RESPONSE MATRIX DECOMPOSITION ON SINGLE CONNECTION EDGES

SIMON PAI

Abstract. This paper gives a method of computing new response matrix from the old one when adding a boudary spike, and it proves a response matrix can be decomposed into a weighted average of two response matrices in some specific cases called single connections. In its conclusion, it also gives a realization of how a single connection affects the response matrix in a sense of geometry.

1. Introduction

The inverse problem of electrical networks has beed studied for more than ten years. According to the pass research, the recoverability on a circular planar network can be clearly realized thought out a corresponding medial graph. However, we have very little knowledge on non-circular-planar cases. This paper provides a simple view on the relationship between a conductance in a network and its response matrix in the easiest case - *single connections*. In order to state the idea explictly, we shall define the terminologies as the following:

Definition 1.1. A *graph with boundary* is a graph $G(V, V_B, E)$ with a set of vertices V (also called nodes in the sense of electrical network), a set of edges E, where some of the vertices are set as boundary nodes V_B, see [1] p11. A vertex is called an interior node if it is not a boundary node. For convienence, a graph always denotes a graph with boundary in this paper. Addtionally, let a *network* denotes a graph with a conductance value assigned on each edge, and *conductivity* refers to an assignment of conductance values on all the edges in a graph in this paper.

Definition 1.2. A *subgraph with boudary* is a subgraph $G'(V', {V_B}', E')$ of $G(V, V_B, E)$ such that any node v in V' is set to ${V_B}'$ if and only if $v \in V_B$ or not all of its adjacent edges are in E'. It's easy to see the compliment of a subgraph with boundary is also a subgraph with boundary. Similarly, we simply use subgraph to denote subgraph with boundary thoughout this paper. For more interesting results about subgraph, see Jeff Russell's work [2]. The word *subnetwork* follows the same analogy.

Definition 1.3. A *boundary spike* is an edge connecting a boundary node and an interior node, such that it is the only edge adjacent to that boundary node. A *boundary edge* is an edge connecting two boundary nodes [1] p55-57.

Definition 1.4. A *single connection edge* is an edge in a connected graph such that removing it breaks the connectedness. A boundary spike is always a single connection edge. Any boundary edge doesn't count for a single connection.

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Remark 1.5. For *response matrix, medial graph, circular-planarity*, see [1]. Let $\Lambda(\Gamma)$ donotes the response matrix of a network Γ thourghout this paper.

Remark 1.6. In this paper, a graph doesn't need to be circular-planar, and we shall consider all-positive conductivities. However, the concept in this paper also works for mix-signed cases with some minor problems (i.e. sigularities). For signed conductivities, see also [3].

2. Boundary Spike Computation

In this section, we work out how the response matrix changes when adding a boundary spike onto the network. Precisely, we shall pharse our statement as the following:

Lemma 2.1. *Suppose $G(V, V_B, E)$ is a network with $V_B = \{V_1, V_2, \ldots, V_n\}$, and V_1 has a boundary spike S connecting to an interior node V_0. Let $G'(V', {V_B}', E')$ be a subgraph of G, with $V' = V - \{V_1\}, E' = E - \{S\}$, and $V'_B = \{V_0, V_2, \ldots, V_n\}$, then we have:*

$$\Lambda = \Lambda' \begin{pmatrix} \frac{c}{c+\Lambda'_{11}} & -\frac{\Lambda'_{12}}{c+\Lambda'_{11}} & -\frac{\Lambda'_{13}}{c+\Lambda'_{11}} & \cdots & -\frac{\Lambda'_{1n}}{c+\Lambda'_{11}} \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \tag{1}$$

where c is the conductance on S, Λ'_{ij} denotes the i_{th} row, j_{th} column entry of Λ' (also throughout this paper).

