AN EXTENTION OF THE CLASS OF RECOVERABLE RESISTOR NETWORKS

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Abstract. We show that the set of recoverable resistor networks is much larger than it was previously known to be. We also give a new necessary condition for the recoverability of such a network.

1. Introduction

The primary purpose of this note is to show that many more networks of electrical resistors are recoverable from boundary measurements than was previously known.

The problem of recovering the conductances of a network from measurements of current and voltage taken at the boundary is a specialized, slightly modified, case of the inverse problem for manifolds described in [Ca]. The network recoverability problem has been studied extensively by Curtis and Morrow, who present a summary of their results in [CM]. The central question in [CM] as well as here is the following: whether conductances on the edges of a known graph $\mathfrak{G}$ with a designated set of boundary nodes $\partial\mathfrak{G} \subset V(G)$ can be recovered from the Dirichlet-to-Neumann (*i.e.* voltage-to-current) map.

To date the greatest advance on this problem was made by Ingerman, Curtis and Morrow who gave in [CIM] formulae and an algorithm of sorts for the recovery of the conductances of a critical circular planar graph. Many of the important definitions and some other necessary preliminaries were available to these authors from Colin de Verdiere [CdV1, CdV2]. We recall that a graph is *critical* if the removal or contraction of any edge in the graph breaks a *connection*, where a connection is set of vertex-disjoint paths from boundary nodes $(\alpha_1, ..., \alpha_n)$ to boundary nodes $(\beta_{\pi(1)}, ..., \beta_{\pi(n)})$, with the futher restriction that path k can hit no boundary nodes other than α_k and $\beta_{\pi(k)}$. (The connection is said to be broken if there is no longer a set of paths satisfying these criteria.) Circular planar graphs are the special graphs which can be embedded in the plane in such a way that all of their boundary nodes lie in the unit circle.

Nonetheless, there are various examples which can be recovered by essentially the same methods as had been applied to the circular planar graphs, but which did not fall into the same topological class. The examples include certain graphs which can be embedded in the plane in such a way that their boundary nodes lie in the boundary of an annulus but which are not circular planar. Also, the complete graphs, and more generally any graph with only boundary-to-boundary edges, is recoverable by a formula given in [CIM 4.2, CM 3.15] (which is a fact that will be relied upon heavily in our work).

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Ours is a "clear box" inverse problem: we make measurements only at the boundary, but we know the graph structure of the network we are attempting to recover. The task we have set is to understand the necessary and sufficient conditions on the structure of a graph for its recoverability from boundary measurements. This note falls short of such a characterization. What we provide is a new framework for the problem.

The strategy is to use the recoverability of complete graphs to reduce the problem of recovering a network made up of numerous smaller recoverable networks to a local problem that we know how to solve. The difficulty lies in the fact that we can not make measurements on subnetworks directly because current flows everywhere. Theorem 1 shows how to circumvent this difficulty in a setting in which the graphs have been glued together in a special way. Theorem 2 shows, in the form of a necessary condition for recoverability, that this sort of gluing is actually the only one that works.

The proofs of these theorems are simple. The real difficulty was to find the right viewpoint, since many of the graphs studied here look, at first, to be nothing like the those studied by previous researchers. The simplicity of the methods may also point to their usefulness in other settings. For instance, Alex Postnikov [P] has worked out the theory of recoverable *directed* networks (which correspond to diodes) in a circular planar setting. An extention similar ours in the directed case looks to be much more difficult to obtain, because cycles do not behave nearly as well in space as they do in the plane. It is clear that similar methods to ours may also be brought to bear in some special cases of the analogous problem for manifolds.

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2. Many more recoverable networks

Here we give an extension of the results of [CIM, CM], and discuss some corollaries.

Theorem 1. *Any graph $\mathfrak{G}$ that is decomposable into a collection of distinct recoverable graphs $\{G_i\}$, disjoint except that G_i and G_j may share up to one boundary node, $\forall\ i \neq j$. Then $\mathfrak{G}$ is recoverable.*

Proof. We will focus our attention on recovering one of the G_i's; if we can do this, then we can iterate and recover $\mathfrak{G}$. Let $\mathcal{N}(G_i)$ be the collection $\{G_j : |\partial G_j \cap \partial G_i| = 1\}$.

We restrict ourselves to making measurements at nodes $\beta \in \partial\mathcal{N}(G_i) = \{\partial G_j : G_j \in \mathcal{N}(G_i)\}$. (This set of course includes the boundary nodes of G_i.) We write $\mathfrak{G} = G_i \cup \mathcal{N}(G_i) \cup R$. Each pair of vertices $\alpha \in \partial G_j - \partial G_i, \beta \in \partial G_k - \partial G_i$, where G_j, G_k are in $\mathcal{N}(G_i)$ may (or may not) be connected through R.

