Connections and Determinants

Mark Blunk
Sam Coskey

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Abstract

The relationship between connections and determinants in conductivity networks is discussed. We paraphrase Lemma 3.12, by Curtis and Morrow [1], infamous in smaller circles as “the page 50 proof.”

1 Introduction

A conductivity network G is given by a graph Γ, and some arrangement of conductivities. The vertices of the graph are made up of boundary ($\mathcal{B}$) and interior ($\mathcal{I}$) nodes, and are ordered with the boundary nodes listed first. The graph may be directed; in special cases, the conductivities will be assigned to bidirectional edges [1] or vertices [2].

Vectors of potentials u and currents I are defined over the vertices of Γ. The Kirchhoff matrix K is defined so that $K \cdot u = I$. The information in this matrix is equivalent to G.

The response matrix Λ is defined so that $\Lambda \cdot u|_{\mathcal{B}} = I(u)|_{\mathcal{B}}$. Λ is thus the Schur complement of $K(\mathcal{I};\mathcal{I})$ in K, that is $\Lambda = K/K(\mathcal{I};\mathcal{I})$. For now, we will assume that $K(\mathcal{I};\mathcal{I})$ is invertible; we will later impose certain hypotheses to guarantee that this is true.

We take the definition of connection from [1]. Given two disjoint sets of k boundary nodes, P and R, a connection from P to R is a set of vertex disjoint paths through Γ from the vertices of P to the vertices of R through the interior nodes $\mathcal{I}$.

1.1 Connections and permutations

Definition 1.1. Let S_m represent the permutation group on m symbols.

Suppose P and R are given as above. Let $M = K(P+\mathcal{I}; R+\mathcal{I})$. Then

$$\det M = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} M(i;\sigma(i)) \tag{1}$$

In the above formula, a permuation σ of n elements may be interpreted as a possible connection from P to R. The path can be constructed as follows. Suppose that all of the nodes in P are numbered $1, 2, \ldots, k$, the nodes of R are also numbered $1, 2, \ldots, k$, and that the interior nodes denoted $\mathcal{I}$ are numbered $k+1, \ldots, n$.

Begin with the ordered set $T_0 = (1, \ldots, k)$, the nodes of P. This is the beginning of the path. The first step in the path, the set T_1, is obtained by permuting each element of the set by σ. In this new set, mark each element that is less than or equal to k. To obtain the second step in the path, T_2, let σ act on each unmarked element of T_1, keeping the marked elements unchanged. Continue this process, until every element of the newest set is marked. This set is the last step.

Suppose the last step of the path is T_m. These are the elements of R. In all of the ordered sets T_0 up to T_m, no element greater than k is repeated. Thus, each σ represents a single vertex-disjoint connection from P to R. However, σ is only an actual connection through a network if every vertex i in $P \cup \mathcal{I}$ is connected by a directed edge to the vertex $\sigma(i)$.

2 The Determinental Expansion

It is clear that their is a relationship between connections through a network and the determinant of the Kirchhoff matrix for that network. It is natural to seek a representation of paths within corresponding determinants.

Definition 2.1. Suppose S is a set. Then $\pi(\mathcal{S})$ is the set of all orderings of $\mathcal{S}$.

Definition 2.2. Suppose A is a matrix, and $I = (i_1, i_2, \ldots, i_m)$ and $J = (j_1, j_2, \ldots, j_n)$ are subsets of the rows and columns of A, respectively.

- Let $D[I, J]$ represent the determinant of an $m \times m$ sub-matrix of A obtained by deleting the entries of A in rows $i_1, \ldots, i_m$ and in columns $j_1, \ldots, j_n$.
- Let $D(I, J)$ represent the determinant of an $m \times m$ sub-matrix of A obtained by including only the entries of A in rows $(i_1, \ldots, i_m)$ and in columns $(j_1, \ldots, j_m)$, in listed order. Of course, this definition and the previous only make sense when $m = n$.
- Let $a_{p \mapsto I \mapsto q} = a_{ps_1} a_{s_1 s_2} \cdots a_{s_m q}$, where $-a_{ij}$ is the entry in the i^{th} row and j^{th} column of A, for $i \neq j$.
- Let the symbol $I + J$ denote the concatenation $(i_1, \ldots, i_m, j_1, \ldots, j_n)$.

We now proceed to describe an expansion of the determinant of a general matrix. When the matrix is interpreted as a subdeterminant of a Kirchhoff matrix, the terms of the expansion may be interpreted as boundary to boundary connections.

2.1 One-expansion

We first examine the case that we wish to examine a one-connection, a single path. The sets P and R would thus have one element each. The following lemma is stated very generally, but the matrix A is of the form of, and will later be interpreted as, the submatrix $K(P + \mathcal{I}; R + \mathcal{I})$.

