INFINITE NETWORKS

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ABSTRACT. In this paper, I consider infinite rectangular resistor networks in m dimensions. I show that the power dissipated due to a single source of current is infinite in dimensions 1 and 2, and finite in dimension 3.

1. General Definitions

I consider a rectangular network of resistors, denoted by Ω_{∞_m}, which extends infinitely in m dimensions. The weights on each edge are assumed to be real and positive.

In the one-dimensional case, the underlying graph of the network Ω_{∞_1} lies on the x-axis, with each point $(b, 0)$ for $b \in \mathbf{Z}$ as a node. Each node is joined by an edge to the two other nodes which are nearest to it; each node lies a Euclidean distance one away from each of its neighbors.

Similarily, the set of the nodes of Ω_{∞_2} is the two-dimensional lattice consisting of the points with integer coordinates:$\{(a, b) | a, b \in \mathbf{Z}\}$. Each node is joined by an edge to four other nodes; again, each node lies a Euclidean distance one away from each of its neighbors.

This leads to the following definition:

DEFINITION

An m-dimensional infinite rectangular network of resistors, denoted by Ω_{∞_m}, is a function defined on each edge of a rectangular graph, together with this graph, which is defined as follows: The nodes of Ω_{∞_m} are the points $\{(a_1, a_2, \dots, a_m) | a_1, a_2, \dots, a_m \in \mathbf{Z}\}$. Each node is joined by an edge to 2m other nodes; each node is a Euclidean distance of one away from each of its neighbors. The function defined on the edges of Ω_{∞_m} is positive and real.

DEFINITION

On Ω_{∞_m}, the class of networks defined above, impose a unit source at the node which lies at the origin. By imposing certain boundary conditions upon Ω_{∞_m}, a unique electric potential function, denoted by g, will be generated throughout the network in response to the unit source. This function will be called the Green's function.

2. The Green's Function

Let us construct the Green's function for these networks. For simplicity, initially assume that all conductivities γ_{ij} have the same value, denoted by γ in each case.

I will first consider the case of dimension one:

Theorem 2.1. *Let all conductivities of Ω_{∞_1} be constant. Assign the value v_o to the potential of the node which lies at $(0,0)$. Then the electric potential g at each node is a function of its lattice point $(x,0)$, and is given by*

$$g(x) = v_o - |x|/2\gamma \tag{2.1}$$

Proof. By symmetry, the current which flows from the source node in each direction is equal, so .5 units of current flow in both directions away from the unit source. By repeated applications of Kirchhoff's law, it is easily seen that .5 units of current flow through every conductor in the network, directed away from the source node. Thus the potential change across each conductor, $\Delta V = IR$, is constant, equalling $1/2\gamma$. Furthermore, since current flows away from the source node, the potential at each node must decrease as a function of its distance from the source node. Assign the value v_o to the potential at the source node. Then the potential at the nth node from the source is given by $v_o - n\,(1/2\gamma)$. The nth node from the source lies at either $(n,0)$ or $(-n,0)$. Therefore $g(x) = v_o - |x|/2\gamma$. □

In two dimensions, the Green's function is much more complicated. In [3], it is shown that the Green's function, $g(a,b)$, for Ω_{∞_2} is uniquely determined by the following two conditions:

$$g(0,0) = 0 \tag{2.2}$$

(2.3)

As $\sqrt{a^2+b^2} \to \infty$, $g(a,b) - g(a-1,b)$ and $g(a,b) - g(a,b-1) \to 0$.

Theorem 2.2. *Let all conductivities of Ω_{∞_2} be constant. The electric potential g for Ω_{∞_2} determined by (2.2) and (2.3) is given by*

$$g(a,b) = \frac{1}{\gamma(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{i(ax+by)}-1}{4(\sin^2\frac{x}{2}+\sin^2\frac{y}{2})}dxdy \tag{2.4}$$

Proof. Let γ=1. Then, as shown in [1], the Green's function is given by

$$g_1(a,b) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{i(ax+by)}-1}{4(\sin^2\frac{x}{2}+\sin^2\frac{y}{2})}dxdy \tag{2.5}$$

Let D_γ be the operator which takes a potential function to the corresponding current function for a given value of γ:

$$\begin{aligned} D_\gamma(u(a,b)) = w(a,b) = \gamma\,[u(a+1,b) + u(a-1,b) \\ + u(a,b+1) + u(a,b-1) - 4u(a,b)] \end{aligned} \tag{2.6}$$

Let g_γ denote the Green's function for a given value of γ. Since there is a unit source at the origin,

$$w(0,0) = D_\gamma(g_\gamma(0,0)) = -1$$

Let $g_\gamma = \frac{g_1}{\gamma}$. Then

$$D_\gamma(g_\gamma(0,0)) = \gamma D_1(g_\gamma(0,0)) = \gamma D_1(\frac{g_1(0,0)}{\gamma} = D_1(g_1(0,0)) = -1$$

from (2.5). Also, $Dg_1 = 0$ for all other points, so $Dg_\gamma = \frac{Dg_1}{\gamma}$ does as well. Since g_1 satisfies conditions (2.2) and (2.3), so $g_\gamma = \frac{g_1}{\gamma}$ does also. Therefore $g_\gamma = \frac{g_1}{\gamma}$ satisfies all conditions for the Green's function. □

It will become very useful to know how the Green's function behaves at infinity. By expanding the Green's function as r approaches infinity, we obtain the following theorem:

Theorem 2.3. *As $r \to \infty$, the Green's function for Ω_{∞_2}, given by (2.4), has the asymptotic expansion*

$$g(a,b) = \frac{1}{2\pi\gamma}\left(-\ln r - \gamma^* - \frac{3\ln 2}{2} + \frac{\cos 4\phi}{12r^2} - O\left(\frac{1}{r^4}\right)\right) \tag{2.7}$$

where $r^2 = a^2 + b^2$, γ^* is Euler's constant, and $\phi = \arctan(b/a)$.

This theorem is proved in [3].