Proof. Use Schur complements, see [1] p57. □

3. Contraction and Deletion

Definition 3.1. Suppose there is an edge E connecting two vertices A_1 and A_2 in graph G. The *contraction* of E removes E and merges A_1 and A_2 into a vertex A' so that A' owns all the degrees of A_1 and A_2 originally. The *deletion* just removes E, and keeps A_1 and A_2 seperated. See [1] p16-17.

When we consider a conductance in an inclusive sense, the possible greatest value is *positive infinite* and the possible least value is *zero*. A positive infinite conductance on a network is equivalent to the corresponing edge contracted in the graph. Similarly, a zero conductance is equivalent to the edge deleted. Contraction and deletion therefore represent the two extreme conditions of the graph when a conductance varies from zero to positive infinite.

Given a conductance c in a network Γ, let the networks after contraction and deletion of c called Γ^C and Γ^D respectively, throughout this paper. If we consider the response matrix $\Lambda(\Gamma)$ as a point in the "response matrix space" (i.e. think each non-trivial entry in the matrix as a dimention in a geometric space.) when c varies from zero to positive infinite, the trace should be a curve connecting from $\Lambda(\Gamma^D)$ to $\Lambda(\Gamma^C)$. Conceptionally, we shall think of contraction and deletion as two endpoints of a dimention in the conductivity space.

4. Decomposition on Boundary Spike

Based on the concept of contraction and deletion, now we shall be able to rewrite the formula given by lemma 2.1 to get the following result:

Theorem 4.1. *Let c be a boundary spike on the first boundary node in a network Γ, while Γ^C and Γ^D be the networks after contraction and deletion of c. Then we have:*

$$\Lambda(\Gamma) = p\Lambda(\Gamma^C) + q\Lambda(\Gamma^D), \tag{2}$$

where $p = \frac{c}{c+\Lambda(\Gamma^C)_{11}}, p + q = 1$

Proof. In lemma 2.1 we have:

$$\Lambda = \Lambda' \begin{pmatrix} \frac{c}{c+\Lambda'_{11}} & -\frac{\Lambda'_{12}}{c+\Lambda'_{11}} & -\frac{\Lambda'_{13}}{c+\Lambda'_{11}} & \cdots & -\frac{\Lambda'_{1n}}{c+\Lambda'_{11}} \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$= \frac{c}{c+\Lambda'_{11}}\Lambda' \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} + \frac{\Lambda'_{11}}{c+\Lambda'_{11}}\Lambda' \begin{pmatrix} 0 & -\frac{\Lambda'_{12}}{\Lambda'_{11}} & -\frac{\Lambda'_{13}}{\Lambda'_{11}} & \cdots & -\frac{\Lambda'_{1n}}{\Lambda'_{11}} \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$= \frac{c}{c+\Lambda'_{11}}\Lambda' + \frac{\Lambda'_{11}}{c+\Lambda'_{11}}\Lambda_A$$

Notice that Λ' itself is just $\Lambda(\Gamma^C)$, and Λ_A is the result of letting $C = 0$ in lemma 2.1, which is also $\Lambda(\Gamma^D)$. Therefore we got the theorem. □

For convienence, we shall call the coefficient $\frac{c}{c+\Lambda'_{11}}$ the *p-value* of c in the network Γ, denoted as $p(c;\Gamma)$. As c varies from zero to positive infinite, $p(c;\Gamma)$ goes from 0 to 1.

Corollary 4.2. *Suppose $\Gamma, \Gamma_1, \Gamma_2$ are three networks with all the same conductances except a boundary spike, whose conductances are c, c_1, c_2 respectively. If $p(c;\Gamma) = kp(c_1;\Gamma_1) + (1-k)p(c_2;\Gamma_2)$, then we have $\Lambda(\Gamma) = k\Lambda(\Gamma_1) + (1-k)\Lambda(\Gamma_2)$.*