Lemma 2.3. *Let A be an $n \times n$ matrix, with negative entries everywhere except on the second through n^{th} diagonals.*

$$A = \left[\begin{array}{c|cccc} -a_{11} & -a_{12} & -a_{13} & & -a_{1n} \\ \hline -a_{21} & +a_{22} & -a_{23} & \cdots & -a_{2n} \\ -a_{31} & -a_{32} & +a_{33} & & -a_{3n} \\ & \vdots & & \ddots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & +a_{nn} \end{array}\right]$$

Let the ordered set $\mathcal{B} = (1)$, and the ordered set $\mathcal{I} = (2, \ldots, n)$. Suppose that J be a subset of $\mathcal{I}$. Suppose that $\sigma \in \pi(J)$. Then the determinant of A may be written as follows.

$$\det A \quad = \quad -\sum_{\sigma} a_{1 \mapsto \sigma \mapsto 1} \cdot D[1, J; 1, J] \tag{2}$$

Proof. First, note that the expression (2) may be expanded as follows.

$$\begin{aligned} \det A \quad = \quad -a_{11} \cdot D[1; 1] \quad & - \quad \sum_{j \neq 1} (a_{1j} a_{j1}) \cdot D[1, j; 1, j] \\ & - \sum_{\substack{j_1, j_2 \neq 1 \\ j_1 \neq j_2}} (a_{1j_1} a_{j_1 j_2} a_{j_2 1}) \cdot D[1, j_1, j_2; 1, j_1, j_2] \\ & - \cdots \\ & - \sum_{\substack{j_i \neq 1 \\ j_i \neq j_k}} (a_{1j_1} a_{j_1 j_2} \cdots a_{j_n 1}) \cdot 1 \end{aligned}$$

We proceed by induction on n. First, note that when $n = 1$, J is null, so the determinant of A is equal to its lone element. Hence (2) would correctly give $-a_{11} \cdot 1$ for $\det A$.

Now, suppose that given a matrix of the form A, but of size $(n-1) \times (n-1)$, its determinant may be computed using the above expansion. We will show that the determinant of A can be computed using the above expansion.

The cofactor expansion of the determinant gives us the following expression.

$$\det A = -a_{11} \cdot D[1;1] + a_{12} \cdot D[1;2] - a_{13} \cdot D[1,3] + a_{14} \cdot D[1;4] - + \cdots + (-1)^n a_{1n} \cdot D[1;n]$$

In each subdeterminant in the above expansion (except the first), perform $i-2$ row interchanges so that the i^{th} row is now the top row of the subdeterminant. These row interchanges result in multiplying each term by $(-1)^{i-2}$, thus changing the sign of each odd term.

$$\begin{aligned} = -a_{11} \cdot D[1;1] \quad &+ \quad a_{12} \cdot D(2,3,4,5,\ldots,n;1,3,4,5,\ldots,n) \\ &+ \quad a_{13} \cdot D(3,2,4,5,\ldots,n;1,2,4,5,\ldots,n) \\ &+ \quad a_{14} \cdot D(4,2,3,5,\ldots,n;1,2,3,5,\ldots,n) \\ &+ \quad \cdots \\ &+ \quad a_{1n} \cdot D(n,2,3,4,\ldots,n-1;1,2,3,4,\ldots,n-1) \end{aligned}$$

Each determinant in the above expression is now of the form of A, but of size $(n-1) \times (n-1)$. Thus, by the inductive assumpsion, we may rewrite each of them using (2).

$$\begin{aligned} = -a_{11} \cdot D[1;1] \quad & - a_{12}\Big\{a_{21} \cdot D[1,2;1,2] && + \sum_{j \neq 1,2} (a_{2j}a_{j1}) \cdot D[1,2,j;1,2,j] \\ &&& + \cdots \\ &&& + \sum_{\substack{j_i \neq 1,2 \\ j_i \neq j_k}} (a_{2j_1} \cdots a_{j_{n-2}1}) \cdot 1\Big\} \\ & - a_{13}\Big\{a_{31} \cdot D[1,3;1,3] && + \sum_{j \neq 1,3} (a_{3j}a_{j1}) \cdot D[1,3,j;1,3,j] \\ &&& + \cdots \\ &&& + \sum_{\substack{j_i \neq 1,3 \\ j_i \neq j_k}} (a_{3j_1} \cdots a_{j_{n-2}1}) \cdot 1\Big\} \\ & - \cdots \\ & - a_{1n}\Big\{a_{n1} \cdot D[1,n;1,n] && + \sum_{j \neq 1,n} (a_{nj}a_{j1}) \cdot D[1,n,j;1,n,j] \\ &&& + \cdots \\ &&& + \sum_{\substack{j_i \neq 1,n \\ j_i \neq j_k}} (a_{nj_1} \cdots a_{j_{n-2}1}) \cdot 1\Big\} \end{aligned}$$

Each expansion in brackets contains the same number of terms. Defactoring (carrying through) the multipliers in front of each group (the a_{1i} terms), and combining the corresponding terms (those of equal length) from each group, gives the result. $\square$

2.2 Two-expansion

We now consider the case that P and R each have two elements. The proof will proceed along similar lines.

Lemma 2.4. *Let A be an $n \times n$ matrix, with negative entries everywhere except on the third through n^{th}*

diagonals.

$$A = \left[\begin{array}{cc|cccc} -a_{11} & -a_{12} & -a_{13} & & -a_{1n} \\ -a_{21} & -a_{22} & -a_{23} & \cdots & -a_{2n} \\ \hline -a_{31} & -a_{32} & +a_{33} & & -a_{3n} \\ & \vdots & & \ddots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & +a_{nn} \end{array}\right]$$

Let the ordered set $\mathcal{B} = (1, 2)$, and the ordered set $\mathcal{I} = (3, \ldots, n)$. Suppose that J is a subset of $\mathcal{I}$. Suppose that the sets $\{J_1, J_2\}$ partition J into two subsets. Further suppose that $\sigma_1 \in \pi(J_1)$ and that $\sigma_2 \in \pi(J_2)$. Then the determinant of A may be written as follows.