Now, let us consider Ω_{∞_m} for $m \geq 3$. Impose the following condition upon the Green's function:

$$\text{As } \sqrt{a_1^2 + a_2^2 \ldots a_m^2}\ , g(a_1, a_2, \ldots, a_m) \to 0 \tag{2.8}$$

Theorem 2.4. *Let all conductivities of Ω_{∞_m} be constant. For $m \geq 3$, the Green's function g as a function of its lattice point is given by*

$$g(a_1, \ldots, a_m) = \frac{1}{\gamma(2\pi)^m}\underbrace{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}}_{m}\frac{e^{i\sum_{n=1}^m a_n x_n}}{4\sum_{n=1}^m \sin^2(\frac{x_n}{2})}\,dx_1\ldots dx_m \tag{2.9}$$

Proof. Let D be the operator which takes a potential node function to the corresponding current node function, i.e.,

$$D[u(a_1, \ldots, a_m)] = w(a_1, \ldots, a_m)$$

$$= \gamma[u(a_1+1, \ldots, a_m) + \ldots + u(a_1, \ldots, a_m+1) + u(a_1-1, \ldots, a_m) +$$

$$\ldots + u(a_1, \ldots, a_m - 1) - 2mu(a_1, \ldots, a_m)]$$

Let the lattice functions u and w be regarded as the Fourier coefficients of the functions U and W. Then the Fourier series U and W are related to u

and w by the following formulas:

$$(2.10)\qquad U(x_1, \dots, x_m) = \underbrace{\sum_{a_m=-\infty}^{\infty} \cdots \sum_{a_1=-\infty}^{\infty}}_{m} u(a_1, \dots, a_m) e^{-i\sum_{n=1}^{m} a_n x_n}$$

$$(2.11)\qquad W(x_1, \dots, x_m) = \underbrace{\sum_{a_m=-\infty}^{\infty} \cdots \sum_{a_1=-\infty}^{\infty}}_{m} w(a_1, \dots, a_m) e^{-i\sum_{n=1}^{m} a_n x_n}$$

(2.12)

$$u(a_1, \dots, a_m) = \frac{1}{(2\pi)^m} \underbrace{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi}}_{m} e^{i\sum_{n=1}^{m} a_n x_n} U(x_1, \dots, x_m)\, dx_1 \dots dx_m$$

(2.13)

$$w(a_1, \dots, a_m) = \frac{1}{(2\pi)^m} \underbrace{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi}}_{m} e^{i\sum_{n=1}^{m} a_n x_n} W(x_1, \dots, x_m)\, dx_1 \dots dx_m$$

Operate on (2.12) with D. Then

(2.14)

$$D(u) = w = \frac{1}{(2\pi)^m} \underbrace{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi}}_{m} D[e^{i\sum_{n=1}^{m} a_n x_n}] U(x_1, \dots, x_m)\, dx_1 \dots dx_m$$

Now, by a simple calculation,

$$(2.15)\qquad D(e^{i\sum_{n=1}^{m} a_n x_n}) = \gamma e^{i\sum_{n=1}^{m} a_n x_n} [\sum_{n=1}^{m} e^{ix_n} + \sum_{n=1}^{m} e^{-ix_n} - 2m]$$

$$= -4\gamma e^{i\sum_{n=1}^{m} a_n x_n} \sum_{n=1}^{m} \sin^2\left(\frac{x_n}{2}\right)$$

Now, by (2.13) and (2.14),

$$(2.16)\qquad W(x_1, \dots, x_m) = -4\gamma \sum_{n=1}^{m} \sin^2(\frac{x_n}{2}) U(x_1, \dots, x_m)$$

since the two functions with identical Fourier coefficients are equal. In our case, $W(x_1, \dots, x_m) = -1$ by (2.10) since Kirchhoff's law holds at all nodes besides the unit source. Thus

$$(2.17)\qquad U(x_1, \dots, x_m) = \frac{1}{4\gamma \sum_{n=1}^{m} \sin^2(\frac{x_n}{2})}$$

Substituting into (2.11) gives (2.8). By a simple calculation, it is seen that $u \to 0$ as $r \to \infty$.

Now it must be shown that (2.8) is the unique solution of condition (2.7). Suppose that v is another solution of the above conditions. Then $s = g - v$ is a γ-harmonic everywhere, and $s \to 0$ as $r \to \infty$. Let A be the least upper bound of s. If $A > 0$, then $v = A$ at some point p. Since $s(p)$ is the average of the values of s at the neighbors of p, then $s = A$ at all its neighbors. Continuing the same argument shows $v = A$ everywhere. But $s \to 0$, so s cannot be A everywhere. Thus $A \not> 0$.

Let B be the greatest lower bound of s. Then the same argument holds for B, so $B \not< 0$. Thus $s \equiv 0$, or $g \equiv v$. Thus g is the unique solution of (2.7). □

The above proof is a generalization of Duffin's work in 3-space in [2].

In 3-space, by again expanding the Green's function as r approaches infinity, another asymptotic expansion is found:

Theorem 2.5. *As $r \to \infty$, the Green's function for Ω_{∞_3}, given by (2.8) has the asymptotic expansion*

$$g(a,b,c) = \frac{1}{4\gamma\pi r} + \frac{1}{32\gamma\pi r^3}[-3 + \frac{5(a^4 + b^4 + c^4)}{r^4}] + \frac{O(\frac{1}{r^5})}{\gamma} \tag{2.18}$$

where $r^2 = a^2 + b^2 + c^2$.

The proof is due to Duffin [2].

3. Power Dissipated

The power P, or energy per unit time, dissipated over a single resistor is defined to be $P = I^2R$. Similarily, the power dissipated over an entire network of resistors is the sum of the dissipations over each individual resistor, or

$$P = \sum_{i<j} r_{ij}(w_{ij})^2 \tag{3.1}$$

where w_{ij} denotes the current from node i to node j if i and j are neighbors (otherwise $w_{ij} = 0$), and r_{ij} is the resistance of the edge between node i and node j.

By Ohm's law, the expression becomes

$$P = \sum_{i<j} \gamma_{ij}(v_i - v_j)^2 \tag{3.2}$$

where v_i is the electric potential at node i , and γ_{ij} is the conductivity of the edge joining i and j. (If i and j are not neighbors, $\gamma_{ij} = 0$.)

In an arbitrary infinite network, given the conductivities of each edge and the voltages at each node, the power dissipated in the entire network can be found. Simply construct an infinite sum over all edges using the above formula. Now, in our case, each Green's function can be used to find the power dissipated in the corresponding infinite network.

Theorem 3.1. *Let all conductivities of Ω_{∞_1} be constant. Then its Green's function (2.1) dissipates infinite power in the network.*

Proof. The current in every edge is constant, at .5, so

$$P = \sum_{i<j} r_{ij}(w_{ij})^2 = .5^2 \sum_{i<j} r_{ij} \tag{3.3}$$

Since r_{ij} is a positive constant for all $i \sim j$, and there are an infinite number of edges in Ω_{∞_1}, P is infinite. □

In the two dimensional case, the increased complexity of the network requires the introduction of a few new concepts:

Let Ω_{n_2} be the subnetwork of Ω_{∞_2} with vertices

$$\{(a,b) | a,b \in \mathbf{Z}\ ,\ |a| \le n, |b| \le n\}$$

All connections within Ω_{∞_2} between nodes of Ω_{n_2} are preserved.

