Properties of Vertex Conductivity and Directed Edge Conductivity Networks

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Abstract

Inverse problems for directed edge conductivity networks and vertex conductivity networks are discussed. Non-recoverability of certain types of networks is established.

1 Introduction

A directed edge conductivity network $\Gamma = (G, \gamma)$, as posed in [3] is a directed graph with boundary $G = (V, V_B, E)$, where $\emptyset \neq V_B \subseteq V$ and $E \subseteq \{(u, v) \mid u, v \in V, u \neq v\}$, together with a positive real-valued function γ (the conductance function) on E. Here, $\gamma(u, v)$ can be viewed as the conductance from u to v, as seen by u. Unlike [3], we define current flow into a node to be positive, and the current (due to a potential function u on the vertices) at a node p which has edges to neighbors q (designated $p \to q$) is thus:

$$\sum_{q|p\to q} \gamma(p, q)(u(q) - u(p))$$

We additionally require that each vertex that is not a boundary vertex (called an interior vertex) has a directed path to some boundary vertex.

In this case our Kirchoff matrix is as follows:

1. $K_{i,j} = \gamma(i, j)$ for $i \to j$
2. $K_{i,j} = 0$ for other $i \neq j$
3. $K_{i,i} = -\sum_{j|i\to j} \gamma(i, j)$

The response matrix will again be the Schur complement of the rows and columns corresponding to the interior vertices (the required invertibility of the submatrix is established in [3]).

We see then that the conductivity networks discussed in [2] may be viewed (with an appropriate change of sign) as directed edge conductivity networks with the additional restrictions that $(u, v) \in E$ iff $(v, u) \in E$ and $\gamma(u, v) = \gamma(v, u)$ $\forall (u, v) \in E$. That is, between any two vertices, if there is any edge between them then there are two edges between them, and they have the same conductances.

Furthermore, we may also view the vertex conductivity networks seen in [4] and [1] as directed edge conductivity networks with the additional restrictions that $(u, v) \in E$ iff $(v, u) \in E$ and $\gamma(p, q) = \gamma(r, q)\ \forall q \in V$ and where $p \to q$ and $r \to q$. We also assume that the graph G is connected. That is, between any two vertices, if there is any edge between them then there are two edges between them, and all edges pointing toward a given vertex have the same conductances. For this reason, in the vertex conductivity case we can define γ to be not on the edges but on the vertices, and our conductance equation at node p becomes (where $q \sim p$ iff q and p are neighbors):

$$\sum_{q \sim p} \gamma(q)(u(q) - u(p))$$

The Kirchoff matrix becomes:

1. $K_{i,j} = \gamma(j)$ for $i \sim j$
2. $K_{i,j} = 0$ for other $i \neq j$
3. $K_{i,i} = -\sum_{j \sim i} \gamma(j)$

We find the response matrix just as we did in the undirected case.

Note that both the undirected edge conductivity and vertex conductivity cases are strong restrictions on the values of the conductors of the network, and it is usually the information that results from this restriction (and not simply data gleaned from viewing this as an undirected network) that allows for recovery of the network.

As in [2], the interesting functions u are those for which the conductivity equation is 0 on all interior vertices, and these are called γ-harmonic functions. It is established in [3] and [4] that for the directed edge conductivity case and the vertex conductivity case, given a set of potentials u on the boundary, there is a unique γ-harmonic extension of u to the interior vertices. These papers also establish the existence of the response matrix Λ_γ for a network, which, when applied to a vector of boundary potentials, returns the corresponding vector of resulting boundary currents.

The inverse problem for these networks is thus, given a graph G and a response matrix Λ_γ, to recover γ everywhere (i.e. on all directed edges, or on all vertices).

We see immediately that if a vertex conductivity network (or undirected edge conductivity network) is not recoverable, then the corresponding directed edge conductivity network is not recoverable. The converse is clearly not the case, as will become apparent.

2 Recoverability of Directed Edge Conductivity Networks

Theorem 2.1 *Let G be a directed graph with boundary, and $\Gamma = (G, \gamma)$ a directed edge conductivity network with response matrix Λ_γ. Suppose G has at least one interior vertex. Then γ is not recoverable from Λ_γ.*

Proof: Since we have at least one interior vertex, and it must have a directed path to some boundary node, there exists at least one interior vertex (we shall call it k) such that $\{j \mid k \to j\} \neq \emptyset$.