$$\det A \quad = \quad \sum_{\sigma_1, \sigma_2} (a_{1\mapsto\sigma_1\mapsto 1} \cdot a_{2\mapsto\sigma_2\mapsto 2} - a_{1\mapsto\sigma_1\mapsto 2} \cdot a_{2\mapsto\sigma_2\mapsto 1}) \cdot D[1, 2, J; 1, 2, J] \tag{3}$$

Proof. First, note that the equation (3) may be expanded into the following.

$$\begin{array}{rcll} \det A & = & (a_{11}a_{22} - a_{12}a_{21}) \cdot D[1, 2; 1, 2] & \\ & + & \sum_{j_1 \in \mathcal{I}} & (a_{11}a_{2j_1}a_{j_1 2} - a_{12}a_{2j_1}a_{j_1 1} \\ & & & a_{1j_1}a_{j_1 1}a_{22} - a_{1j_1}a_{j_1 2}a_{21}) \cdot D[1, 2, j_1; 1, 2, j_1] \\ & + & \sum_{\substack{j_1, j_2 \in \mathcal{I} \\ j_1 \neq j_2}} & (a_{11}a_{2j_1}a_{j_1 j_2}a_{j_2 2} - a_{12}a_{2j_1}a_{j_1 j_2}a_{j_2 1} \\ & & & + a_{1j_1}a_{j_1 1}a_{2j_2}a_{j_2 2} - a_{1j_1}a_{j_1 2}a_{2j_2}a_{j_2 1} \\ & & & + a_{1j_1}a_{j_1 j_2}a_{j_2 1}a_{22} - a_{1j_1}a_{j_1 j_2}a_{j_2 2}a_{21}) \cdot D[1, 2, j_1, j_2; 1, 2, j_1, j_2] \\ & + & \cdots & \\ & + & \sum_{\substack{j_i \in \mathcal{I} \\ j_i \neq j_k}} & (a_{11}a_{2j_1}a_{j_1 j_2} \cdots a_{j_{n-2} 2} - a_{12}a_{2j_1}a_{j_1 j_2} \cdots a_{j_{n-2} 1} \\ & & & + a_{1j_1}a_{j_1 1}a_{2j_2} \cdots a_{j_{n-2} 2} - a_{1j_1}a_{j_1 2}a_{2j_2} \cdots a_{j_{n-2} 1} \\ & & & + \cdots \\ & & & + a_{1j_1}a_{j_1 j_2} \cdots a_{j_{n-2} 1}a_{22} - a_{1j_1}a_{j_1 j_2} \cdots a_{j_{n-2} 1}a_{22}) \cdot 1 \end{array}$$

We proceed by induction on n. When $n = 2$, J_1 and J_2 are null. Hence, (3) reduces to $(a_{11}a_{22} - a_{12}a_{21}) \cdot 1$, which is the determinant of A. When $n = 3$, (3) reduces to

$$\det A = (a_{11}a_{22} - a_{12}a_{21}) \cdot a_{33} + (a_{13}a_{31}a_{22} - a_{13}a_{32}a_{21} + a_{11}a_{23}a_{32} - a_{12}a_{23}a_{31}) \cdot 1$$

which is equivalent to determinant of A. Now suppose that given a matrix of the form A, but of size $(n-2) \times (n-2)$, its determinant may be computed using the above expansion. We will show that the Laplace expansion of the determinant of A can be manipulated to derive (3).

Taking the Laplace expansion of the determinant of A along the first two rows gives the following expression.

$$
\begin{aligned}
\det A = \quad & \Big\{(a_{11}a_{22} - a_{12}a_{21}) \cdot D(3,4,5,6,\ldots,n;3,4,5,6,\ldots,n) \\
& - (a_{11}a_{23} - a_{13}a_{21}) \cdot D(3,4,5,6,\ldots,n;2,4,5,6,\ldots,n) \\
& + (a_{11}a_{24} - a_{14}a_{21}) \cdot D(3,4,5,6,\ldots,n;2,3,5,6,\ldots,n) \\
& - (a_{11}a_{25} - a_{15}a_{21}) \cdot D(3,4,5,6,\ldots,n;2,3,4,6,\ldots,n) \\
& + - \cdots \\
& + (-1)^{n+4}(a_{11}a_{2n} - a_{1n}a_{21}) \cdot D(3,4,5,6,\ldots,n;2,3,4,\ldots,n-1)\Big\} \\
+ \quad & \Big\{(a_{12}a_{23} - a_{13}a_{22}) \cdot D(3,4,5,6,\ldots,n;1,4,5,6,\ldots,n) \\
& - (a_{12}a_{24} - a_{14}a_{22}) \cdot D(3,4,5,6,\ldots,n;1,3,5,6,\ldots,n) \\
& + (a_{12}a_{25} - a_{15}a_{22}) \cdot D(3,4,5,6,\ldots,n;1,3,4,6,\ldots,n) \\
& - + \cdots \\
& + (-1)^{n+5}(a_{12}a_{2n} - a_{1n}a_{22}) \cdot D(3,4,5,6\ldots,n;1,3,4,5,\ldots,n-1)\Big\} \\
+ \quad & \Big\{(a_{13}a_{24} - a_{14}a_{23}) \cdot D(3,4,5,6,\ldots,n;1,2,5,6,\ldots,n) \\
& - (a_{13}a_{25} - a_{15}a_{23}) \cdot D(3,4,5,6,\ldots,n;1,2,4,6,\ldots,n) \\
& + - \cdots \\
& + (-1)^{n+6}(a_{13}a_{2n} - a_{1n}a_{23}) \cdot D(3,4,5,6,\ldots,n;1,2,4,5\ldots,n-1)\Big\} \\
+ \quad & \cdots \\
+ \quad & (a_{1(n-1)}a_{2n} - a_{1n}a_{2(n-1)}) \cdot D(3,4,5,6,\ldots,n;1,2,3,4,\ldots,n-2)
\end{aligned}
$$