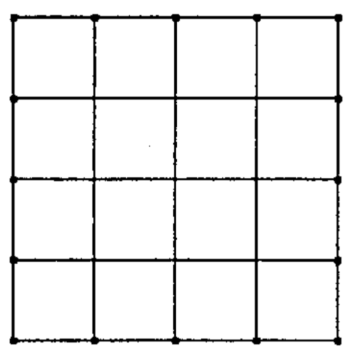
Let B_n be the exterior subnetwork of Ω_{n_2}, i.e., the subnetwork which is composed of nodes

$$\{(a,b) | a,b \in \mathbf{Z}\ ,\ |a| = n \quad or\ |b| = n\}$$

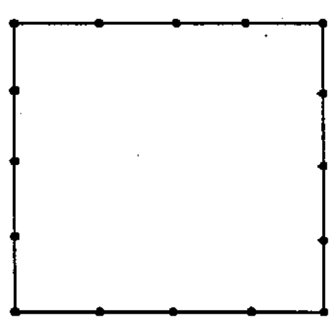
and all edges in Ω_{n_2} between any two of these nodes.

Let I_n be the subnetwork of Ω_{n_2} with edges which adjoin vertices in B_n to vertices not in B_n.

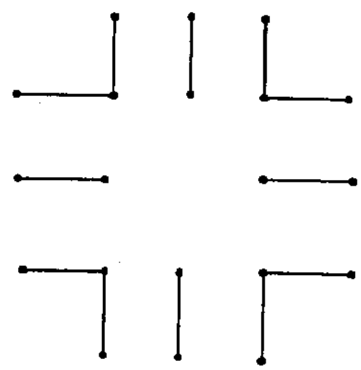
For example, below is the graph of Ω_{2_2}:



And the graph of B_2:



And the graph of I_2:



Using the subnetworks B_n and I_n, an explicit equation for power dissipated is found:

(3.4) $$P(\Omega_{\infty_2}) = \sum_{n=1}^{\infty} P(B_n) + \sum_{n=1}^{\infty} P(I_n)$$

since $\bigcup_1^\infty B_n \cup \bigcup_{n=1}^\infty I_n = \Omega_{\infty_2}$, and since this union is disjoint.

Now since the asymptotic expansion (2.6) converges to the Green's function (2.4), the power dissipated in Ω_{∞_2} will be finite if and only if the approximation, the power dissipated due to (2.7) converges. Using the asymptotic expansion in the infinite sums above, we can find their convergence/divergence, and hence whether finite or infinite power is dissipated in the network.

Theorem 3.2. *Let all conductivities of Ω_{∞_2} be constant. Then its Green's function dissipates infinite power in the network.*

Proof. Consider B_n.

(3.5) $$P(B_n) = \sum_{i \sim j; i,j \in B_n} \gamma(v_i - v_j)^2$$

Let p and q be two adjacent nodes of B_n with $r_p \geq r_q$. Then using (2.7),

(3.6)
$$g(p) - g(q) = \frac{1}{2\pi\gamma}\left[-\ln\left(\frac{r_p}{r_q}\right) + \frac{\cos 4\phi_p}{12r_p^2} - \frac{\cos 4\phi_q}{12r_p^2} - O\left(\frac{1}{r_p^4}\right) + O\left(\frac{1}{r_q^4}\right)\right]$$

for large n. Now

(3.7) $$\left|\frac{\cos 4\phi_q}{12r_q^2} - \frac{\cos 4\phi_p}{12r_p^2}\right| \leq \frac{4\,|\phi_q - \phi_p|}{12r_q^2} + \frac{|\cos 4\phi_p|}{12}\left(\frac{1}{r_q^2} - \frac{1}{r_p^2}\right)$$

Since p and q are neighbors,

(3.8) $$\frac{4\,|\phi_q - \phi_p|}{12r_q^2} \leq \frac{4\tan^{-1}(\frac{1}{n})}{12n^2}$$

and

$$\frac{|\cos 4\phi_p|}{12}\left(\frac{1}{r_q^2}-\frac{1}{r_p^2}\right) \le \frac{1}{12}\left(\frac{1}{n^2}-\frac{1}{n^2+1}\right) \tag{3.9}$$

The sum of the right-hand sides of (3.8) and(3.9) is less than or equal to $\frac{1}{3n^3}+\frac{1}{12n^4}$. And since $\left|O(\frac{1}{r_p^4})-O(\frac{1}{r_q^4})\right| \le \frac{1}{n^4}$,

$$|g(p)-g(q)| \ge \frac{1}{2\pi\gamma}\left|\ln(\frac{r_p}{r_q})-\frac{1}{3n^3}-\frac{2}{n^4}\right| \tag{3.10}$$

Therefore,

(3.11)

$$(g(p)-g(q))^2 \ge (\frac{1}{2\pi\gamma})^2\left[\ln^2(\frac{r_p}{r_q})+\frac{1}{9n^6}+\frac{4}{n^8}+\frac{2}{3n^7}-\frac{\ln(\frac{r_p}{r_q})}{3n^3}-\frac{2\ln(\frac{r_p}{r_q})}{n^4}\right]$$

Now, summing over all edges in B_n, by (3.5),

$$\begin{aligned} P(B_n) &\ge \gamma(\frac{1}{2\pi\gamma})^2\left[8\sum_{a=1}^{n}\ln^2\frac{\sqrt{n^2+a^2}}{\sqrt{n^2+(a-1)^2}}\right.\\ &\left.+8n\left(\frac{1}{9n^6}+\frac{1}{n^8}+\frac{1}{3n^7}-\frac{\ln\sqrt{2}}{3n^3}-\frac{2\ln\sqrt{2}}{n^4}\right)\right] \end{aligned} \tag{3.12}$$

since $\frac{r_p}{r_q} \le \sqrt{2}$. Now, we will sum B_n from $n=2$ to ∞, skipping $n=1$ because the Green's function is finite there (and the asymptotic expansion is not), and so will not affect the convergence of B_n.

