a conductor to solve for, such as $\gamma_{1,4}$. The first and second of the above ratios are used to express $\gamma_{2,4}$ and $\gamma_{3,4}$ in terms of $\gamma_{1,4}$. Specifically, $\gamma_{2,4} = \gamma_{1,4}\lambda_{2,3}/\lambda_{1,3}$, and $\gamma_{3,4} = \gamma_{1,4}\lambda_{2,3}/\lambda_{1,2}$. These expressions can then be substituted into σ_4, which then factors down to $\gamma_{1,4}(1 + \lambda_{2,3}/\lambda_{1,3} + \lambda_{2,3}/\lambda_{1,2})$.

This reformulated σ_4, and the expression for $\gamma_{2,4}$ in terms of $\gamma_{1,4}$ are then substituted into Equation 2 above. That equation is now strictly in terms of $\gamma_{1,4}$ and the three lambdas. A little more algebra yields

$$\gamma_{1,4} = (\lambda_{1,2}\lambda_{1,3} + \lambda_{1,2}\lambda_{2,3} + \lambda_{1,3}\lambda_{2,3})/\lambda_{2,3}. \quad (5)$$

By the same method, it is found that

$$\gamma_{2,4} = (\lambda_{1,2}\lambda_{1,3} + \lambda_{1,2}\lambda_{2,3} + \lambda_{1,3}\lambda_{2,3})/\lambda_{1,3} \quad (6)$$

and

$$\gamma_{3,4} = (\lambda_{1,2}\lambda_{1,3} + \lambda_{1,2}\lambda_{2,3} + \lambda_{1,3}\lambda_{2,3})/\lambda_{1,2}. \quad (7)$$

Thus, this inverse problem has yielded the conductances of the three conductors of T_1 from information strictly about boundary currents. (The same can be done for any similar network with at least 3 spokes, provided it has only one conductor per spoke.) The same technique could have been applied to T_2, but there is only enough information in Λ to find the sums $(\gamma_{1,4}+\gamma_{4,7})$, $(\gamma_{2,5}+\gamma_{5,7})$, and $(\gamma_{3,6}+\gamma_{6,7})$. Specific values for the six conductances can not be determined because there are an infinite number of combinations that could yield each sum.

With the addition of two capacitors, possibly unknown, across the middle nodes of T_2, the resulting network is the 3-prong, 2-layer network of Figure 1 (now called Ω_1). Lambda matrices *can* be used to uniquely determine the six conductances of Ω_1.

2. 2-Layer Networks vs. Spoked Networks. All 2-layer networks, as defined, are structured essentially the same as the spoked networks T_1 and T_2, with the addition of $n-1$ capacitors. (n is the number of spokes in T.) All examples of these networks could have included a last, nth, capacitor, but the algorithm to be outlined works just as well if it is not present. Furthermore, by using the 2-layer networks as will be done, a geophysical application is suggested. The network Ω_1, or perhaps better the network of Figure 2, could be used to model the structure of a section of the Earth, with conductivities being analogous to depth, and capacitances being the analog of magnetic induction[1].

The property of capacitors that allows inverse problems containing them to be solved is this: when admittance is expressed in terms of capacitance, it is dependent on frequency. That is, $a = f(C,\omega)$ or, in this case, $a = i\omega C$. This implies that the Kirkhoff matrix of a network containing capacitors, as well as its Lambda matrix, also become functions of ω. (In the previous example of T_1, Λ had a guaranteed existence because A, being composed of one real entry, was invertible. An A matrix containing capacitances is not so obviously invertible, but will soon be shown to be so.)

A network consisting strictly of conductors is analyzed with the use of direct currents; therefore, there is no ω dependence and the network has only one Kirkhoff matrix and one Lambda matrix. A network containing

[1] These interpretations were brought to the author's attention by a paper titled *The Inverse Problem of Electromagnetic Induction: Existence and Construction of Solutions Based On Incomplete Data* by Robert L. Parker. Published in the *Journal of Geophysical Research, vol. 85, no. B8*, pp. 4421-4428, 1980.

capacitors and their resulting ω dependence, on the other hand, has as many K's and Λ's as an analyst has frequencies to choose from.