For every $(n-2) \times (n-2)$ subdeterminant, let j_1, j_2 be the indices of the first and second columns deleted by the Laplace expansion. Now perform the following row interchanges. If $j_1 \notin \mathcal{I}$ and $j_2 \in \mathcal{I}$, perform $j_2 - 3$ row interchanges so that row j_2 is now the first row in the subdeterminant. If $j_1, j_2 \in \mathcal{I}$, perform $j_1 - 3$ row intechanges so that row j_1 is now in the first row, and perform $j_2 - 4$ row interchanges so that row j_2 is now in the second row of the subdeterminant. No row changes are peformed on the subdeterminant where columns 1 and 2 have been deleted. The result of these row interchanges is to negate every subdeterminant where row 1 is deleted, and to change the sign of every other term to positive. Continuing from above,

$$
\begin{aligned}
= \quad & \Big\{ \ (a_{11}a_{22} - a_{12}a_{21}) \cdot D(3,4,5,6,\ldots,n;3,4,5,6\ldots,n) \\
& - (a_{11}a_{23} - a_{13}a_{21}) \cdot D(3,4,5,6,\ldots,n;2,4,5,6\ldots,n) \\
& - (a_{11}a_{24} - a_{14}a_{21}) \cdot D(4,3,5,6,\ldots,n;2,3,5,6,\ldots,n) \\
& - (a_{11}a_{25} - a_{15}a_{21}) \cdot D(5,3,4,6,\ldots,n;2,3,4,6,\ldots,n) \\
& - \cdots \\
& - (a_{11}a_{2n} - a_{1n}a_{21}) \cdot D(n,3,4,5,\ldots,n-1;2,3,4,5\ldots,n-1)\Big\} \\
+ \quad & \Big\{ \ (a_{12}a_{23} - a_{13}a_{22}) \cdot D(3,4,5,6,\ldots,n;1,4,5,6\ldots,n) \\
& + (a_{12}a_{24} - a_{14}a_{22}) \cdot D(4,3,5,6,\ldots,n;1,3,5,6\ldots,n) \\
& + (a_{12}a_{25} - a_{15}a_{22}) \cdot D(5,3,4,6\ldots,n;1,3,4,6,\ldots,n) \\
& + \cdots \\
& + (a_{12}a_{2n} - a_{1n}a_{22}) \cdot D(n,3,4,5,\ldots,n-1;1,3,4,5,\ldots,n-1)\Big\} \\
+ \quad & \Big\{ \ (a_{13}a_{24} - a_{14}a_{23}) \cdot D(3,4,5,6,\ldots,n;1,2,5,6,\ldots,n) \\
& + (a_{13}a_{25} - a_{15}a_{23}) \cdot D(3,5,4,6,\ldots,n;1,2,4,6,\ldots,n) \\
& + \cdots \\
& + (a_{13}a_{2n} - a_{1n}a_{23}) \cdot D(3,n,4,5,\ldots,n-1;1,2,4,5,\ldots,n-1)\Big\} \\
+ \quad & \cdots \\
+ \quad & (a_{1(n-1)}a_{2n} - a_{1n}a_{2(n-1)}) \cdot D(n-1,n,3,4,\ldots,n-2;1,2,3,4,\ldots,n-2)
\end{aligned}
$$

More simply, this can be stated

$$
\begin{array}{rl}
= & (a_{11}a_{22} - a_{12}a_{21}) \cdot D[1,2;1,2] \\
- & \sum_{j_1 \in \mathcal{I}} (a_{11}a_{2j_1} - a_{1j_1}a_{22}) \cdot D(j_1, \mathcal{I}\backslash\{j_1\}; 2, \mathcal{I}\backslash\{j_1\}) \\
+ & \sum_{j_1 \in \mathcal{I}} (a_{12}a_{2j_1} - a_{1j_1}a_{21}) \cdot D(j_1, \mathcal{I}\backslash\{j_1\}; 1, \mathcal{I}\backslash\{j_1\}) \\
+ & \sum_{j_1, j_2 \in \mathcal{I}} (a_{1j_1}a_{2j_2} - a_{1j_2}a_{2j_1}) \cdot D(j_1, j_2, \mathcal{I}\backslash\{j_1, j_2\}; 1, 2, \mathcal{I}\backslash\{j_1, j_2\})
\end{array}
$$

By the inductive assumption, the subdeterminant of size $(n-2) \times (n-2)$ in the third sum can be expressed by (3). The subdeterminants of size $(n-2) \times (n-2)$ in the first two sums contain only one negative diagonal entry, so they refer to a one connection, and can be expressed by (2). Hence the subdeterminants in all three sums can be expanded by the assumption to produce the following expression.