Proof. Apply theorem 4.1. □

5. Decomposition on Single Connection

Theorem 5.1. *Let c be a single connection in a network Γ, while Γ^C and Γ^D be the networks after contraction and deletion of c. Then we have:*

$$\Lambda(\Gamma) = p\Lambda(\Gamma^C) + q\Lambda(\Gamma^D), \tag{3}$$

where $p = \frac{c}{c+\frac{\Lambda^A_{11}\Lambda^B_{11}}{\Lambda^A_{11}+\Lambda^B_{11}}}, p + q = 1$, *and Λ^A, Λ^B are the response matrices of the two subnetworks connected by c.*

Proof. We shall temporarily see the conductance c as two conductances c_A and c_B connected in series. Thus $\frac{1}{c} = \frac{1}{c_A} + \frac{1}{c_B}$. Let the vertex connecting c_A and c_B be V. Consider seperating the network into two subnetworks by cutting off the wire at the point V, so that c_A and c_B are boudary spikes in subnetworks Γ^A and Γ^B respectively. Apply theorem 4.1 on each of c_A and c_B, and wiggle the distribution of c_A and c_B until $p(c_A;\Gamma^A) = p(c_B;\Gamma^B)$. This can always be done as long as we have all-positive conductivity, because the set of equations

$$\begin{aligned} \tfrac{1}{c} &= \tfrac{1}{c_A} + \tfrac{1}{c_B} \\ \tfrac{c_A}{c_A+\Lambda^A_{11}} &= \tfrac{c_B}{c_B+\Lambda^B_{11}} \end{aligned} \tag{4}$$

always has a unique solution

$$\begin{aligned} c_A &= (\tfrac{\Lambda^A_{11}+\Lambda^B_{11}}{\Lambda^B_{11}})c \\ c_B &= (\tfrac{\Lambda^A_{11}+\Lambda^B_{11}}{\Lambda^A_{11}})c \end{aligned} \tag{5}$$

and $p(c;\Gamma) = p(c_A;\Gamma^A) = p(c_B;\Gamma^B)$.

After decomposition on each part, we just "glue" back two subgraphs after contractions together, and two after deletions together, respectively. Nick Addington's work ensures the correctness of this step, see [4] p1-Lemma 1.7. □

Corollary 5.2. *Suppose* $\Gamma, \Gamma_1, \Gamma_2$ *are three networks with all the same conductances except a single connection, whose conductances are* c, c_1, c_2 *respectively. If* $p(c;\Gamma) = kp(c_1;\Gamma_1) + (1-k)p(c_2;\Gamma_2)$*, then we have* $\Lambda(\Gamma) = k\Lambda(\Gamma_1) + (1-k)\Lambda(\Gamma_2)$*.*

Proof. Apply theorem 5.1. □

6. A Geometric View

When we mention a *response matrix space*, it refers to a geometric space fromed by setting each non-trivial entry in the response matrix as a dimention. Similarly, a *conductivity space* means seeing each conductance in a network as a dimention. Thus the recoverability problem becomes a study on how a graph maps a conductivity space to a response matrix space. Precisely, a network (conductivity) is a point in a conductivity space, while a response matrix is a point in a response matrix space.

Consider a single connection c in a network Γ. When c varies from zero to positive infinite, the point $\Lambda(\Gamma)$ should moves from $\Lambda(\Gamma^D)$ to $\Lambda(\Gamma^C)$ along some path in the response matrix space. However, since $\Lambda(\Gamma)$ is a weighted average of $\Lambda(\Gamma^C)$ and $\Lambda(\Gamma^D)$, it always lies on the segement connecting these two points, which means the path is a straight line segement. This gives a geometric realization about how a single connection acts in the response matrix.

References

[1] Curtis, Edward B. and James A. Morrow. "Inverse Problems for Electrical Networks." Series on Applied Mathematics, Vol 13; 2000.
[2] Russell, Jeff. "Recoverability of Subgraphs"; 2003.
[3] Goff, Michael. "Recovering Networks with Signed Conductivities"; 2003.
[4] Addington, Nicolas. "Stars, Eigenvalues, and Negative Conductivities"; 2003.