The algorithm to follow is a method of comparing these multitudes of Lambda matrices to determine the composition of 2-layer networks.

3. Forward Problem. Before dealing with the inverse problem of a 2-layer network, it needs to be shown that the forward problem can be solved. This, and all future calculations and proofs, will be demonstrated using the 3-prong, 2-layer network of Figure 9, to be called Ω_2:

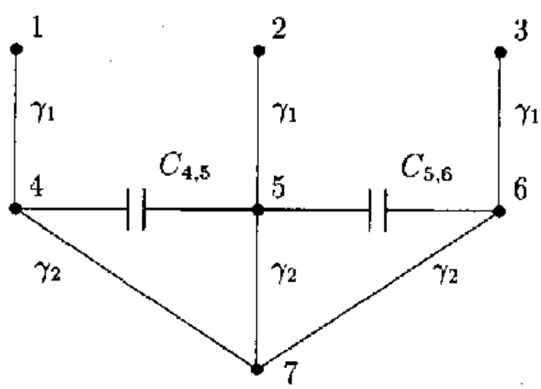

FIG. 9

Ω_2 has the same structure as Ω_1, except that rather than having six different conductors, it has only two types, with each layer consisting of one type of conductor. Anything to be done with this network can be similarly done with Ω_1. Ω_2 is preferable strictly because it will have fewer unknowns to solve for. This model is perhaps a little simple, but will suffice for our purposes.

The Kirkhoff matrix for Ω_2 is shown in Figure 10:

$$K(\omega)=\left[\begin{array}{ccc|cccc}
\gamma_1 & 0 & 0 & -\gamma_1 & 0 & 0 & 0\\
0 & \gamma_1 & 0 & 0 & -\gamma_1 & 0 & 0\\
0 & 0 & \gamma_1 & 0 & 0 & -\gamma_1 & 0\\
\hline
-\gamma_1 & 0 & 0 & \sigma_4 & -i\omega C_{4,5} & 0 & -\gamma_2\\
0 & -\gamma_1 & 0 & -i\omega C_{4,5} & \sigma_5 & -i\omega C_{5,6} & -\gamma_2\\
0 & 0 & -\gamma_1 & 0 & -i\omega C_{5,6} & \sigma_6 & -\gamma_2\\
0 & 0 & 0 & -\gamma_2 & -\gamma_2 & -\gamma_2 & 3\gamma_2
\end{array}\right]$$

FIG. 10

where:

$$\sigma_4 = \gamma_1+\gamma_2+i\omega C_{4,5} \tag{8}$$
$$\sigma_5 = \gamma_1+\gamma_2+i\omega(C_{4,5}+C_{5,6}) \tag{9}$$
$$\sigma_6 = \gamma_1+\gamma_2+i\omega C_{5,6} \tag{10}$$

The lines in $K(\omega)$ represent the borders between K', B^T, B, and A, as in Figure 7.

The question of whether or not $\Lambda(\omega)$ exists for Ω_2, or any other 2-layer network, comes down to the existence or non-existence of A^{-1}. In the example of T_1, this was easily verified because A was simply one real entry. Here, this question of $A(\omega)$'s invertibility is not so straightforward.

THEOREM 1. *$A(\omega)$ is invertible for any 2-layer network.*

Proof. Recall that $A(\omega)$ is the lower-right block of $K(\omega)$, as is shown by Figure 11:

$$A(\omega)=\begin{bmatrix}
\sigma_4 & -i\omega C_{4,5} & 0 & -\gamma_2\\
-i\omega C_{4,5} & \sigma_5 & -i\omega C_{5,6} & -\gamma_2\\
0 & -i\omega C_{5,6} & \sigma_6 & -\gamma_2\\
-\gamma_2 & -\gamma_2 & -\gamma_2 & 3\gamma_2
\end{bmatrix}$$

FIG. 11

To prove the existence of $A^{-1}(\omega)$, it is sufficient to show that the only solution to the equation:

$$A(\omega)u(\omega) = 0 \tag{11}$$

is $u = 0$; that is, $A(\omega)$ is nonsingular. Assume that $A(\omega)u(\omega) = 0$.