$$
\begin{array}{lll}
 & (a_{11}a_{22} - a_{12}a_{21}) \cdot D[1,2;1,2] & \\
- & \sum_{j_1 \in \mathcal{I}} (a_{11}a_{2j_1} - a_{1j_1}a_{21}) & \Big\{ - a_{j_1 2} \cdot D[1,2,j_1;1,2,j_1] \\
 & & - \quad \sum_{j_1, j_2 \in \mathcal{I}\backslash\{j_1\}} (a_{j_1 j_2} a_{j_2 2}) \cdot D[1,2,j_1,j_2;1,2,j_1,j_2] \\
 & & - \quad \cdots \\
 & & - \quad \sum_{j_i \in \mathcal{I}\backslash\{j_1\}} (a_{j_1 j_2} \cdots a_{j_{n-3} j_{n-2}} a_{j_{n-2} 2}) \cdot 1 \Big\} \\
+ & \sum_{j_1 \in \mathcal{I}} (a_{12}a_{2j_1} - a_{1j_1}a_{22}) & \Big\{ - a_{j_1 1} \cdot D[1,2,j_1;1,2,j_1] \\
 & & - \quad \sum_{j_2 \in \mathcal{I}\backslash\{j_1\}} (a_{j_1 j_2} a_{j_2 1}) \cdot D[1,2,j_1,j_2;1,2,j_1,j_2] \\
 & & - \quad \cdots \\
 & & - \quad \sum_{j_i \in \mathcal{I}\backslash\{j_1\}} (a_{j_1 j_2} \cdots a_{j_{n-3} j_{n-2}} a_{j_{n-2} 1}) \cdot 1 \Big\} \\
+ & \sum_{j_1, j_2 \in \mathcal{I}} (a_{1j_1}a_{2j_2} - a_{1j_2}a_{2j_1}) & \Big\{ (a_{j_1 1} a_{j_2 2} - a_{j_1 2} a_{j_2 1}) \cdot D[1,2,j_1,j_2;1,2,j_1,j_2] \\
 & & + \quad \sum_{j_3 \in \mathcal{I}\backslash\{j_1, j_2\}} (a_{j_1 1} a_{j_2 j_3} a_{j_3 2} - a_{j_1 2} a_{j_2 j_3} a_{j_3 1} \\
 & & \qquad + a_{j_1 j_3} a_{j_3 1} a_{j_2 2} - a_{j_1 j_3} a_{j_3 2} a_{j_2 1}) \cdot D[1,2,j_1,j_2,j_3;1,2,j_1,j_2,j_3] \\
 & & + \quad \cdots \\
 & & + \quad \sum_{j_i \in \mathcal{I}\backslash\{j_1, j_2\}} (a_{j_1 1} a_{j_2 j_3} a_{j_3 j_4} \cdots a_{j_{n-3} j_{n-2}} a_{j_{n-2} 2} \\
 & & \qquad - a_{j_1 2} a_{j_2 j_3} a_{j_3 j_4} \cdots a_{j_{n-3} j_{n-2}} a_{j_{n-2} 1} \\
 & & \qquad + a_{j_1 j_3} a_{j_3 1} a_{j_2 j_4} \cdots a_{j_{n-3} j_{n-2}} a_{j_{n-2} 2} \\
 & & \qquad - a_{j_1 j_3} a_{j_3 2} a_{j_2 j_4} \cdots a_{j_{n-3} j_{n-2}} a_{j_{n-2} 1} \\
 & & \qquad + - \cdots \\
 & & \qquad + a_{j_1 j_3} a_{j_3 j_4} \cdots a_{j_{n-3} j_{n-2}} a_{j_{n-2} 1} a_{j_2 2} \\
 & & \qquad - a_{j_1 j_3} a_{j_3 j_4} \cdots a_{j_{n-3} j_{n-2}} a_{j_{n-2} 2} a_{j_2 1}) \cdot 1 \Big\}
\end{array}
$$

As was the case with the one connection, defactoring the multipliers of each expression in curly braces gives the result. ☐

2.3 k-expansion

Next, we will generalize the above expansions to represent a connection of any size. But first, some junk about permutations.

Definition 2.5. Let $\binom{I}{J} = \binom{i_1, i_2, \ldots, i_m}{j_1, j_2, \ldots, j_m}$ be the mapping from the elements of I to the elements of J in their listed order. Suppose p is the number of interchanges required to sort the elements $(j_1, j_2, \ldots, j_m)$ into increasing order. Then let $\operatorname{sign}\binom{I}{J} = (-1)^p$. Let the binary infix operator $\circ$ act on two mappings, and represent the new mapping obtained by composing the two argument mappings.

Lemma 2.6. *Suppose that S is an ordered set of integers, that I and J partition S, and that I and J are sorted in increasing order. Then* $\operatorname{sign}\binom{S}{I+J} = (-1)^{\sum I - \sum_1^{|I|} i}$. *Simplifying the sum, and changing its sign (which does not affect its value in modulus 2) gives us the expression* $\operatorname{sign}\binom{S}{I+J} = (-1)^{\sum I + \frac{1}{2}|I|(|I|+1)}$.

