Discrete inverse problems for triangular Schrödinger and Resistor networks

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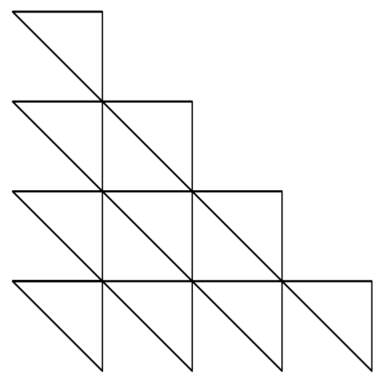

1 Introduction

For each positive integer n, construct an undirected triangle graph with boundary $\Gamma = (V, V_B, E)$ as follows. V is the set of vertices in the graph and consists of the integer lattice points (x, y) where $0 \le x$, $0 \le y$, and $x + y \le n + 1$, excluding the three corner points $(0,0), (0, n+1)$, and $(n+1, 0)$. $V_B \subseteq V$ is the set of boundary vertices and consists of the vertices in V where x or y is equal to 0, or where $x + y = n + 1$. The interior vertices are denoted int$V$ and consist of $V - V_B$. E is the set of edges. Every vertex (i, j) is connected by exactly one edge to each of $(i+1, j), (i-1, j), (i, j+1), (i, j-1), (i+1, j-1), (i-1, j+1)$, whenever those vertices exist. These edges are the only edges in E. Given vertices p and q, if there is an edge in E connecting p and q we say that p neighbors q, and denote this $p \sim q$.

A vertex conductivity network is a graph with boundary $\Gamma = (V, V_B, E)$ together with a positive real-valued function γ defined on V. A Schrödinger network is a graph with boundary $\Gamma = (V, V_B, E)$ together with a real-valued function q defined on V.

We use here vertex discretizations of the conductivity equation L_γ and Schrödinger equation S_q as follows:

$$L_{\gamma_d} u(i) = \sum_{j \sim i} \gamma(j)(u(j) - u(i))$$

$$S_{q_d} u(i) = \left(\sum_{j \sim i} (u(j) - u(i)) \right) - q(i)u(i).$$

Using these discretizations, if u is a solution to $L_{\gamma_d} = 0$ then $w = \gamma u$ is a solution to $S_{q_d} = 0$ with $q(i) = \frac{\sum_{j \sim i} \gamma(j) - \gamma(i)}{\gamma(i)}$. We call u *γ-harmonic* if $L_{\gamma_d} u = 0$ for all interior vertices, and we call u a *q-state* if $S_{q_d} u = 0$ for all interior vertices.

This paper refers to several properies of conductivity and Schrödinger networks established by Richard Oberlin. Though he took as a premise that the networks he was working with were square, the properties we quote do not utilize this in their proofs.

Given a conductivity network (Γ, γ), let (Γ, q) be the Schrödinger network with q such that $q(i) = \frac{\sum_{j \sim i} \gamma(j) - \gamma(i)}{\gamma(i)}$. Then our Schrödinger response matrix Ψ_q (with Neumann data $\sum_{j \sim i} (u(j) - u(i))$), where Λ_γ is our conductivity response matrix (with Neumann data $\sum_{j \sim i} \gamma(j)(u(j) - u(i))$) , is

$$\Psi_q = \Lambda_\gamma I_\gamma(B; B)^{-1} - I_q(B; B).$$

Here, $I_\gamma(B; B)$ is the diagonal matrix with the values of γ at the boundary nodes on its diagonal, and $I_q(B; B)$ is the diagonal matrix with the values of q at the boundary nodes on its diagonal. In general, $M(E; F)$ is the matrix

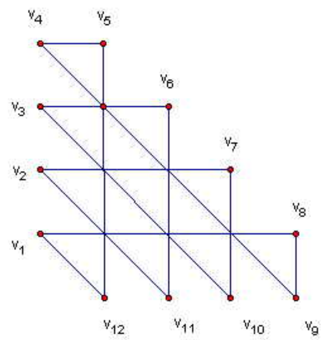

Figure 1: Triangle network, $n = 4$

defined by rows E and columns F of matrix M, and we use B to refer to the rows or columns corresponding to the boundary nodes of a graph, and N to the interior.

The inverse problem is, given a conductivity response matrix Λ_γ and a triangle graph with boundary $\Gamma = (V, V_B, E)$, to recover γ on all vertices. We give here an algorithm for doing this that utilizes the Schrödinger network.

2 Recovering q on the boundary

We note that the formula for the Schrödinger response matrix requires us to first obtain γ and q on the boundary; to obtain q on the boundary, we must also recover γ on any vertices sharing an edge with a boundary node.

