Stars, Eigenvalues, and Negative Conductivities

Nicolas Addington

July 21, 2003

1 General Definitions, Graph Arithmetic

Definition 1.1. A graph G consists of an ordered set of boundary nodes ∂G, a set of interior nodes $\operatorname{int} G$, and a set of edges. Unless explicitly stated, we shall assume that all graphs are connected.

Definition 1.2. A network Γ is a graph G together with a real-valued conductivity function γ on the edges of G. We place no restriction on the sign of γ. We call Γ positive if all conductivities are positive, negative if all are negative, or of mixed sign otherwise. We do not distinguish between an edge with conductivity zero and the absence of an edge.

Definition 1.3. We define the response matrix of a network Γ as in [1] and denote it by $\operatorname{Resp}(\Gamma)$. We say that Γ_1 is response-equivalent to Γ_2 and write $\Gamma_1 \approx \Gamma_2$ if $\operatorname{Resp}(\Gamma_1) = \operatorname{Resp}(\Gamma_2)$.

Definition 1.4. Given a network Γ with n boundary nodes, we define $\overline{\Gamma}$ to be the unique network on the complete graph with n nodes such that $\overline{\Gamma} \approx \Gamma$.

Definition 1.5. Given a real number α and a Γ, γ, and G as above, we define $\alpha\Gamma$ to be the network on G with conductivity function $\alpha\gamma$. Clearly $\operatorname{Resp}(\alpha\Gamma) = \alpha \operatorname{Resp}(\Gamma)$.

Definition 1.6. Given networks Γ_1 and Γ_2 with n boundary nodes, we define $\Gamma_1 + \Gamma_2$ to be the network obtained by identifying the first boundary node of Γ_1 with the first boundary node of Γ_2, the second with the second, and so on. We will sometimes refer to this construction as gluing Γ_1 and Γ_2 together.

Lemma 1.7. *Let Γ_1 and Γ_2 be networks with n boundary nodes. Then* $\operatorname{Resp}(\Gamma_1+\Gamma_2) = \operatorname{Resp}(\Gamma_1)+\operatorname{Resp}(\Gamma_2)$.

Proof. Let $\Lambda = \operatorname{Resp}(\Gamma_1 + \Gamma_2)$, and let $\Gamma = \overline{\Gamma_1} + \overline{\Gamma_2}$. The reader may convince him or herself that since $\Gamma_1 \approx \overline{\Gamma_1}$ and $\Gamma_2 \approx \overline{\Gamma_2}$, $\Gamma_1 + \Gamma_2 \approx \overline{\Gamma_1} + \overline{\Gamma_2} = \Gamma$. Thus $\Lambda = \operatorname{Resp}(\Gamma)$. Now any pair of nodes in Γ is connected by two edges in parallel, so we may add their conductivities and replace them with a single edge. Thus $\Lambda = \operatorname{Resp}(\Gamma_1) + \operatorname{Resp}(\Gamma_2)$. □

Remark 1.8. It may be observed that our definitions of addition and scalar multiplication make the set of networks with n boundary nodes (mod response-equivalence) into a vector space over $\mathbb{R}$. This seems interesting until it is observed that this space is just isomorphic to the space of $n \times n$ response matrices.

2 Nullity Lemma

Lemma 2.1. *Let Γ be a (not necessarily connected) positive network with n boundary nodes, and let $\Lambda =$ $\operatorname{Resp}(\Gamma)$. Then the nullity of Λ is the number of connected components of Γ.*

Proof. Let $\{C_i\}$ be the connected components of Γ, and for each C_i, let $x_i \in \mathbb{R}^n$ be the vector of potentials which is 1 on ∂C_i and 0 on the rest of $\partial\Gamma$. Clearly, the x_i's are independent, and $\Lambda x_i = 0$ for each x_i. We wish to show that $\{x_i\}$ spans $\ker \Lambda$. Suppose $\Lambda y = 0$, and for each C_i let $y_i \in \mathbb{R}$ be the value of y on some node of ∂C_i. By harmonic continuation y is constant on all of ∂C_i, so $y = \sum y_i x_i$. Thus $\{x_i\}$ is a basis for $\ker \Lambda$. □

3 Diagonalization

Let Λ and Λ' be real symmetric $n \times n$ matrices which share at least n (orthogonal) eigenvectors $\{x_1, \ldots, x_n\}$. Then they can be diagonalized:

$$\Lambda = QDQ^{-1} \qquad \text{and} \qquad \Lambda' = QD'Q^{-1}$$

with the same orthogonal matrix

$$Q = \begin{pmatrix} \\ x_1 & \cdots & x_n \\ \\ \end{pmatrix}$$

Thus we can add their eigenvalues:

$$\Lambda + \Lambda' = Q(D + D')Q^{-1}$$

Remark 3.1. For this manipulation, it is not necessary that *all* eigenvectors of Λ be eigenvectors of Λ' (or vice versa), only that they share at least n eigenvectors. If, for example, Λ has repeated eigenvalues, it may be that one choice of eigenbasis for Λ is also an eigenbasis for Λ', while another is not.

4 Stars

4.1 General Stars

Let the graph G be a star with n rays, that is, let G have n boundary nodes, one interior node, and one edge between each boundary node and the interior node. Let $x \in \mathbb{R}^n$ such that no component x_i of x is 0, and let $\bigstar_x$ be the network on G where the conductivities on the rays are given by the components x_i. The reader may verify that

$$S_x = \mathrm{Resp}(\bigstar_x) = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix} - \frac{xx^\top}{x^\top \vec{1}} \tag{1}$$

where $\vec{1}$ is the vector in $\mathbb{R}^n$ with all entries equal to 1.