Suppose for the sake of contradiction that γ is recoverable. For some positive real constant c not equal to 1, consider another network $\Gamma' = (G, \gamma')$, where $\gamma'(k,j) = c\gamma(k,j)$ for $k \to j$, and $\gamma'(i,j) = \gamma(i,j)$ for all other i, j, as well as its response matrix $\Lambda_{\gamma'}$. Since this graph is recoverable and $\gamma \neq \gamma'$ we must have that $\Lambda_\gamma \neq \Lambda_{\gamma'}$. So $\exists\, u$ such that $\Lambda_\gamma u \neq \Lambda_{\gamma'} u$. Let v be the γ-harmonic extension of u in Γ. We have then that v is also γ'-harmonic in Γ': consider an interior vertex i. If i is not k,

$$\sum_{j|i\to j} \gamma'(i,j)(v(j) - v(i)) = \sum_{j|i\to j} \gamma(i,j)(v(j) - v(i)) = 0$$

since v is γ-harmonic. In the other case, we consider when i is k. Then,

$$\sum_{j|i\to j} \gamma'(i,j)(v(j) - v(i)) = \sum_{j|i\to j} c\gamma(i,j)(v(j) - v(i)) = c \sum_{j|i\to j} \gamma(i,j)(v(j) - v(i)) = 0$$

again since v is γ-harmonic. Thus v is γ'-harmonic as well.

But then, since $k \notin V_B$, we have that

$$(\Lambda' u)_i = \sum_{j|i\to j} \gamma'(i,j)(v(j) - v(i)) = \sum_{j|i\to j} \gamma(i,j)(v(j) - v(i)) = (\Lambda u)_i$$

for each $i \in V_B$, contradicting $\Lambda_\gamma u \neq \Lambda_{\gamma'} u$. Thus, G is not a recoverable graph for directed edge conductivity networks. □

Thus, the only graphs that might possibly be recoverable are those that contain only boundary nodes. For these graphs, however, the response matrix is the same as the Kirchoff matrix, and so they are recoverable. Thus, by Theorem 2.1, we have the following characterization:

Corollary 2.1 *A directed edge conductivity network is recoverable if and only if its graph contains no interior vertices.*

3 Two-colorings and recoverability

We can use a similar argument to establish that certain vertex conductivity networks are not recoverable.

Definition 3.1 *Call a graph G* **2-colorable** *if there exists a function $s : V \longrightarrow \{0, 1\}$ such that for all $i, j \in V$, if there is an edge from i to j, then $s(i) \neq s(j)$.*

Clearly, not all graphs are 2-colorable. Because we are dealing only with connected graphs, it is evident that if a graph is 2-colorable, there are exactly two such functions s; moreover, they differ only in that the preimage of 0 in one of them is equal to the preimage of 1 in the other, and vice versa. Certain 2-colorable graphs are not recoverable, as is shown in the following theorem.

Theorem 3.1 *Suppose a graph G is 2-colorable, and that $s(i) = s(j)\ \forall i, j \in V_B$. Let $\Gamma = (G, \lambda)$ be a vertex conductivity network with response matrix Λ_γ. Then γ is not recoverable from Λ_γ.*

Proof: Assume, WLOG, that $s(i) = 0\ \forall i \in V_B$. Let $S = s^{-1}(0)$ and $T = V - S = s^{-1}(1)$. Note that $V_B \subseteq S$.

Suppose for the sake of contradiction that γ is recoverable. For some positive real constant c not equal to 1, consider another network $\Gamma' = (G, \gamma')$, where $\gamma'(i) = \gamma(i)$ for $i \in T$, and $\gamma'(i) = c\gamma(i)$ for $i \in S$, as well as its response matrix $\Lambda_{\gamma'}$. Since this graph is recoverable and $\gamma \neq \gamma'$ we must have that $\Lambda_\gamma \neq \Lambda_{\gamma'}$. So $\exists u$ such that $\Lambda_\gamma u \neq \Lambda_{\gamma'} u$. Let v be the γ-harmonic extension of u in Γ. We have then that v is also γ'-harmonic in Γ': consider an interior vertex i. If $i \in S$, we know that $j \in T$ for all $j \sim i$. So

$$\sum_{j\sim i} \gamma'(i)(v(j) - v(i)) = \sum_{j\sim i} \gamma(i)(v(j) - v(i)) = 0$$

since v is γ-harmonic. In the other case, we have that $i \in T$, so $j \in S$ for all $j \sim i$. So

$$\sum_{j\sim i} \gamma'(i)(v(j) - v(i)) = \sum_{j\sim i} c\gamma(i)(v(j) - v(i)) = c \sum_{j\sim i} \gamma(i)(v(j) - v(i)) = 0$$

again since v is γ-harmonic. Thus v is γ'-harmonic as well.