We quote a theorem from Oberlin:

Theorem 2.1 *Let (Γ, γ) be a conductivity network with boundary potential f. There exists a unique γ-harmonic function u such that $u\,|_{V_B} = f$.*

The triangle graph with boundary has three edges: we call them West, South, and Hypotenuse. Label the boundary vertices in clockwise order, with v_1 at the southmost position of the West edge and v_{3n} at the leftmost position of the South edge.

Lemma 2.1 *Given a triangular conductivity network with boundary potential u defined on the West and South edges and Neumann data defined on the West , there is a uniquely determined γ-harmonic extension of u to the boundary vertices on the Hypotenuse and the interior vertices.*

Proof: Consider the interior vertices connected by edges to the West edge (vertices (1, 1) through (1, $n-1$)). The potential of each vertex i in this collection is determined by the Neumann data on the vertex to its west, the potential on the vertex to its west, the value of γ at i, and the potential and value of γ on the vertex to its south. This is so inductively: the southmost vertex, (1, 1) in this group is determined; once we know its potential, we know that of its neighbor to the north, and so on. So we also now know the extension of u on the hypotenuse where it meets these vertices ((1, n)) by the same argument. Now, proceeding inductively from south to north, at each step looking at node i we can calculate the potential of the vertex just east of i, since $L_{\gamma_d} u = 0$ at i, and the potentials at all of i's neighbors but its eastern neighbor are known (and all needed values of γ are known). The boundary vertex of the Hypotenuse here is also determined by the same argument. We're now done by induction, since this now determines the values of potentials of the vertices to the east, and so on. □

Corollary 2.1 *Let (Γ, γ) be a triangular conductivity network. Let u be a γ-harmonic function on it which is 0 on the West and South edges, with corresponding Neumann data which is 0 on the West edge. Then for each remaining vertex i, u(i) is also 0.*

Lemma 2.2 *The submatrix of Λ_γ consisting of the rows corresponding to the boundary vertices on the West edge and the columns corresponding to the boundary vertices on the Hypotenuse is nonsingular.*

Proof: This submatrix has the following interpretation: given a boundary potential u which is 0 on the West and South edges, and equal to g on the Hypotenuse, $\Lambda_\gamma(W; H)g$ is the resulting Neumann data on the West edge from a γ-harmonic extension of u. By our corollary, if $\Lambda_\gamma(W; H)g = 0$ then $g = 0$. □

Corollary 2.2 *Given Λ_γ and a vector of potentials u defined on the West and South edges and Neumann data p on the West edge, there is a unique γ-harmonic extension of u to the Hypotenuse.*

Proof: Let g be the boundary potential on the Hypotenuse. $\Lambda_\gamma(W; S + W)u + \Lambda_\gamma(W; H)g = p$. By the previous lemma, $\Lambda_\gamma(W; H)$ is invertible, so we may solve here for g. Hence our extension. □

Theorem 2.2 *Given a triangular conductivity network (Γ, γ) we can recover γ on the boundary vertices and interior vertices adjacent to the boundary vertices.*

Proof: Let the potentials on the West and South edges be 0. Specify corresponding Neumann data of 1 on v_n and 0 elsewhere on the West edge. Note that all interior vertices have potential 0: consider the interior vertices attached by some edge to the West edge ((1, 1) through (1, $n-1$)). By the Neumann data of v_1 and the potentials of v_1 and v_{3n}, we see that the southmost of these vertices must have potential 0. But then by the Neumann data of v_2, and the potential on the previous vertex, the vertex just to the north of it must also have potential 0. Similarly, each such interior vertex has potential 0. Since u is γ-harmonic, however, the next batch of interior vertices ((2, 1) through

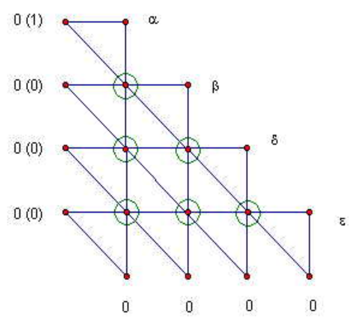

Figure 2: Recovering γ near the boundary

(2, $n-2$)) must also be 0 (starting at the bottom vertex and proceeding by induction). Similarly, all interior vertices must have potential 0.

