Convergence of solutions to discrete inverse problems

J.T. Russell

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1 Definitions (the cool part)

Definition 1.1. A cell-conductivity network Γ (in Ω) is an ordered pair (G, γ), where $G = (C, C_B, E)$ is a graph and $\gamma : C \to \mathbf{R}$ is a conductivity function.

- $C = \{C_1, C_2, \ldots, C_n\}$ is a set of cells, which are disjoint open subsets of Ω such that $\bigcup \operatorname{cl} C_i = \Omega$.
- C_B consists of the boundary cells, which are those cells such that $\partial C_i \cap \partial\Omega$ is non-empty.
- E is the edge relation on C, which marks two cells as being adjacent iff $\partial C_i \cap \partial C_j$ is non-empty.

We will often write G instead of C and ∂G instead of C_B, and denote adjacency by $\sim$.

Each cell has an associated conductivity γ_i, and current flow into a cell is determined by

$$(Kv)_i = \sum_{j \sim i} \gamma_j (v_j - v_i) \tag{1}$$

A cell-conductivity network thus defined is identical to a vertex-conductivity network, but with a new geometric association: the 'vertices' now fill space. (In fact, given any vertex-conductivity network on a graph $\hat{G}$ embedded in a region, we can construct a corresponding cell-conductivity network by considering the faces of the dual graph of $\hat{G}$.) With this understanding, we can extend a function on G in a natural way to a step-function on Ω, simply by painting the value at each cell across the part of Ω covered by that cell.

For $x \in \operatorname{int}\Omega$, $cell(x)$ will denote the cell containing x, and for $x \in \partial\Omega$, $cell(x)$ will denote the cell whose boundary contains x. We will write $|C_i|_\Omega$ for the measure of C_i in Ω, and $|C_i|_{\partial\Omega}$ for the $\partial\Omega$-measure of $\partial C_i \cap \partial\Omega$.

Definition 1.2. For a function ϕ defined on G or ∂G, define the extension $\tilde{\phi}$ on Ω or $\partial\Omega$ (resp.) such that $\tilde{\phi}(x) = \phi(cell(x))$.

Next we will define extensions of operators that act on functions defined on graphs to operators acting on functions defined on continuous region.

Definition 1.3. Let $(X, \widetilde{X})$ and $(Y, \widetilde{Y})$ each be either (G, Ω) or $(\partial G, \partial\Omega)$. Consider L, a linear operator from $Map(X, \mathbf{R})$ to $Map(Y, \mathbf{R})$, which we will extend to $\widetilde{L} : Map(\widetilde{X}, \mathbf{R}) \to Map(\widetilde{Y}, \mathbf{R})$. Let K be the matrix representation of L. We can also think of K as an integration kernel for L, where the integration is with respect to a uniform discrete measure. That is,

$$L\phi(y) = (L\phi)_i = \int_X K(y, x)\phi(x)\, dx = \sum_j K_{ij}\phi_j \tag{2}$$

So we define the extension $\widetilde{K}$ of the kernel K by

$$\widetilde{K}(y, x) = K(cell(y), cell(x)) \tag{3}$$

and finally we define the extended operator $\widetilde{L}$ by

$$\widetilde{L}\phi(y) = \int_{\widetilde{X}} \frac{\widetilde{K}(y, x)\phi(x)}{|cell(x)|_{\widetilde{X}}}\, dx \tag{4}$$

Note that under this definition $\widetilde{L}\widetilde{\phi} = \widetilde{L\phi}$. We will refer to operators acting on functions on G or ∂G as "discrete operators" and to those acting on Ω or $\partial\Omega$ as "continuous operators".

Definition 1.4. For a sequence of functions $\phi_k : G_k \to \mathbf{R}$ and a function $\phi : \Omega \to \mathbf{R}$, we say ϕ_k converges to ϕ iff $\widetilde{\phi}_k \to \phi$. Likewise, a sequence of discrete operators (i.e. matrices) L_k converges to a continuous linear operator L iff $\widetilde{L}_k \to L$.

We can think of convergence of conductivity functions on G_k to a function on Ω in any of the traditional senses of convergence: uniform, pointwise, L^p, etc. We will use the L^2 norm for functions on $\operatorname{int}\Omega$ and the L^2 norm with respect to boundary-measure on $\partial\Omega$. For convergence of operators we use the 'natural' norm $\|A\| = \sup\{\|Av\| : \|v\| = 1\}$.