But then, since $V_B \subseteq S$, we know that

$$(\Lambda' u)_i = \sum_{j\sim i} \gamma'(i)(v(j) - v(i)) = \sum_{j\sim i} \gamma(i)(v(j) - v(i)) = (\Lambda u)_i$$

for each $i \in V_B$, contradicting $\Lambda_\gamma u \neq \Lambda_{\gamma'} u$. Thus, G is not a recoverable graph for vertex conductivity networks.

□

This two-colorability condition can be expressed in a slightly more useful form as follows.

Corollary 3.1 *Let G be a graph with boundary such that for any path p (not necessarily simple) from any boundary vertex to any (not necessarily distinct) boundary vertex, the length of p is even. Then G is not a recoverable graph for vertex conductivity networks.*

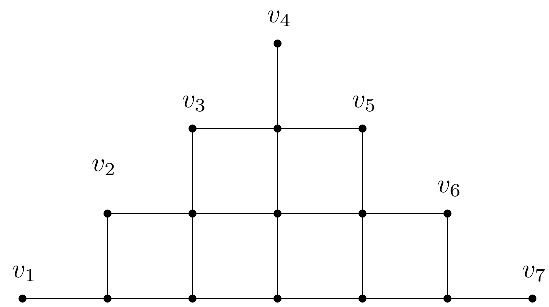

Figure 1:

Proof: It suffices to show that this graph satisfies the conditions of the previous theorem. Let $S = \{i \mid i \text{ is reachable through } G \text{ via a path of even length (possibly 0) from some boundary vertex}\}$. Let $T = V - S$. Suppose $j \sim i$. We cannot have that $i, j \in S$: if that were the case, then there would be some path of even length from a boundary vertex to i, another of even length from some boundary vertex to j, and then by appending these paths with the edge from j to i, we would obtain a path of odd length from a boundary vertex to another boundary vertex, a contradiction. Similarly, we cannot have that $i, j \in T$: because the graph is connected, there must be some path from i to some boundary vertex, and some path from j to some boundary vertex. But then these paths must both be of odd length (otherwise the vertices would not be in T), and as before we could construct a path of odd length from some boundary vertex to another boundary vertex b! y appending the two paths with the edge between i and j. So, we can easily construct a two-coloring of G: let $s(i) = 0$ for $i \in S$ and $s(i) = 1$ for $i \in T$. Furthermore, $s(i) = s(j) = 0 \ \forall i, j \in V_B$. So, by our previous theorem, G is not recoverable for vertex conductivity networks. □

Thus, if G is recoverable, there must be some path of odd length from some boundary vertex to some boundary vertex. It follows from this that Tower of Hanoi graphs are not recoverable for odd n, as Richard Oberlin conjectured in [5]. Figure 1 shows the Tower of Hanoi graph for $n = 7$. Note also that it is another way to demonstrate the non-recoverability of the "Stars" (graphs containing one interior vertex connected to two or more boundary nodes, with no other edges) discussed in [1]. A possible topic of future study may be to look at the recovery of two-colorable graphs that do not meet the above criteria (certainly not all of them are recoverable).

4 The Schrödinger Equation for Vertex Conductivity Networks

In [4], Richard Oberlin extensively studies the technique of using a discretization of the Schrödinger equation to recover vertex conductivity networks. A Schrödinger network is a graph with boundary $\Gamma = (V, V_B, E)$ together with a real-valued function q defined on V.

We define the Schrödinger equation to be

$$\left(\sum_{j\sim i}(u(j)-u(i))\right) - q(i)u(i).$$

Given the response matrix Λ_γ, we can obtain the Schrödinger response matrix (with Neumann data $\sum_{j\sim i}(u(j)-u(i))$) via the following formula established in [4] (although the paper takes as a premise that the network is a square lattice, the derivation of the formula does not use this premise):

$$\Psi_q = \Lambda_\gamma I_\gamma(B;B)^{-1} - I_q(B;B)$$

In the above formula, $I_\gamma(B;B)$ is the diagonal matrix with the values of γ at the boundary on its diagonal. Similarly, $I_q(B;B)$ is the diagonal matrix with the values of q (as defined below) at the boundary on its diagonal.