Consider a column vector $x(\omega)$:

$$x(\omega) = \begin{bmatrix} 0 \\ u(\omega) \end{bmatrix}$$

FIG. 12

whose entries stand for potentials at a network's nodes. The zero in $x(\omega)$ is a column vector representing a network's boundary potentials, all zero, and $u(\omega)$ is another column vector of interior potentials. Now,

$$K(\omega)x(\omega) = \begin{bmatrix} B^T u(\omega) \\ A(\omega)u(\omega) \end{bmatrix}$$
$$= \begin{bmatrix} B^T u(\omega) \\ 0 \end{bmatrix}$$

by our assumption that $A(\omega)u(\omega) = 0$.

It can be verified that:

$$\bar{x}^T(\omega)K(\omega)x(\omega) = 0, \tag{12}$$

with $\bar{x}(\omega)$ being the complex conjugate of $x(\omega)$.

Now consider a matrix $C(\omega)$, for some network, to be constructed as follows. Number its rows according to the network's nodes, and number its columns according to its edges (conducting elements). In each column l, corresponding to edge l, there will be two entries. One will appear in each row corresponding to one of the nodes that edge l connects, differing only by sign. The entries will be of the form $\pm\sqrt{a_{m,n}}$, with $a_{m,n}$ being the value of the admittance of edge l between connected nodes m and n in the network. The sign of each element does not matter, as long as one entry in each column is positive and the other is negative. Any $C(\omega)$ will look something like Figure 13, with appropriate dimensions:

$$C(\omega) = \begin{bmatrix} \sqrt{a_{i,j}} & \ldots\ldots & 0 \\ 0 & \ldots\ldots & -\sqrt{a_{p,q}} \\ -\sqrt{a_{i,j}} & \ldots\ldots & 0 \\ 0 & \ldots\ldots & \sqrt{a_{p,q}} \end{bmatrix}$$

FIG. 13

It can be verified that:

$$K(\omega) = C(\omega)C^T(\omega). \tag{13}$$

This is substituted into Equation 12 above, resulting in:

$$(\bar{x}^T C)(C^T x) = 0. \tag{14}$$

$C^T(\omega)x$ looks like Figure 14:

$$C^T(\omega)x = \begin{bmatrix} \pm\sqrt{a_{i,j}}(x_i - x_j) \\ \ldots \\ \ldots \\ \ldots \\ \pm\sqrt{a_{p,q}}(x_p - x_q) \end{bmatrix}$$

FIG. 14

Therefore,

$$0 = \bar{x}^T K x = \bar{x}^T C C^T x = \Sigma a_{i,j} |x_i - x_j|^2. \tag{15}$$

This can be broken down to:

$$0 = \Sigma Re(a_{i,j})|x_i - x_j|^2 + i\Sigma Im(a_{i,j})|x_i - x_j|^2. \tag{16}$$

All real parts of A are non-negative, since all conductivities were originally assumed to be greater than zero. The same applies to the imaginary parts. From Equation 16, it follows that $x_i = x_j$ for all i, j where $\gamma_{i,j} \neq 0$. Hence, all the x_i's must be equal to all the x_j's (for all i, j), which will then be equal to some constant. But since the x matrix (Figure 12) already has some entries equal to zero, *all* entries of x must be equal to zero. Therefore, $u = 0$ and A is invertible, concluding the proof. •

It is significant that the only conditions necessary for the above to hold is that all conductivities be greater than zero and that likewise all capacitors be greater than zero. The shape of the network and size of A make no difference.

The forward problem can therefore always be solved for 2-layer networks as considered here. $\Lambda(\omega)$ can in fact be computed for Ω_2. The entries of $\Lambda(\omega)$ will consist of ratios of polynomials of degree $n-1$ (where n is the number of spokes in the network) in ω.