$$\begin{aligned} \sum_{n=2}^{\infty}P(B_n) &= \frac{2}{\pi^2\gamma}\sum_{n=2}^{\infty}\left[\sum_{a=1}^{n}\ln^2\frac{\sqrt{n^2+a^2}}{\sqrt{n^2+(a-1)^2}}\right.\\ &\left.+\frac{1}{9n^5}+\frac{1}{n^7}+\frac{1}{3n^6}-\frac{\ln\sqrt{2}}{3n^2}-\frac{2\ln\sqrt{2}}{n^3}\right] \end{aligned} \tag{3.13}$$

Clearly, all sums converge other than the first. Thus consider

$$\sum_{n=2}^{\infty}\sum_{a=1}^{n}\ln^2\sqrt{\frac{n^2+a^2}{n^2+(a-1)^2}} \tag{3.14}$$

By the double integral test, this sum converges if and only if the following double integral converges:

(3.15)

$$\int_2^{\infty}\int_1^{x}\ln^2\sqrt{\frac{x^2+y^2}{x^2+(y-1)^2}}\,dydx = \frac{1}{4}\int_2^{\infty}\int_1^{x}\ln^2\frac{x^2+y^2}{x^2+(y-1)^2}\,dydx$$

Divide this integral into the difference of two integrals:

$$\frac{1}{4}\int_1^{\infty}\int_1^{x}\ln^2\frac{x^2+y^2}{x^2+(y-1)^2}\,dydx-\frac{1}{4}\int_1^{2}\int_1^{x}\ln^2\frac{x^2+y^2}{x^2+(y-1)^2}\,dydx \tag{3.16}$$

The second integral is finite, so consider the first. Converting it to polar coordinates gives

$$(3.17)\qquad \frac{1}{4}\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\sin\theta}}^{\infty}\ln^2\frac{r^2}{r^2-2r\sin\theta+1}\,r\,drd\theta$$

This integrand is positive for all $y > 0$, so the integral (3.17) is greater than the integral over part of the region (3.18):

$$(3.18)\qquad \frac{1}{4}\int_{\frac{1}{4}}^{\frac{\pi}{4}}\int_4^{\infty}\ln^2\frac{r^2}{r^2-2r\sin\theta+1}\,r\,drd\theta$$

For $\frac{1}{4}\leq\theta\leq\frac{\pi}{4}$, $r>4$, $\frac{r^2}{r^2-2r\sin\theta+1}\geq\frac{r^2}{r^2-ar+1}>1$ for $a=2\sin\frac{1}{4}$. Thus (3.17) is greater than

$$(3.19)\qquad \frac{\pi-1}{16}\int_4^{\infty}\ln^2\frac{r^2}{r^2-ar+1}\,r\,dr$$

Now make the substitution $r=\frac{1}{t}$. Then $dr=\frac{-dt}{t^2}$, and the above expression becomes

$$(3.20)\qquad \frac{\pi-1}{16}\int_0^{\frac{1}{4}}\frac{\ln^2(1-at+t^2)}{t^3}\,dt$$

For $0<x<1$, $\ln(1-x)<x<0$, so $\ln^2(1-x)>x^2>0$. For $0<t<\frac{1}{4}$, $0<at-t^2<1$, so $\ln^2(1-at+t^2)>(at-t^2)^2$. Therefore (3.19) is greater than

$$(3.21)\qquad \frac{\pi-1}{16}\int_0^{\frac{1}{4}}\frac{(at-t^2)^2}{t^3}dt=\frac{\pi-1}{16}\left[\int_0^{\frac{1}{4}}\frac{a^2}{t}dt-\frac{a}{4}+\frac{1}{16}\right]$$

a clearly divergent expression. Therefore $\sum_{n=2}^{\infty}P(B_n)$ diverges by comparison with (3.11) through (3.20). $P(\Omega_{n_2})\geq P(B_n)$ for all n, so $P(\Omega_{\infty_2})\geq\sum_{n=2}^{\infty}P(B_n)$. Thus $P(\Omega_{\infty_2})$ is infinite. □

Remark Another very similar proof can be constructed by examining $\sum_{n=2}^{\infty}P(I_n)$ which also diverges.

A similar procedure can be used to find the convergence of the power dissipated in the three dimensional case. Let us create similar subnetworks of Ω_{∞_3}:

Let Ω_{n_3} be the subnetwork of Ω_{∞_3} with vertices

$$\{(a,b,c)|a,b,c\in\mathbf{Z}\ ,\ |a|\leq n,|b|\leq n,|c|\leq n\}$$

All connections within Ω_{∞_3} between nodes of Ω_{n_3} are preserved.

Let b_n be the exterior subnetwork of Ω_{n_3}, i.e., the subnetwork which is composed of nodes

$$\{(a,b,c)|a,b,c\in\mathbf{Z}\ ,\ |a|=n\ \ or\ |b|=n\ \ or\ |c|=n\}$$

and all edges in Ω_{n_3} between any two of these nodes.

Let h_n be the subnetwork of Ω_{n_3} with edges which adjoin vertices in h_n to vertices not in b_n.

Now, let us consider the difference in the Green's function between two adjacent nodes. Let $i \sim j$, $i, j \in \Omega_{n_3}$, and assume $r_i < r_j$. Then the asymptotic expansion (2.17) gives

$$|g(i) - g(j)| = \frac{1}{4\gamma\pi r_i} - \frac{1}{4\gamma\pi r_j} + \frac{3}{32\gamma r_i^3} - \frac{3}{32\gamma r_j^3} + \frac{5}{32\gamma\pi r_i^4} - \frac{5}{32\gamma\pi r_j^4} + \left|\frac{O(r_i^{-5})}{\gamma} - \frac{O(r_j^{-5})}{\gamma}\right| \tag{3.22}$$

for large r_i, r_j. Using this equation, the convergence of the power dissipated can be found.