Hence for *some* edge-weighting R is electrically equivalent to

$$K(\partial\mathcal{N}(G_i) - \partial G_j) - S,$$

where S is the set $\{\partial G_j\text{–to–}\partial G_j \text{ edges}\} \cup \{\text{self edges in } K(\partial\mathcal{N}(G_i) - \partial G_j) \text{ induced by identifications of boundary nodes among the } \partial G_j \text{ with } G_j \in \mathcal{N}(G_i)\}$

As constructed, some of the edges in R may have infinite resistance; this happens when two nodes in $\partial G_j, \partial G_k \in \mathcal{N}(G_i)$ are not connected through $\mathfrak{G} - G_i - \mathcal{N}(G_i)$. Since we know the graph structure, we can forget about these edges, which are of no electrical significance (and we do this now, without changing notation).

We make an important simplification by noting that for our purposes each G_j in $\mathcal{N}(G_i)$ is electrically equivalent to the complete graph on $V(G_j)$. Since for the time being we are not interested in recovering G_j, henceforward when we speak of an edge in G_j, we really mean an edge in $K(V(G_j))$. In addition we forget about $G_j \in \mathcal{N}(G_i)$ if there are no edges between $\partial G_j - \partial G_i$ and the rest of the graph, because then if we make no measurements on $\partial G_j - \partial G_i$, G_j is electrically insignificant.

We now show that removing any edge in R or any edge in $G_j \in \mathcal{N}(G_i)$ breaks some connection in $\mathfrak{G}$. First consider an edge $e = \alpha, \beta$ in $G_j \in \mathcal{N}(G_i)$. Removing e breaks the connection $\{e\}$, because all neighbors of α in $\partial\mathcal{N}(G_i)$ are boundary nodes, and moreover if $\alpha \in \partial G_i$, there is no path to β through G_i that avoids hitting some other boundary node. Similarly, both endpoints of any edge in R have only boundary nodes as neighbors. Hence removing $e \in R$ breaks the connection $\{e\}$.

Hence by [CIM 4.2, CM 3.15] we can recover effective resistances on the edges in $\mathfrak{G} - G_i$ as seen by someone making measurements only at the boundary nodes of $G_j \in \mathcal{N}(G_i)$. We can collapse this data to get resistances on edges in $K(\partial G_i)$, so that $\mathfrak{G}$ is seen, by someone making measurements only at nodes in ∂G_i, as electrically equivalent to $G_i \cup K(\partial G_i)$.

Now the problem of recovering G_i is just the problem of recovering a recoverable graph with added *known* boundary-to-boundary edges. But by [CIM Section 8] this is a recoverable graph, and so the proof is complete. □

Remark. The reason that this works is that the proof that we can recover boundary-to-boundary edges whose removal breaks a connection [CIM 4.2, CM 3.15] is purely linear algebra, and does not rely in any way on the hypothesis that the graph is circular planar.

Theorem 1 implies the existance of some truly new and interesting examples, like the following.

Example. Let M be the 4-regular mesh on the torus $\mathbf{T}^2$ obtained by identifying opposite sides of the rectangle pictured in Figure 1. Let the existing vertices of M be interior nodes, and add a boundary node on each existing edge. By Theorem 1, the resulting graph is recoverable. Furthermore, it has no boundary-to-boundary edges, nor any boundary-to-interior spikes, which were necessary ingredients for [CIM, CM].

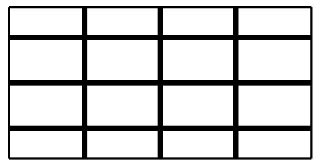

FIGURE 1. A mesh on the torus

Corollary (Interior Condition on Recoverability). If every decomposition of $\mathfrak{G}$ into distinct graphs $\{G_i\}$, where a pair G_i and G_j which can share up to one boundary node, is such that one of the G_i in each decomposition is non-recoverable, then $\mathfrak{G}$ is not recoverable.

This corollary is of course only a modest extension of the trivial result that if one of G_i, G_j is not recoverable, neither is $G_i \sqcup G_j$. But it serves as an example of the sort of local result that can determine the recoverability of a network. The following section has a less obvious result in the same vein.

3. A necessary condition

We show that the intersection conditions of Theorem 1 are in some sense characteristic for all recoverable graphs.

Theorem (Exterior Condition on Recoverability). Let $\mathfrak{G}$ be decomposable into a collection of distinct connected graphs $\{G_i : E(G_i) \neq \phi\}$, disjoint except possibly at the boundary. Suppose that $|\partial G_i \cap \partial G_j| > 1$ for some i, j with $i \neq j$. Then $\mathfrak{G}$ is not recoverable.

Proof. It is necessary to send current between each pair α and β of distinct boundary nodes to recover the graph. For, suppose that we have a recoverable graph, and we try to avoid sending current between α and β as long as possible, recovering the rest of the graph first. We would obtain a single edge between α and β, which would then have to be recovered.

If α and β are common to the boundaries of two distinct graphs G_1 and G_2, we can not send current into α without it flowing through both graphs into β. Hence, we can not get an accurate measurement of the current at β, and so can not recover $\mathfrak{G}$. □

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