Proof outline. This concept is perhaps best explained with an example. Suppose that $S = (1, \ldots, 10)$, and that $I = (3, 4, 9)$. $I + J$ is therefore $(3, 4, 9, 1, 2, 5, 6, 7, 8, 10)$. To sort this set, the third element requires

$9-3$ interchanges to bring it to the 9^{th} spot from the 3^{rd} position. Similarly, the second element must be interchanged $4-2$ times, and the first element must be interchanged $3-1$ times. Summing together all of the interchanges gives

$$(3-1)+(4-2)+(9-3)=\sum I-\sum_{1}^{3} i$$

The result is hopefully an intuitive consequence of understanding this example. $\square$

Lemma 2.7. (The Curtis-Morrow expansion) *Let A be an $n \times n$ matrix, with negative entries everywhere except on the $(k+1)^{\text{st}}$ through n^{th} diagonals.*

$$A = \left[\begin{array}{ccc|ccc} -a_{11} & & -a_{1k} & & & -a_{1n} \\ & \ddots & & & & \\ -a_{k1} & & -a_{kk} & & & \\ \hline & & & +a_{k+1,k+1} & & -a_{k+1,n} \\ & & & & \ddots & \\ -a_{n1} & & & -a_{n,k+1} & & +a_{nn} \end{array}\right]$$

Let the ordered set $\mathcal{B}=(1,\dots,k)$, and the ordered set $\mathcal{I}=(k+1,\dots,n)$. Suppose that J is a subset of $\mathcal{I}$. Suppose that $\{J_1, J_2, \dots, J_k\}$ partition J into k subsets. Suppose that σ is a set of k elements such that $\sigma_i \in \pi(J_i)$, for $1 \le i \le k$. Then the determinant of A may be written as follows.

$$\det A = (-1)^k \sum_{\tau \in S_k} \operatorname{sign}(\tau) \sum_{\sigma} \left\{ \left(\prod_{i=1}^{k} a_{i \mapsto \sigma_i \mapsto \tau(i)} \right) \cdot D[\mathcal{B} \cup J; \mathcal{B} \cup J] \right\} = (P \Rightarrow R) \tag{4}$$

Further, the above expression is to be interpreted as a connection from $P=\mathcal{B}$ to $R=\mathcal{B}$ through the interior nodes $\mathcal{I}$, and is denoted $(P \Rightarrow R)$. A direct connection that does not go through any interior nodes is denoted $(P \to R)$.

Proof. We proceed by induction on k, the size of the upper-left (purely negative) quadrant of A, and on n, the dimension of the matrix. Notice that (4) holds for $k=1$ by Lemma 2.3. Also notice that when $n=k$, the expansion (4) is merely the definition of the determinant of a matrix of size $k \times k$ with all negative entries.

Now suppose inductively that for any matrix of the form of A, with dimension at most $(n-1)\times(n-1)$, and upper left-hand corner of size at most $k \times k$, that its determinant can be expressed by (4) for any size connection less than k. We will show that the Laplace expansion on the first k rows of A is equivalent to the expansion (4).

Taking the Laplace expansion of the determinant of A along the first k rows gives the following expression.

$$\det A = \sum_{|Q|=k} (-1)^{\sum P + \sum Q}\ D(P;Q)\ D[P;Q]$$

In each term of the above sum, some of the elements of Q will be in the set $\mathcal{B}$, and the rest will be in the set $\mathcal{I}$. Let Q_B consist of the elements of Q that are elements of $\mathcal{B}$, and Q_I consist of the elements of Q that are elements of $\mathcal{I}$.

Now perform row interchanges on $D[P;Q]$ (except for when $Q=R$) until it is of the form $D(Q_I + (\mathcal{I}\backslash Q_I);\ (R+\mathcal{I})\backslash Q)$. This will take $\sum Q_I - \sum_1^{|Q_I|}(k+i)$ row interchanges. The new subdeterminant is now of the form of A, but of size $n-k$. Also, notice that $D(P;Q)$ is of the form of A, with $n=k$. Thus, by the inductive assumption, we have

$$\begin{aligned} \det A &= \sum_{|Q|=k} (-1)^{\sum P + \sum Q} \cdot (P \to Q) \cdot (-1)^{\sum Q_I - \sum_1^{|Q_I|}(k+i)}\ (Q_I \Rightarrow R \backslash Q_B) \\ &= \sum_{|Q|=k} (-1)^{\sum Q_B + \frac{1}{2}(k+|Q_I|)(k+|Q_I|+1)} \cdot (P \to Q) \cdot (Q_I \Rightarrow R \backslash Q_B) \end{aligned}$$

In the above expression $|Q_I| + |Q_B| = k$, so we may perform the following rearrangement.

$$\begin{aligned}
&\frac{1}{2}(k + |Q_I|)(k + |Q_I| + 1) = \frac{1}{2}(2k - |Q_B|)(2k - |Q_B| + 1) \\
&= \quad 2(k^2 - kQ_B) + \frac{1}{2}(|Q_B|^2 + |Q_B| + 2|Q_I|) \\
&\equiv \quad \frac{1}{2}(|Q_B|^2 + |Q_B| + 2|Q_I|) = \frac{1}{2}|Q_B|(|Q_B| + 1) + |Q_I| \pmod 2
\end{aligned}$$

Accordingly, substitution of the sign exponent gives us

$$\det A = \sum_{|Q|=k} (-1)^{\sum Q_B + \frac{1}{2}|Q_B|(|Q_B|+1) + |Q_I|} \cdot (P \to Q) \cdot (Q_I \Rightarrow R \backslash Q_B)$$

By Lemma 2.6, the expression $\sum Q_B + \frac{1}{2}|Q_B|(|Q_B| + 1)$ can be represented as the sign of a mapping, giving us

$$\det A = \sum_{|Q|=k} (-1)^{|Q_I|} \operatorname{sign} \left({Q \atop Q_B + (R \backslash Q_B)} \right) \cdot (P \to Q) \cdot (Q_I \Rightarrow R \backslash Q_B) \tag{5}$$

In the above equation, the $(P \to Q)$ is the determinant of a $k \times k$ matrix. Using (4), with σ and J null, we can see that this determinant is the sum of products representing every possible direct connection from the elements of P to the elements of R. Similarly, the $(Q_I \Rightarrow R \backslash Q_B)$ is the sum of products representing every possible connection from the elements of Q_I to the elements of R that are not already reached in the first path (through leftover interior nodes). The product $(P \to Q) \cdot (Q_I \Rightarrow R \backslash Q_B)$ is thus the sum of every path from P to Q appended onto every path from Q_I to $R \backslash Q_B$ through interior; this sum will include every connection from P to R through Q. Summing over every possible Q then gives every possible connection from P to R through the interior of the graph.