Note: This $\Lambda(\omega)$ is actually a "pseudo-Lambda" matrix. The true Λ has an additional coefficient $e^{i\omega t}$ in front of each entry because of the alternating voltage being applied. (In networks with strictly real conductivities, the i, jth entry of Λ signified the current into boundary node i due to a potential of 1 at boundary node j. With the addition of capacitors, $\lambda_{i,j}$ now represents the current into boundary node i caused by a potential of $e^{i\omega t}$ at boundary node j.) The term $e_{i\omega t}$ will be divided out of each entry of $\Lambda(\omega)$ to produce the Lambda matrix to be used. (This has the additional benefit of making$\Lambda(\omega)$ independent of t).

4. Inverse Problem and Algorithm. The before-mentioned polynomials in ω, even those of small degrees, will turn out to have many more terms than are comfortable to work with, such as this entry, $\lambda_{1,2}(\omega)$, from the Lambda matrix of Ω_2:

$$\frac{-(\gamma_1^2(\gamma_1\gamma_2^2 + \gamma_2^3 + 3iC_{4,5}\gamma_1\gamma_2\omega + 2iC_{4,5}\gamma_2^2\omega + 2iC_{5,6}\gamma_2^2\omega - 3C_{4,5}C_{5,6}\gamma_2\omega^2))}{(3(\gamma_1^3\gamma_2 + 2\gamma_1^2\gamma_2^2 + \gamma_1\gamma_2^3 + 2iC_{4,5}\gamma_1^2\gamma_2\omega + 2iC_{5,6}\gamma_1^2\gamma_2\omega + 2iC_{4,5}\gamma_1\gamma_2^2\omega + 2iC_{5,6}\gamma_1\gamma_2^2\omega - 3C_{4,5}C_{5,6}\gamma_1\gamma_2\omega^2))} \quad (17)$$

While this ratio is only quadratic in ω, it is not a very simple expression. It is theoretically possible that if given one Lambda matrix of a 2-layer network, a form like the one above could be used for the $\lambda_{i,j}(\omega)$'s to calculate the network's constituent conductors and capacitors. This is not necessary, given that an alternate method exists.

This method, soon to be described, takes advantage of the fact that Λ is dependent on ω. Since the entries of Λ depend on ω, it is reasonable to think that there may be some values of ω that provide useful results.

The key aspect of the algorithm is that it manipulates these useful frequencies to solve for a network's components. Specifically, it involves taking the limits of $\Lambda(\omega)$ as ω goes to zero and infinity to find the admittances of a 2-layer network of known shape and unknown composition. The steps of the algorithm, as applied to Ω_2, are summarized as follows:

1. Compute $\Lambda(\infty)$. From this, γ_1 can be determined.
2. Look at $\Lambda(0)$. From this, γ_2 can be determined.
3. Calculate $C_{4,5}$ and $C_{5,6}$ from known potentials at Ω's interior nodes (solved from the known conductors).

Each step will now be considered in detail:

4.1. Inspection of Equation 17 reveals that as ω goes to infinity, $\lambda_{1,2}(\omega)$ of Ω_2 looks like:

$$\lambda_{1,2}(\omega) = \frac{-3C_{4,5}C_{5,6}\gamma_1^2\gamma_2}{9C_{4,5}C_{5,6}\gamma_1\gamma_2} \quad (18)$$

which, after canceling terms, leaves:

$$\gamma_1 = -3\lambda_{1,2}(\omega) \tag{19}$$

(as ω approaches ∞).

Physically, what is occurring in Ω_2 as ω approaches infinity is that the two capacitors are short circuiting, and the network takes the form and has the same properties as the tripod networks with three conductors discussed earlier.

In fact, each non-diagonal entry of $\Lambda(\infty)$ can be used to compute γ_1. For any n-prong, 2-layer network with only two types of conductor, one in the top layer and one in the lower, as ω goes to infinity:

$$\gamma_1 = -n\lambda_{i,j}(\omega) \tag{20}$$

with $i \neq j$.

For a network like Ω_2, with three different conductors in the top layer, a situation would occur similar to that of the tripod network previously solved. That is, some algebraic manipulation would still need to be performed to isolate the three top layer conductors from Λ.