By the previous corollary, we can use the response matrix to find the potentials on the hypotenuse. Call the potential at v_{n+1} α. By the Neumann data on v_n, we have that $\gamma(v_{n+1})\alpha = 1$, that is, that $\gamma(v_{n+1}) = \frac{1}{\alpha}$. By symmetry, we can use a similar method to determine γ at all of the "corner" vertices, notably, $\gamma(v_n)$. Once we've discovered $\gamma(v_n)$, let us come back to this assignment again. Consider the diagonal of interior vertices attached by an edge to the Hypotenuse ((1, $n-1$) to ($n-1$, 1)). Let i be the leftmost of these vertices. Now, the Neumann data on v_{n+1}, the potentials on v_n, v_{n+1}, and i, and the value of γ at v_n allow us to solve for the value of γ at i. Next, using that u is γ-harmonic and knowing the potentials of v_{n+1} and v_{n+2}, we can solve for $\gamma(v_{n+2})$. Proceeding in this fashion (using Neumann data at v_{n+2}, then looking at the vertex east of i, etc.) we see that inductively we have solved for γ at every vertex on the Hypotenuse as well as all interior vertices connected to them. By symmetry, we can do the same for the other two edges, and we are done. □

3 Recovering q from the Schrödinger response matrix

Now that we have recovered the conductances on the boundary nodes and all nodes adjacent to them, we can compute q on the boundary nodes, and thus obtain, from our formula Ψ_q. Our task is now to recover q on all interior nodes.

We quote the following lemma from Oberlin without proof.

Lemma 3.1 *Let (Γ, γ) be a conductivity network. Let (Γ, q) be the Schrödinger network with $q(i) = \frac{\sum_{j\sim i}\gamma(j)-\gamma(i)}{\gamma(i)}$. Given a boundary potential f, there is a unique q-state function u with $u\,|_{V_B} = f$.*

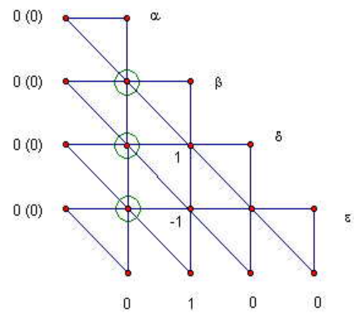

Figure 3: Recovering q

Lemma 3.2 *Given Ψ_q and a vector of potentials u defined on the West and South edges and corresponding Neumann data p on the West edge, there is a unique q-state extension of u to the Hypotenuse.*

Proof: Making the appropriate substitutions, the proof is identical to that for conductivity networks (and utilizes analogous lemmas, also with identical proofs). □

Theorem 3.1 *Given a triangular conductivity network (Γ, γ) and Schrödinger network (Γ, q) where $q(i) = \frac{\sum_{j\sim i}\gamma(j)-\gamma(i)}{\gamma(i)}$, and the response matrix for the Schrödinger network, we can recover q on all interior vertices of the Schrödinger network.*

Proof: We do so inductively. Assume we have recovered the value of q at all interior vertices east of $(i, 1)$ through $(i, n-i)$. Set the potentials on the West edge to be 0, specify corresponding Neumann data of 0 on the West edge, a potential of 1 at $(i, 0)$ and potential of 0 elsewhere on the South edge. Note that proceeding inductively as in our previous theorem, all interior vertices west of $(i, 1)$ through $(i, n-i)$ are of potential 0. Since this assignment extends to a unique q-state, we know that $S_{q_d}u = 0$ at all interior vertices, that is,

$$\left(\sum_{j\sim i}(u(j) - u(i))\right) - q(i)u(i) = 0.$$

At interior vertices where $u(i) = 0$ (as well as boundary vertices where $u(i) = 0$ and our Neumann data is 0) this reduces to

$$\sum_{j\sim i} u(j) = 0$$

Examining this equation at $(i-1,1)$ (or the Neumann data, should that be a boundary node), we see that u at $(i,1)$ is -1. Examining it at $(i-1,2)$, we see that u at $(i,2)$ is 1; and so on, alternatively taking on values 1 and -1. Furthermore, since we know the values of q of all interior vertices east of these vertices, and using the Neumann and Dirichlet data on the boundary, we can compute the potentials at all interior vertices east of $(i,1)$ through $(i,n-i)$ (starting at the bottom right, proceeding left, and then starting up from the right again). Thus, we need only examine $S_{q_d}u = 0$ at each of $(i,1)$ through $(i,n-i)$ to solve for q at each of these vertices. So, we're done by induction (the base case is trivial, since for interior vertex $(n-1,1)$, there are no interior vertices to its east). □

Knowing q on every vertex and knowing γ at least on the boundary, it is a trivial matter now to recover γ on every interior vertex.