It is hopefully clear that the expression (5) contains exactly the same terms as the expansion (4). We will now show that corresponding terms have the same sign.

Fix a permutation α on the elements of Q, and a permutation δ on the elements of $R \backslash Q_B$. The α and δ determine a permutation τ of the elements of R as follows.

$$\left({P \atop \tau(R)} \right) = \left({P \atop \alpha(Q)} \right) \circ \left({Q \atop Q_B + \delta(R \backslash Q_B)} \right)$$

Together, α, δ, and Q map one-to-one onto the terms of (4). Now, take a term from (4), with sign $(-1)^k \operatorname{sign}(\tau)$. The corresponding term in (5) has sign

$$\begin{aligned}
&(-1)^{|Q_I|} \operatorname{sign} \left({Q \atop Q_B + (R \backslash Q_B)} \right) \cdot (-1)^k \operatorname{sign} \left({P \atop \alpha(Q)} \right) \cdot (-1)^{|Q_I|} \operatorname{sign} \left({Q_I \atop \delta(R \backslash Q_B)} \right) \\
&= \quad (-1)^k \operatorname{sign} \left({Q \atop Q_B + (R \backslash Q_B)} \right) \cdot \operatorname{sign} \left({P \atop \alpha(Q)} \right) \cdot \operatorname{sign} \left({Q_B + \quad Q_I \atop Q_B + \delta(R \backslash Q_B)} \right) \operatorname{sign} \left({Q_B + \quad Q_I \atop Q_B + (R \backslash Q_B)} \right) \\
&= \quad (-1)^k \operatorname{sign} \left({P \atop \alpha(Q)} \right) \cdot \operatorname{sign} \left({Q \atop Q_B + \delta(R \backslash Q_B)} \right) \\
&= \quad (-1)^k \operatorname{sign}(\tau)
\end{aligned}$$

This concludes the proof of Lemma 2.7. □

3 Connections Revisited

In this section, we apply Lemma 2.7 in order to establish a relationship between the determinant of a response matrix and connections through its network. What follows is a summary of Theorem 3.13 of [1].

Lemma 3.1. *Suppose that G is a directed network with n interior nodes, and that K is the Kirchhoff matrix for this network. Suppose further that there is a directed path through G from every interior node to the boundary.*

Let $A = K(\mathcal{I};\mathcal{I})$. Thus, A is of the form

$$A = \begin{bmatrix} \sigma_1+\epsilon_1 & -a_{12} & -a_{13} & & -a_{1n} \\ -a_{21} & \sigma_2+\epsilon_2 & -a_{23} & \cdots & -a_{2n} \\ -a_{31} & -a_{32} & \sigma_3+\epsilon_3 & & -a_{3n} \\ & \vdots & & \ddots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & \sigma_n+\epsilon_n \end{bmatrix} \tag{6}$$

where for $1 \le i \le n$, $\epsilon_i \ge 0$ and $\sigma_i = \sum_{j \ne i} a_{ij}$. Then $\det A > 0$. Furthermore, every principal subdeterminant of A is also greater than zero.

Proof. We begin by constructing a new network, $\tilde{G}$, with the same interior nodes as G, but only one boundary node (with index zero). Let $\mathcal{B} = (0)$, and $\mathcal{I} = (1, 2, \ldots, n)$. For $1 \le i \le n$, let ϵ_i be the row-sum of the i^{th} row of A. Then for every interior node i, we let ϵ_i be the conductivity from i to the boundary node. Hence, there is a directed path from every interior node in $\tilde{G}$ to the boundary node.

The Kirchhoff matrix $\tilde{K}$ of $\tilde{G}$ is thus of the form

$$\tilde{K} = \left[\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & & 0 \\ \hline -\epsilon_1 & \sigma_1+\epsilon_1 & -a_{12} & -a_{13} & & -a_{1n} \\ -\epsilon_2 & -a_{21} & \sigma_2+\epsilon_2 & -a_{23} & \cdots & -a_{2n} \\ -\epsilon_3 & -a_{31} & -a_{32} & \sigma_3+\epsilon_3 & & -a_{3n} \\ & & \vdots & & \ddots & \vdots \\ -\epsilon_n & -a_{n1} & -a_{n2} & -a_{n3} & \cdots & \sigma_n+\epsilon_n \end{array}\right] \tag{7}$$

Notice that this matrix has the property that its row-sums are zero. We now will now prove by induction on n that the submatrix $\det \tilde{K}(1, \ldots, n; 1, \ldots, n) = \det A > 0$.

When $n = 1$, K is of the form

$$\tilde{K} = \begin{bmatrix} 0 & 0 \\ -\epsilon_1 & \epsilon_1 \end{bmatrix}$$

where $A = [\epsilon_1]$, and $\epsilon_1 > 0$, since there must be a directed edge from the interior node to the boundary node. Hence, $\det A > 0$. Now, assume that for when a matrix of size $(n-1) \times (n-1)$ is of the form (6), then its determinant is greater than zero.