The problem with this method is that it is a physical impossibility to actually have ω approach infinity. Therefore, an alternate approach is required. Since Equation 17 is quadratic in ω, it can be expressed in the form:

$$\lambda_{1,2}(\omega) = p(\omega) = \frac{1 + b\omega + c\omega^2}{\alpha + \beta\omega + \gamma\omega^2} \tag{21}$$

with b, c, α, β, and γ being five undetermined coefficients. For an n-prong, 2-layer network in general, $\lambda_{i,j}(\omega)$ is a ratio of polynomials of degree $n-1$ with $2n-1$ coefficients. If, in the case of Ω_2, c and γ could be determined from other data, $p(\omega)$ would be known and the $\lambda_{1,2}(\omega)$ (for $\omega = \infty$) would be found as the ratio c/γ.

The values $p(\omega_i)$ for five different frequencies will yield a system of five linear equations in the five unknowns b, c, α, β, and γ. (In the case of the general 2-layer network, $2n-1$ different values of ω would be needed.) That system is:

$$1 + b\omega_i + c\omega_i^2 = p(\omega_i)(\alpha + \beta\omega_i + \gamma\omega_i^2) \quad i = 1, \cdots, 5 \tag{22}$$

Let:

$$M = \begin{bmatrix} \omega_1 & \omega_1^2 & -p(\omega_1) & -\omega_1 p(\omega_1) & -\omega_1^2 p(\omega_1) \\ \omega_2 & \omega_2^2 & -p(\omega_2) & -\omega_2 p(\omega_2) & -\omega_2^2 p(\omega_2) \\ \omega_3 & \omega_3^2 & -p(\omega_3) & -\omega_3 p(\omega_3) & -\omega_3^2 p(\omega_3) \\ \omega_4 & \omega_4^2 & -p(\omega_4) & -\omega_4 p(\omega_4) & -\omega_4^2 p(\omega_4) \\ \omega_5 & \omega_5^2 & -p(\omega_5) & -\omega_5 p(\omega_5) & -\omega_5^2 p(\omega_5) \end{bmatrix}$$

FIG. 15

Unique solutions for b,c,α,β, and γ can be found if the determinant of M does not equal zero.

The ratio of quadratic expressions $p(\omega_i)$ can be rewritten as:

$$p(\omega_i) = tq(\omega_i) = \frac{t(\omega_i - \delta_1)(\omega_i - \delta_2)}{(\omega_i - \epsilon_1)(\omega_i - \epsilon_2)} \tag{23}$$

(with some non-zero constant t).

M can therefore be written as:

$$M = \begin{bmatrix} \omega_1 & \omega_1^2 & -tq(\omega_1) & -\omega_1 tq(\omega_1) & -\omega_1^2 tq(\omega_1) \\ \omega_2 & \omega_2^2 & -tq(\omega_2) & -\omega_2 tq(\omega_2) & -\omega_2^2 tq(\omega_2) \\ \omega_3 & \omega_3^2 & -tq(\omega_3) & -\omega_3 tq(\omega_3) & -\omega_3^2 tq(\omega_3) \\ \omega_4 & \omega_4^2 & -tq(\omega_4) & -\omega_4 tq(\omega_4) & -\omega_4^2 tq(\omega_4) \\ \omega_5 & \omega_5^2 & -tq(\omega_5) & -\omega_5 tq(\omega_5) & -\omega_5^2 tq(\omega_5) \end{bmatrix}$$

FIG. 16

The determinant of M can be shown to be:

$$t^3 \frac{\Pi_{i>j}(\omega_i - \omega_j)\Pi_{i=1}^2\Pi_{j=1}^2(\delta_i - \epsilon_j)}{\Pi_{i=1}^5\Pi_{j=1}^2(\omega_i - \epsilon_j)} \tag{24}$$

If $\omega_i \neq \epsilon_j$, $\delta_i \neq \epsilon_j$, and $\omega_i \neq \omega_j$, for all i, j, $\det(M) \neq 0$. Therefore, as long as these conditions hold, b, c, α, β, and γ can be determined. This allows the limit of $\lambda_{1,2}$ as ω approaches infinity to be calculated. Equation 19 then allows γ_1 to be computed.