Remark 4.1. Note that in the case where $x \perp \vec{1}$, that is, when the sum of all the conductances around the star is 0, the second term of (1) blows up. If we consider $\overline{\bigstar_x}$, we find that all the conductivities are infinite. This is not meaningless: a network containing such a star is response-equivalent to one where the star is replaced with a single node. In general, an edge with infinite conductivity has the effect of identifying its endpoints.

Conjecture 4.2. The map $x \mapsto S_x$ is linear.

4.2 Constant Stars

S_x has 0 as an eigenvalue and $\vec{1}$ as a 0-eigenvector (like any response matrix), but the other eigenvalues and eigenvectors are not elegant in general. Since this paper is largely concerned with eigenvalues and eigenvectors, we restrict our discussion to constant stars, that is, $\bigstar = \bigstar_{\vec{1}}$ and multiples of it, so (1) becomes

$$S = \mathrm{Resp}(\bigstar) = I - \frac{\vec{1}\vec{1}^\top}{n}$$

The reader may verify that $Sx = 0$ if $x = \vec{1}$ and $Sx = x$ if $x \perp \vec{1}$, so if $\{\vec{1}, x_2, \ldots, x_n\}$ is an orthogonal basis for $\mathbb{R}^n$, we may diagonalize S as

$$S = \begin{pmatrix} 1 & & & \\ \vdots & x_2 & \cdots & x_n \\ 1 & & & \end{pmatrix} \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ \vdots & x_2 & \cdots & x_n \\ 1 & & & \end{pmatrix}^{-1}$$

4.3 Gluing Stars

Fix a positive connected network Γ with n boundary nodes, and let $\Lambda = \mathrm{Resp}(\Gamma)$. From [1] we know that Λ is symmetric and positive semi-definite, and its 0-eigenspace is $\mathbb{R}\vec{1}$. Let $\{0, \lambda_2, \ldots, \lambda_n\}$ be eigenvalues of Λ and $\{\vec{1}, x_2, \ldots, x_n\}$ be a corresponding orthogonal basis of eigenvectors, and diagonalize Λ:

$$\Lambda = \begin{pmatrix} 1 & & & \\ \vdots & x_2 & \cdots & x_n \\ 1 & & & \end{pmatrix} \begin{pmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} 1 & & & \\ \vdots & x_2 & \cdots & x_n \\ 1 & & & \end{pmatrix}^{-1}$$

If we glue a constant star of conductivity α to Γ, we obtain the following diagonalization:

$$\mathrm{Resp}(\Gamma + \alpha\bigstar) = \begin{pmatrix} 1 & & & \\ \vdots & x_2 & \cdots & x_n \\ 1 & & & \end{pmatrix} \begin{pmatrix} 0 & & & \\ & \lambda_2 + \alpha & & \\ & & \ddots & \\ & & & \lambda_n + \alpha \end{pmatrix} \begin{pmatrix} 1 & & & \\ \vdots & x_2 & \cdots & x_n \\ 1 & & & \end{pmatrix}^{-1}$$

That is, gluing $\alpha\bigstar$ to Γ leaves the eigenvectors of Λ unchanged and adds α to all the non-zero eigenvalues.

4.4 Eigenstars

Let $\lambda \neq 0$ be an eigenvalue of Λ, and glue a star $-\lambda\bigstar$ to Γ. Then by the previous section, we subtract λ from each non-zero eigenvalue of Λ, and in particular the λ-eigenspace of Λ is in the nullspace of $\Lambda - \lambda S$. We verify this with a calculation: if $\Lambda x = \lambda x$, then

$$(\Lambda - \lambda S)x = \Lambda x - (\lambda I - \frac{\vec{1}\vec{1}^\top}{n})x = \lambda x - \lambda x + 0 = 0$$

since $\vec{1}^\top x = 0$.

Remark 4.3. This gives a strange answer to the question, what does it mean for λ and x to be eigenvalues and eigenvectors of Λ? If we glue a $-\lambda$ star to Γ and apply the (non-constant) potentials x to the boundary, no current will flow in or out of the network. In Figure 1, the current flows in loops. Clearly the analogy to a physical network of resistors breaks down in this case, because the construction employs negative conductances.

In view of our nullity lemma, we might imagine that the addition of the eigenstar has the effect of chopping Γ into several components. Figure 2 affirms this interpretation. In Figure 3, however, we still have loops, but Γ is not disconnected.

Our eigenstar construction circumvents the nullity lemma because $\Gamma - \lambda\bigstar$ is of mixed sign, but not all mixed-sign networks misbehave this way: [2] exhibits a circular planar network of mixed sign which is response-equivalent to a positive non-planar network. We present two conjectures:

Conjecture 4.4. A connected network has nullity greater than 1 if and only if it is of mixed sign and contains loops in some sense, like those in the examples above. (The definition of loop must be made precise.)

Conjecture 4.5. Let $\Gamma_\pm$ be a critical circular planar network of mixed sign. Then there is no positive critical circular planar network Γ_+ such that $\Gamma_\pm \approx \Gamma_+$.

Remark 4.6. There are several possible approaches to proving this. Morrow suggests using determinental identites. Another approach might involve medial graphs: the conjecture is easily proved if we consider only $Y - \Delta$ equivalences rather than general response-equivalences, so any counterexample to the conjecture would necessarily involve changing the connections of $\Gamma_{\pm}$ or, equivalently, permuting the endpoints of the geodesics of its medial graph (see [1], page 8).

References

[1] Curtis, M., and James A. Morrow. "Inverse Problems for Electrical Networks." Series on applied mathematics – Vol. 13. World Scientific, ©2000.

[2] Schrøder, Konrad. "Negative Conductors and Network Planarity." 1993.

Figure 1: Current flowing in loops.

Figure 2: Starred kite equivalent to disconnected graph.

Figure 3: Problematic example.