Now, consider a matrix of the form (6) where there are n interior nodes. Perform Gaussian elimination on the entry $\sigma_n + \epsilon_n$ of the augmented matrix $\tilde{K}$. This produces a new matrix, $\tilde{K}'$.

$$\tilde{K}' = \left[\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & & 0 \\ \hline -\epsilon_1' & \sigma_1'+\epsilon_1' & -a_{12}' & -a_{13}' & & 0 \\ -\epsilon_2' & -a_{21}' & \sigma_2'+\epsilon_2' & -a_{23}' & \cdots & 0 \\ -\epsilon_3' & -a_{31}' & -a_{32}' & \sigma_3'+\epsilon_3' & & 0 \\ & & \vdots & & \ddots & \vdots \\ -\epsilon_n & -a_{n1} & -a_{n2} & -a_{n3} & \cdots & \sigma_n+\epsilon_n \end{array}\right]$$

In this matrix, $-a_{ij}' = -a_{ij} - \frac{a_{in}a_{nj}}{\sigma_n+\epsilon_n} \le -a_{ij}$. Hence, if a_{ij} is nonzero, then a_{ij}' is also nonzero. Similarly, if a_{in} and a_{nj} are nonzero, then a_{ij}' is nonzero. Thus, the elimination of node n in this manner does not eliminate any paths in the graph. Therefore, the submatrix consisting of the rows and columns $(1, 2, \ldots, n-1)$ of $\tilde{K}'$ is a matrix of the form (6) and hence, by our inductive assumption, its determinant is greater than zero. The cofactor expansion of A along the last column yields

$$\det A = \det \tilde{K}(\mathcal{I};\mathcal{I}) = (\sigma_n + \epsilon_n) \cdot \det \tilde{K}'(1, 2, \ldots, n-1; 1, 2, \ldots, n-1)$$

Both terms in the final product are greater than zero, so $\det A > 0$

We now verify that the principal minors of A satisfy the same hypothesis as A. Let $J \subseteq \mathcal{I}$, and $A(J; J)$ be a principal minor of A. Let $\tilde{G}_J$ be the network constructed from the interior nodes J, and one boundary node (with index zero). We will show that if there was a directed path from $j_i \in J$ to the boundary of $\tilde{G}$ through $\mathcal{I}$, then there is a directed path from j_i to the boundary of $\tilde{G}_J$ through J. We rewrite each diagonal entry of $A(J;J)$ $\sigma_{j_i} + \epsilon_{j_i} = \hat{\sigma}_{j_i} + \hat{\epsilon}_{j_i}$, where $\hat{\sigma}_{j_i}$ is the sum of the off-diagonal terms of the j_i^{th} row of $A(J;J)$.

Given a node j_i, Let s be a step in the path from j_i to the boundary of $\tilde{G}$ through $\mathcal{I}$. If s goes from an element of J to the boundary of $\tilde{G}$, this step also goes to the boundary of $\tilde{G}_J$. If s is from a node in J to another node in J, then this portion of the path will also be unaffected. If s is from a node in J through an interior node that is not in J, then $\hat{\epsilon}_{j_i} > \epsilon_{j_i}$. Thus, node j_i is now connected directly to the boundary.

Thus, every node $j_i \in J$ has a directed path to the bounday through J, and hence $A(J;J)$ is of the same form as A. $\square$

Theorem 3.2. *Let G be a directed conductivity network with a circular planar graph. Suppose that $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_k)$ are disjoint subsets of the boundary of G. Also, suppose that the sequence of boundary nodes $(p_1, \ldots, p_k, q_k, \ldots, q_1)$ is in circular order. Finally, suppose that for every interior node, there is a directed path to the boundary. Then $(-1)^k \det \Lambda(P;R) > 0$ if and only if there is a directed connection from P to R through G.*

Proof. For any square matrix M of the form

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

where M_4 is a non-singular principal submatrix of M. Then

$$\det M = \det(M/M_4) \cdot \det M_4$$

Setting $M = K(P \cup \mathcal{I}; R \cup \mathcal{I})$, and $M_4 = K(\mathcal{I};\mathcal{I})$, we have

$$\det K(P \cup \mathcal{I}; R \cup \mathcal{I}) = \det \Lambda(P;R) \cdot \det K(\mathcal{I};\mathcal{I})$$

By Lemma 3.1 $\det K(J;J) > 0$ for $J \subseteq \mathcal{I}$. Therefore, if we compute the determinant of $K(P \cup \mathcal{I}; R \cup \mathcal{I})$ by (4), every weight determinant of the form $D[\mathcal{B} \cup J; \mathcal{B} \cup J]$ in that expression will be greater than zero. In addition, when G is restricted to be circular planar, every nonzero term in the expression (4) will have the sign $(-1)^k$. Furthermore, $\det K(P \cup \mathcal{I}; R \cup \mathcal{I}) = 0$ if and only if there is no connection from P to R through G. Hence, $(-1)^k \det \Lambda(P;R) > 0$ if and only if there is a connection from P to R. $\square$

References

[1] Curtis, Edward B., and James A. Morrow. *Inverse Problems for Electrical Networks.*

[2] Oberlin, Richard. *Discrete inverse problems for Schrödinger and Resistor networks.*

[3] Blunk, Mark, and Sam Coskey. *Vertex Conductivity Networks.*