Theorem 3.3. *Let all conductivities of Ω_{∞_3} be constant. Then its Green's function dissipates finite power in the subnetwork $\bigcup_{n=1}^{\infty} b_n$.*

Proof. For $i, j \in b_n$,

$$|g(i) - g(j)| \le \frac{1}{4\gamma\pi}\left(\frac{1}{r_i} - \frac{1}{r_j}\right) + \frac{3}{32\gamma n^3} + \frac{5}{32\gamma\pi n^4} + \frac{1}{\gamma n^5} \tag{3.23}$$

where the inequality holds for large n. Since $0 < \frac{1}{r_i} - \frac{1}{r_j} < \frac{1}{n}$,

$$\gamma(g(i) - g(j))^2 \le \frac{1}{16\gamma\pi^2}\left(\frac{1}{r_i} - \frac{1}{r_j}\right)^2 + \frac{9}{1042\gamma n^6} + \frac{25}{1024\gamma\pi^2 n^8} + \frac{1}{\gamma n^{10}} + \frac{3}{64\gamma\pi n^4} + \frac{5}{64\gamma\pi^2 n^5} + \frac{1}{2\gamma\pi n^6} + \frac{15}{512\gamma\pi n^7} + \frac{3}{16\gamma n^8} + \frac{5}{16\gamma\pi n^9} \tag{3.24}$$

Summing over all edges in b_n gives

$$P(b_n) \le \frac{1}{16\gamma\pi^2} \sum_{i\sim j; i,j \in B_n} \left(\frac{1}{r_i} - \frac{1}{r_j}\right)^2 + 48n^2\left[\frac{a}{n^4} + \frac{b}{n^5} + \frac{c}{n^6} + \frac{d}{n^7} + \frac{e}{n^8} + \frac{f}{n^9} + \frac{g}{n^{10}}\right] \tag{3.25}$$

since there are $48n^2$ edges in b_n (where a through g are the constants given above). Summing b_n over all n gives

$$\sum_{n=2}^{\infty} P(b_n) \le \sum_{n=2}^{\infty}\left[\frac{1}{16\gamma\pi^2} \sum_{i\sim j; i,j \in B_n} \left(\frac{1}{r_i} - \frac{1}{r_j}\right)^2 + 48\left[\frac{a}{n^2} + \frac{b}{n^3} + \frac{c}{n^4} + \frac{d}{n^5} + \frac{e}{n^6} + \frac{f}{n^7} + \frac{g}{n^8}\right]\right] \tag{3.26}$$

Obviously, all terms converge other than the first, so consider

$$\sum_{i\sim j;i,j\in B_n}\left(\frac{1}{r_i}-\frac{1}{r_j}\right)^2 \tag{3.27}$$

Let us examine a face of the cube upon which b_n lies. On any face of the cube, the above sum is given by

$$\begin{aligned}8\sum_{b=1}^{n}\sum_{a=1}^{n}&\left(\frac{1}{\sqrt{n^2+a^2+b^2}}-\frac{1}{\sqrt{n^2+(a-1)^2+b^2}}\right)^2\\&+4\sum_{a=1}^{n}\left(\frac{1}{\sqrt{n^2+a^2}}-\frac{1}{\sqrt{n^2+(a-1)^2}}\right)^2\end{aligned} \tag{3.28}$$

Thus $\sum_{n=2}^{\infty}P(b_n)$ converges if

$$\begin{aligned}\sum_{n=2}^{\infty}\Bigg[48\sum_{b=1}^{n}\sum_{a=1}^{n}&\left(\frac{1}{\sqrt{n^2+a^2+b^2}}-\frac{1}{\sqrt{n^2+(a-1)^2+b^2}}\right)^2\\&+24\sum_{a=1}^{n}\left(\frac{1}{\sqrt{n^2+a^2}}-\frac{1}{\sqrt{n^2+(a-1)^2}}\right)^2\Bigg]\end{aligned} \tag{3.29}$$

converges. By the double and triple integral tests, this sum converges if and only if the following integrals converge:

$$\begin{aligned}\int_2^{\infty}\int_1^{z}\int_1^{z}&\left(\frac{1}{\sqrt{x^2+y^2+z^2}}-\frac{1}{\sqrt{x^2+(y-1)^2+z^2}}\right)^2 dydxdz\\&+\int_2^{\infty}\int_1^{x}\left(\frac{1}{\sqrt{x^2+y^2}}-\frac{1}{\sqrt{x^2+(y-1)^2}}\right)^2 dydx\end{aligned} \tag{3.30}$$

Examine the first integral. Since the integrand is always positive, we can integrate over a larger region and bound the integral. Converting to spherical coordinates gives

$$\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}}\int_2^{\infty}\left(\frac{1}{\rho}-\frac{1}{\sqrt{\rho^2-2\rho\sin\phi\sin\theta+1}}\right)^2\rho^2\sin\phi\, d\rho d\theta d\phi \tag{3.31}$$

as an upper bound for this integral. Consider the integrand a function of ϕ and θ. Then in the range $0\le\phi\le\frac{\pi}{4}$, $0\le\theta\le\frac{\pi}{2}$, it attains its maximum value at $\phi=\frac{\pi}{4}$, $\theta=\frac{\pi}{2}$. Therefore, an upper bound for (3.31) is

$$\frac{2\pi^2}{16}\int_2^{\infty}\left(\frac{1}{\rho}-\frac{1}{\sqrt{\rho^2-\sqrt{2}\rho+1}}\right)^2\rho^2 d\rho \tag{3.32}$$

The integrand simplifies to $\left(1-\frac{\rho}{\sqrt{\rho^2-\sqrt{2}\rho+1}}\right)^2$. Now for $\rho \geq 2$, $(p-1)\sqrt{\rho^2-\sqrt{2}\rho+1} < \rho^2$, and this implies that

$$\rho \geq \sqrt{\rho^2-\sqrt{2}\rho+1} - \frac{\sqrt{\rho^2-\sqrt{2}\rho+1}}{\rho} \tag{3.33}$$

$$\sqrt{\rho^2-\sqrt{2}\rho+1} - \rho \leq \frac{\sqrt{\rho^2-\sqrt{2}\rho+1}}{\rho} \tag{3.34}$$

$$\left(1-\frac{\rho}{\sqrt{\rho^2-\sqrt{2}\rho+1}}\right) < \frac{1}{\rho} \tag{3.35}$$

or that the integral (3.32) is less than

$$\frac{\sqrt{2}\pi^2}{16}\int_2^\infty \frac{d\rho}{\rho^2} \tag{3.36}$$

This converges, so (3.31) and (3.32) also converge, and by comparison, the first integral of (3.30) does as well. Now convert the second integral of (3.30) to polar coordinates. An upper bound for the second integral, found by converting the integrand to polar coordinates and integrating over a larger region, is given by

$$\int_0^{\frac{\pi}{4}}\int_{\sqrt{5}}^\infty \left(\frac{1}{r}-\frac{1}{\sqrt{r^2-2r\sin\theta+1}}\right)^2 r\sin\theta\, dr d\theta \tag{3.37}$$