When u is such that this equation is 0 for all interior vertices, we say that u is a *q-state*. Oberlin establishes that when $q(i) = \frac{\sum_{j\sim i}\gamma(j)-\gamma(i)}{\gamma(i)}$, it is the case that, given a vector of potentials u on the boundary, there is a unique q-state extension of u to the interior (and this allows for the existence of the Schrödinger response matrix, which he also establishes).

Theorem 4.1 *Given q everywhere, and γ on the boundary vertices, where $q(i) = \frac{\sum_{j\sim i}\gamma(j)-\gamma(i)}{\gamma(i)}$, we can recover γ everywhere.*

Proof: That there is a unique solution to γ is evident: let $u(i) = \gamma(i)$ on the boundary. Then $u = \gamma$ is a q-state; and by [4], it is the unique q-state agreeing with γ on the boundary. We can find its values as follows: let m be the number of interior vertices in the graph G, and n the number of boundary vertices. Assume that the boundary vertices are labeled $v_1, \ldots, v_n$, and the interior vertices are labeled $v_{n+1}, \ldots, v_{n+m}$. We then have, since γ is a q-state, m equations:

$$\left(\sum_{j\sim v_{n+1}}(\gamma(j)-\gamma(v_{n+1}))\right) - q(v_{n+1})\gamma(v_{n+1}) = 0$$

$$\left(\sum_{j\sim v_{n+2}}(\gamma(j)-\gamma(v_{n+2}))\right) - q(v_{n+2})\gamma(v_{n+2}) = 0$$

$$\vdots$$

$$\left(\sum_{j\sim v_{n+m}}(\gamma(j)-\gamma(v_{n+m}))\right) - q(v_{n+m})\gamma(v_{n+m}) = 0$$

In addition, we have the n equations corresponding to the Neumann data, where Ψ_q is the Schrödinger response matrix, and u is the vector of boundary potentials equal to γ (these are also needed, since q might be 0 for one of those first m equations, for some interior vertex that only neighbors boundary vertices):

$$\sum_{j\sim v_1} (\gamma(j) - \gamma(v_1)) = (\Psi_q u)_1$$
$$\sum_{j\sim v_2} (\gamma(j) - \gamma(v_2)) = (\Psi_q u)_2$$
$$\vdots$$
$$\sum_{j\sim v_n} (\gamma(j) - \gamma(v_n)) = (\Psi_q u)_n$$

Note here that the problem as formulated by Oberlin takes the Neumann data for the Schrödinger response matrix not to involve q; thus, in a sense, in solving the inverse problem, we do not actually solve for q everywhere, since we begin with q known on the boundary. A problem for future study may be to explore an inverse problem where the Neumann data is actually the value of the Schrödinger equation.

Since γ is known for v_1 through v_n, and q is known everywhere, this is a linear system with up to m unknowns and $m + n$ equations. Furthermore, it must have rank m: the rank is at most m since there are at most m unknowns here, and any solution to this equation yields a q-state agreeing with our known γ on the boundary, so there is only one such solution. So we can simply solve this linear system to recover γ. □

Some obvious conclusions that can be drawn from this are that if γ is recoverable on the boundary vertices and all of their neighbors, and γ is not recoverable everywhere, then q is not recoverable everywhere. Also, if q is recoverable everywhere, and γ is not recoverable, then γ is not recoverable on some boundary vertex or some adjacent interior vertex.

It should be noted, however, that γ must be recovered not only on the boundary vertices but on all of their neighbors to construct Ψ_q as in [4]. The problem as formulated by Oberlin takes the Neumann data for the Schrödinger response matrix not to involve q; thus, in a sense, in solving the inverse problem, we do not actually solve for q everywhere, since we begin with q known on the boundary. A problem for future study may be to explore an inverse problem where the Neumann data is actually the value of the Schrödinger equation. Such a problem would require only the recovery of γ on the boundary to form the response matrix, and, by the above theorem, to recover all of γ once all of q is found.

References

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