In the general case of the n-prong, 2-layer network (where all conductors are different), each entry of $\Lambda(\omega)$ would have a different form, and consist of a ratio of different polynomials. Here, enough coefficients (such as b,c,α, β, and γ) would have to be found to find each entry. From this point, the conductor in the network's top layer can be recovered in a similar way to those of the earlier tripod example.

4.2. Another look at Equation 17 shows that the limit of $\lambda_{1,2}(\omega)$ as ω approaches zero will be:

$$\lambda_{1,2}(0) = \frac{-\gamma_1^2(\gamma_1\gamma_2^2 + \gamma_2^3)}{3(\gamma_1^3\gamma_2 + 2\gamma_1^2\gamma_2^2 + \gamma_1\gamma_2^3)} \tag{25}$$

This reduces to:

$$\lambda_{1,2}(0) = \frac{-\gamma_1\gamma_2}{3(\gamma_1 + \gamma_2)} \tag{26}$$

This equation can be solved for γ_2:

$$\gamma_2 = \frac{-3\lambda_{1,2}(0)\gamma_1}{\gamma_1 + 3\lambda_{1,2}(0)} \tag{27}$$

Taking the limit of Λ as ω goes to zero is *not* a physical impossibility–this is the equivalent of applying direct current to the network. Therefore $\lambda_{1,2}(0)$ can be found much easier than $\Lambda(\infty)$. As long as we have $\Lambda(0)$, γ_2 is found simply by solving Equation 27, using the previously obtained γ_1. (In the case where not all lower layer conductors are the same, a substitution method similar to that used for solving T_1 (Figure 3) will have to be used to arrive at an equation like Equation 27.)

A more general form exists for γ_2 in the n-prong, 2-layer network with all top-layer conductors the same and all lower-layer conductors the same:

$$\gamma_2 = \frac{-n\lambda_{i,j}(0)\gamma_1}{\gamma_1 + n\lambda_{i,j}(0)} \tag{28}$$

with $i \neq j$.

At this point, the algorithm has recovered γ_1 and γ_2. Therefore, we know all of the conductors in the network. All that remains to be done is to find the capacitances.

4.3. All of the conductors in Ω_2, shown in Figure 9, are known. The same is true for the ω-dependent boundary currents, by the nature of inverse problems. Ohm's Law can now be used to find interior potentials throughout the network. With the potentials at each node known, the currents across each conductor are also known, again via Ohm's Law.

To solve for $C_{4,5}$, consider the current from node 1 to node 4, $I_{1,4}$. Then consider the current from node 4 to node 7, $I_{4,7}$. The difference between $I_{1,4}$ and $I_{4,7}$ must go across $C_{4,5}$ (by Kirkhoff's Law). With this, and knowing the potential difference across nodes 4 and 5, Ohm's Law will give the admittance between nodes 4 and 5. This then allows $C_{4,5}$ to be calculated from $a = i\omega C$, with any positive, finite ω. The same method is used to find $C_{5,6}$, and is also used to find the capacitances of a more general network such as Ω_1 once all of its conductors are known.

This completes the reconstruction algorithm for n-prong, 2-layer networks.

5. Commentary. This reconstruction algorithm works only for 2-layer networks. The inverse problem for a 3-layer (or more) network, with an extra layer (or more) of conductors and one additional row (or more) of capacitors, can *not* be solved this way. The limits of Λ as ω approaches zero and infinity provide only enough information to solve for the conductors in the top layer, and a combined conductivity for each pair of connected conductors in the bottom two layers–the same problem facing the original tripod with 2 conductors on each spoke.

It may be that knowing $\Lambda(\omega)$ for more values of ω allows the inverse problems of such networks to be solved. However, it is beyond the scope of this article to examine this possibility[2].

[2] For a more detailed look at the inverse resistor problem, see *Finding the Conductors in Circular Networks From Boundary Measurements* by Edward Curtis and James Morrow, published in the *SIAM Journal of Applied Mathematics, vol. 50, no. 3*, pp. 918-930, June 1990.