Consider the integral as a function of θ. Then in the range $0 \leq \theta \leq \frac{\pi}{4}$, it attains its maximum value at $\theta = \frac{\pi}{4}$. Therefore, an upper bound for (3.37) is

$$\frac{\sqrt{2}\pi}{8}\int_{\sqrt{5}}^\infty \left(\frac{1}{r}-\frac{1}{\sqrt{r^2-\sqrt{2}r+1}}\right)^2 r\, dr \tag{3.38}$$

From (3.35),

$$\frac{1}{r}-\frac{1}{\sqrt{r^2-\sqrt{2}r+1}} \leq \frac{1}{r^2} \tag{3.39}$$

Thus the integrand is less than $\frac{1}{r^3}$, and so the integral converges. Therefore both integrals of (3.30) converge, and thus $\sum_{n=2}^\infty P(b_n)$ converges as well. □

Theorem 3.4. *Let all conductivities of Ω_{∞_3} be constant. Then its Green's function dissipates finite power in the subnetwork $\bigcup_{n=2}^\infty h_n$.*

Proof. From (3.22), for $i \sim j$, $r_i < r_j$, $i, j \in h_n$,

(3.40)

$$|g(i) - g(j)| \leq \frac{1}{4\gamma\pi}\left(\frac{1}{r_i} - \frac{1}{r_j}\right) + \frac{3}{32\gamma(n-1)^3} + \frac{5}{32\gamma\pi(n-1)^4} + \frac{1}{\gamma(n-1)^5}$$

where the inequality holds for large n. Since $\frac{1}{r_i} - \frac{1}{r_j} < \frac{1}{n-1}$,

(3.41)

$$\gamma(g(i) - g(j))^2 \leq \frac{1}{16\gamma\pi^2}\left(\frac{1}{r_i} - \frac{1}{r_j}\right)^2 + \frac{9}{1042\gamma(n-1)^6} + \frac{25}{1024\gamma\pi^2(n-1)^8}$$

$$+\frac{1}{\gamma(n-1)^{10}} + \frac{3}{64\gamma\pi(n-1)^4} + \frac{5}{64\gamma\pi^2(n-1)^5} + \frac{1}{2\gamma\pi(n-1)^6}.$$

$$+\frac{15}{512\gamma\pi(n-1)^7} + \frac{3}{16\gamma(n-1)^8} + \frac{5}{16\gamma\pi(n-1)^9}$$

Summing over all edges in h_n gives

(3.42)

$$P(h_n) \leq \frac{1}{16\gamma\pi^2} \sum_{i\sim j; i,j \in h_n} \left(\frac{1}{r_i} - \frac{1}{r_j}\right)^2 + 6(2n-1)^2 \left[\frac{a}{(n-1)^4}\right.$$

$$\left.+\frac{b}{(n-1)^5} + \frac{c}{(n-1)^6} + \frac{d}{(n-1)^7} + \frac{e}{(n-1)^8} + \frac{f}{(n-1)^9} + \frac{g}{(n-1)^{10}}\right]$$

since there are $6(2n-1)^2$ edges in h_n (where a through g are the constants given above). Summing $P(h_n)$ from $n = 2$ to ∞ gives

(3.43)

$$\sum_{n=2}^{\infty} P(h_n) \leq \left[\frac{1}{16\gamma\pi^2} \sum_{i\sim j; i,j \in h_n} \left(\frac{1}{r_i} - \frac{1}{r_j}\right)^2 + 6(2n-1)^2 \left[\frac{a}{(n-1)^4}\right.\right.$$

$$\left.\left.+\frac{b}{(n-1)^5} + \frac{c}{(n-1)^6} + \frac{d}{(n-1)^7} + \frac{e}{(n-1)^8} + \frac{f}{(n-1)^9} + \frac{g}{(n-1)^{10}}\right]\right]$$

Obviously, all terms converge except the first, so consider

(3.44)

$$\sum_{i\sim j; i,j \in b_n} \left(\frac{1}{r_i} - \frac{1}{r_j}\right)^2$$

Let us examine the sixth of h_n which is adjacent to one face of the $2n \times 2n \times 2n$ cube centered at the origin. The above sum over this portion of h_n is

(3.45)

$$\left(\frac{1}{n} - \frac{1}{n-1}\right)^2 + 4\sum_{b=1}^{n-1}\sum_{a=0}^{n-1}\left(\frac{1}{\sqrt{n^2 + a^2 + b^2}} - \frac{1}{\sqrt{(n-1)^2 + a^2 + b^2}}\right)^2$$

Thus $\sum_{n=2}^{\infty} P(h_n)$ converges if

$$\sum_{n=2}^{\infty}\left[6\left(\frac{1}{n}-\frac{1}{n-1}\right)^2 \right.$$

(3.46)

$$\left. +24\sum_{b=1}^{n-1}\sum_{a=0}^{n-1}\left(\frac{1}{\sqrt{n^2+a^2+b^2}}-\frac{1}{\sqrt{(n-1)^2+a^2+b^2}}\right)^2\right]$$

converges. The first term simplifies to $\sum_{n=2}^{\infty}\frac{1}{n^2(n-1)^2}$, which converges. Consider the second term:

$$\sum_{n=2}^{\infty}\sum_{b=1}^{n-1}\sum_{a=0}^{n-1}\left(\frac{1}{\sqrt{n^2+a^2+b^2}}-\frac{1}{\sqrt{(n-1)^2+a^2+b^2}}\right)^2 \tag{3.47}$$

By the triple integral test, this sum converges if and only if the following triple integral converges:

(3.48)

$$\int_2^{\infty}\int_1^{z-1}\int_0^{z-1}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}-\frac{1}{\sqrt{x^2+y^2+(z-1)^2}}\right)^2 dxdydz$$

Again, converting to spherical coordinates and integrating over a larger region gives

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_{\sqrt{5}}^{\infty}\left(\frac{1}{\rho}-\frac{1}{\sqrt{\rho^2-2\rho\cos\phi+1}}\right)^2\rho^2\sin\phi\, d\rho d\phi d\theta \tag{3.49}$$

as an upper bound for (3.48). In the range $0\le\phi\le\frac{\pi}{4}$, $\sin\phi$ attains its maximum at $\phi=\frac{\pi}{4}$. Thus

$$\frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_{\sqrt{5}}^{\infty}\left(\frac{1}{\rho}-\frac{1}{\sqrt{\rho^2-2\rho\cos\phi+1}}\right)^2\rho^2\, d\rho d\phi d\theta \tag{3.50}$$

bounds (3.49). This integral has already been shown to converge by comparison with $\frac{1}{\rho^2}$. Therefore both terms of (3.46) converge, and so $P(\bigcup_{n=2}^{\infty})$ is finite. □

Theorem 3.5. *Let all conductivities of Ω_{∞_3} be constant. Then $P(\Omega_{\infty_3})$ is finite.*

Proof.

$$P(\Omega_{\infty_3})=\sum_{n=2}^{\infty}P(b_n)+\sum_{n=2}^{\infty}P(h_n)+P(b_1)+P(h_1) \tag{3.51}$$

By Theorem 2.4, g is finite everywhere. Thus the power dissipated in the eight edges of b_1 and the four edges of h_1 is finite. By Theorems 3.3 and 3.4, $\sum_{n=2}^{\infty}P(b_n)$ and $\sum_{n=2}^{\infty}P(h_n)$ are finite as well. Therefore $P(\Omega_{\infty_3})$ is the sum of four finite quantities, so it is finite. □

Conjecture 3.6. *Let all conductivities of Ω_{∞_m} be constant for $m \geq 3$. Then $P(\Omega_{\infty_m})$ is finite.*

Remark The first term in the asymptotic expansions of the Green's functions of Ω_{∞_2} and Ω_{∞_3} are the same as the Green's functions in the continuous case in dimensions two and three. If this pattern holds, as it should, the conjecture is true.

4. The Finite Green's Function

I now consider an arbitrary connected network with positive weights assigned to each edge. By examining the Kirchhoff matrix K under appropriate conditions, an important result regarding the power dissipated by a Green's function will be derived.

First, a determinantal identity is needed. Let a be an $n \times n$ matrix, and let $A_{i;j}$ denote the $(n-1) \times (n-1)$ matrix constructed by deleting row i and column j. Similarily, $A_{hi;jk}$ will denote the $(n-2) \times (n-2)$ matrix constructed by deleting rows h and i and columns j and k.

Lemma 4.1. *For any indices h, i, j, and k with $1 \leq h < i \leq n$, and $1 \leq j < k \leq n$,*

$$|A||A_{hi;jk}| = |A_{h;j}||A_{i;k}| - |A_{h;k}||A_{i;j}| \tag{4.1}$$

The proof is given in [1].

Lemma 4.2. *Let Ω be a connected network with positive conductivities assigned to each edge and with boundary $\partial\Omega$. If a unit source is placed at some node in $int\Omega = \Omega - \partial\Omega$, and zero potential is imposed at every boundary node, then the power dissipated in Ω is identically the potential at s, the source node.*

Proof. Under the correct ordering of nodes, the Kirchhoff matrix K has the following structure:

$$\begin{array}{cc} & \begin{array}{cc} \partial & int \end{array} \\ & \left[\begin{array}{cc} A & B \\ B^T & C \end{array}\right] \begin{array}{c} \partial \\ int \end{array} \end{array} \tag{4.2}$$

Also, $Kv = w$ where v is the vector of potentials at each node, and w is the vector of currents at each node (under the same ordering of nodes). Number the nodes such that the node with the unit source is numbered first among the interior nodes. Then under the above conditions, v and w are given by the vectors below:

$$\begin{array}{c} \partial \\ \\ int \end{array} \left[\begin{array}{c} 0 \\ \\ \mathbf{v}^* \end{array}\right] \qquad \begin{array}{c} \partial \\ \\ int \end{array} \left[\begin{array}{c} \mathbf{w}^* \\ 1 \\ 0 \end{array}\right] \tag{4.3}$$

where $\mathbf{w}^*$ is the vector of currents at boundary nodes, and $\mathbf{v}^*$ is the vector of potentials at interior nodes. The power dissipated in the network is given by

$$(4.4) \qquad P = v^T K v = v^T w = \begin{bmatrix} 0 & \mathbf{v}^* \end{bmatrix} \begin{bmatrix} \mathbf{w}^* \\ 1 \\ 0 \end{bmatrix} = v_s$$

the value of the potential at the source node. □

Now

$$(4.5) \qquad \left[\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right] \begin{bmatrix} 0 \\ \mathbf{v}^* \end{bmatrix} = \begin{bmatrix} \mathbf{w}^* \\ 1 \\ 0 \end{bmatrix}$$

which implies that

$$(4.6) \qquad C\ \mathbf{v}^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore, by Cramer's Rule,

$$(4.7) \qquad v_s = \frac{|C_{1;1}|}{|C|} = P(\Omega)$$

This leads to the following theorem:

Theorem 4.3. *Let Ω be a network under the conditions described in Lemma (4.2). Construct Ω^* by increasing γ_{ij}, the conductivity of one edge of Ω. Then $P(\Omega) \geq P(\Omega^*)$.*

Proof. The entries of matrix K are as follows: If $i \neq j$, $k_{ij} = -\gamma_{ij}$. Otherwise, $k_{ii} = \sum_{j \neq i} \gamma_{ij}$. Let us divide the proof into several cases.

Case I: $i, j \in \partial\Omega$

In K, only submatrix A is affected by a change in γ_{ij}. Since C remains unchanged for both Ω and Ω^*, the power dissipated does not change, by (4.7). Thus $P(\Omega) = P(\Omega^*)$.

Case II: $i \in \partial\Omega$, $j \in int\Omega$

Divide into two subcases:

Case IIa: $j \neq s$

In matrix K, four entries are affected by a change in γ_{ij}. However, the only changed entry in C is γ_{jj}. Let us change γ_{ij} by $+x$. Then k_{ij} changes by $-x$, so k_{jj} will change by $+x$. Let us interpret $|C|$ and $|C_{1;1}|$ as functions of x. Then we can take the derivative of v_s with respect to x.

$$(4.8) \qquad \frac{dv_s}{dx} = \frac{d}{dx}\left(\frac{|C_{1;1}(x)|}{|C(x)|}\right) = \frac{1}{|C|^2}\left(|C||C_{1;1}|' - |C|'|C_{1;1}|\right)$$

Now

$$|C_{1;1}(x)| = |C_{1;1}(0)| + x|C_{1j;1j}| \text{ and} |C(x)| = |C(0)| + x|C_{j;j}|$$

which implies that

$$|C_{1;1}|' = |C_{1j;1j}| \text{ and} |C|' = |C_{j;j}|$$

Thus

$$\frac{dv_s}{dx} = \frac{|C||C_{1j;1j}| - |C_{j;j}||C_{1;1}|}{|C|^2} \tag{4.9}$$

By Lemma 4.1,

$$|C||C_{1j;1j}| = |C_{1;1}||C_{j;j}| - |C_{1;j}||C_{j;1}|$$

Therefore, since C is symmetric, $|C_{i;1}| = |C_{1;i}|$, and

$$\frac{dv_s}{dx} = \frac{-|C_{1;j}|^2}{|C|^2} < 0 \tag{4.10}$$

Case IIb: $j = s$

By the same argument above,

$$\frac{dv_s}{dx} = \frac{1}{|C|^2}\left(|C||C_{1;1}|' - |C|'|C_{1;1}|\right) \tag{4.11}$$

However, the derivatives change. By a similar calculation, it is seen that the derivatives change:

$$|C_{1;1}|' = 0 \text{ and} |C|' = |C_{1;1}|$$

Thus

$$\frac{dv_s}{dx} = \frac{-|C_{1;1}|^2}{|C|^2} < 0 \tag{4.12}$$

Case III: $i \in int\Omega, j \in int\Omega$

Again, subcases are needed:

Case IIIa: $i, j \neq s$

Assume $i < j$. Now, all four entries which change, γ_{ij} and γ_{ji}, which change by $-x$, and γ_{jj} and γ_{ii}, which change by $+x$, are in C. Rewrite $|C|$:

(4.13)

$$|C(x)| = |[C_1, \ldots, C_i + xv, \ldots, C_j - xv, \ldots, C_m]| \quad \text{where } v = \begin{matrix} \\ i \rightarrow \\ \\ j \rightarrow \\ \\ \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} &= |[C_1, \ldots, C_i, \ldots, C_j - xv, \ldots, C_m]| + x|[C_1, \ldots, v, \ldots, C_j - xv, \ldots, C_m]| \\ &= |C(0)| - x|[C_1, \ldots, C_i, \ldots, v, \ldots, C_m]| + x|[C_1, \ldots, v, \ldots, C_j, \ldots, C_m]| \\ &\qquad -x^2|[C_1, \ldots, v, \ldots, v, \ldots, C_m]| \end{aligned}$$

where C_k denotes the kth column of matrix C. From this expression, we obtain an explicit formula for $|C|'$. Now evaluate the derivative at $x = 0$:

$$|C|'\Big|_{x=0} = -(-1)^{i+j}|C_{i;j}| + |C_{j;j}| + |C_{i;i}| - (-1)^{i+j}|C_{j;i}| \tag{4.14}$$

Similarily,

$$(4.15)\quad |C_{1;1}|'\Big|_{x=0} = -(-1)^{i+j}|C_{1i;1j}| + |C_{1j;1j}| + |C_{1i;1i}| - (-1)^{i+j}|C_{1j;1i}|$$

Since C is symmetric,

$$(4.16)\quad \begin{aligned} \frac{dv_s}{dx}\Big|_{x=0} = \frac{1}{|C|^2}\Big\{&|C||C_{1i;1i}| + |C||C_{1j;1j}| - 2(-1)^{i+j}|C||C_{1i;1j}| \\ &-|C_{1;1}||C_{i;i}| - |C_{1;1}||C_{j;j}| + 2(-1)^{i+j}|C_{1;1}||C_{i;j}|\Big\}\end{aligned}$$

By Lemma 4.1,

$$(4.17)\quad |C||C_{1i;1i}| = |C_{1;1}||C_{i;i}| - |C_{1;i}||C_{i;1}|$$

$$(4.18)\quad |C||C_{1j;1j}| = |C_{1;1}||C_{j;j}| - |C_{1;j}||C_{j;1}|$$

$$(4.19)\quad |C||C_{1i;1j}| = |C_{1;1}||C_{i;j}| - |C_{1;j}||C_{i;1}|$$

Thus (4.16) simplifies to

$$(4.20)\quad \frac{dv_s}{dx}\Big|_{x=0} = \frac{-|C_{1;i}|^2 + 2(-1)^{i+j}|C_{1;i}||C_{1;j}| - |C_{1;j}|^2}{|C|^2}$$

Now if $i+j$ is even, then

(4.21)

$$\frac{dv_s}{dx}\Big|_{x=0} = \frac{-|C_{1;i}|^2 + 2|C_{1;i}||C_{1;j}| - |C_{1;j}|^2}{|C|^2} = \frac{-(|C_{1;i}| - |C_{1;j}|)^2}{|C|^2} < 0$$

If $i+j$ is odd, then

(4.22)

$$\frac{dv_s}{dx}\Big|_{x=0} = \frac{-|C_{1;i}|^2 - 2|C_{1;i}||C_{1;j}| - |C_{1;j}|^2}{|C|^2} = \frac{-(|C_{1;i}| + |C_{1;j}|)^2}{|C|^2} < 0$$

Therefore $\frac{dv_s}{dx}\big|_{x=0} < 0$. However, because this relation holds for all γ_{ij}, it holds for all x as well. So $\frac{dv_s}{dx} < 0$.

Case IIIb: $i = s$

Again,

$$(4.23)\quad |C|'\Big|_{x=0} = |C_{j;j}| + |C_{i;i}| - 2(-1)^{i+j}|C_{i;j}|$$

But

$$(4.24)\quad |C_{1;1}|' = |C_{1j;1j}|$$

Thus

$$(4.25)\quad \frac{dv_s}{dx}\Big|_{x=0} = \frac{|C||C_{1j;1j}| - |C_{1;1}|^2 - |C_{1;1}||C_{j;j}| - 2(-1)^{i+j}|C_{1;1}||C_{1;j}|}{|C|^2}$$

By Lemma 4.1,

$$(4.26)\quad |C||C_{1j;1j}| = |C_{1;1}||C_{j;j}| - |C_{1;j}|^2$$

So

$$\left.\frac{dv_s}{dx}\right|_{x=0} = \frac{-|C_{1;1}|^2 - 2(-1)^{i+j}|C_{1;1}||C_{1;j}| - |C_{1;j}|^2}{|C|^2} \tag{4.27}$$

If $i+j$ is even, then

$$\left.\frac{dv_s}{dx}\right|_{x=0} = \frac{-(|C_{1;1}| + |C_{1;j}|)^2}{|C|^2} < 0 \tag{4.28}$$

If $i+j$ is odd, then

$$\left.\frac{dv_s}{dx}\right|_{x=0} = \frac{-(|C_{1;1}| - |C_{1;j}|)^2}{|C|^2} < 0 \tag{4.29}$$

By the same argument in Case IIIa, $\frac{dv_s}{dx} < 0$ for all x. Thus in every case, $\frac{dv_s}{dx} \leq 0$. By Lemma 4.2, this implies that $P(\Omega) \geq P(\Omega^*)$. □

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