It can also be useful to think of a function on a finite set as a vector. From this perspective, the L^2 norm $\|\widetilde{v}\|_2$ is identical to the weighted Euclidean norm

$$\|v\|_\omega^2 = \sum_i \omega_i v_i^2 \tag{5}$$

where ω_i is the measure of cell i. Denote the corresponding inner product by $\langle , \rangle_\omega$. Note that $\langle u, v\rangle_\omega = \int_\Omega \widetilde{u}\widetilde{v}$.

There are a few facts to check about extensions and extensional convergence. These are all uninteresting. Pay no attention.

1. $\widetilde{A\phi} = \widetilde{A}\widetilde{\phi}$

2. If A and B are discrete operators, then $\widetilde{AB} = \widetilde{A}\widetilde{B}$
3. $\widetilde{\phi + \psi} = \widetilde{\phi} + \widetilde{\psi}$
4. If A and B are linear operators, then $\|AB\| \leq \|A\|\|B\|$

Suppose that $A_k \to A$, $B_k \to B$, $\phi_k \to \phi$, and $\psi_k \to \psi$.

1. $cA_k \to cA$
2. $A_k B_k \to AB$
3. $I_k \to I$, where I_k and I are the identity operators on their respective domains.
4. $A_k A_k^{-1} \to I$
5. $\phi_k + \psi_k \to \phi + \psi$
6. $A_k \phi_k \to A\phi$

2 Thus saith the mathematician

Theorem 2.1. *Suppose that $\Gamma_k = (G_k, \gamma_k)$ is a sequence of cell-conductivity networks embedded in a region Ω. Let Λ_k be the response matrix for Γ_k. Furthermore, let the sequence Γ_k satisfy certain* HYPOTHESES. *Then if $\Lambda_k \to \Lambda$ and $\gamma_k \to \gamma$, then Λ is the response matrix for (Ω, γ); in other words, γ is a solution of the inverse problem on Ω determined by Λ.*

Proof. For $\phi : \partial\Omega \to \mathbf{R}$ with the right continuity/differentiability conditions, we need to demonstrate some γ-harmonic function $u : \Omega \to \mathbf{R}$ such that on the boundary $u = \phi$ and $\mathbf{n} \cdot \nabla u = \psi = \Lambda\phi$.

Kids, don't try this at home. Initiating hand-waving.

Let K_k be the Kirchhoff matrix for Γ_k. Also, let the operator K be defined piecewise as $\mathbf{n}\cdot\gamma\nabla$ on $\partial\Omega$ and as $\nabla\cdot\gamma\nabla$ on $\operatorname{int}\Omega$. We will show that K_k converges in operator-space to K.

If G_k is a lattice then we can do this in pieces: differencing converges to a partial derivative, multiplication by γ_i goes to multiplication by γ, and summation goes to divergence. The limit of a composition is the composition of the limits, so this gives $K_k \to K$.

K_k is not an invertible operator, but we can make it injective by restricting the domain, and bijective by restricting the codomain. Let V_k be the range of K_k and V be the range of K; these are subspaces of one less dimension than the original function spaces, and $V_k \subset V$. The Neumann problem has a unique solution up to the addition of a constant, and restricting to a space of codimension one effectively 'grounds' the system, yielding a unique solution. Let $L_k = K_k|_{V_k}$, and let $L = K|_V$. Consider as functions onto their ranges, L_k and L are bijective. L_k and L are invertible.

Our next goal is to prove that L_k^{-1} converges to L^{-1}. We will do this by way of

Lemma 2.2. *Let A_k, B_k, A, and B live in an operator-space with a natural norm $\|\cdot\|$ and identity I. Suppose that A_k converges to A, $A_kB_k \to I$, and $AB = BA = I$. Then B_k converges to B.*

Proof. First we will show that B_k converges to B whenever $\|B_k\|$ is bounded.

$$\|(B - B_k) - (BA_kB_k - B_k)\| = \|B(I - A_kB_k\| \leq \|B\|\|I - A_kB_k\|$$

Since the norm of B is some fixed number, and A_kB_k goes to I, the quantity goes to zero. In particular, $\|B - B_k\|$ vanishes iff $\|BA_kB_k - B_k\|$ does. Denote this quantity by M_k. Then we have

$$M_k = \|(BA_k - I)B_k\| \leq \|(BA_k - I)\|\|B_k\|$$

Since $A_k \to A$, $BA_k \to BA = I$. So if $\|B_k\|$ is bounded, say by N, then

$$M_k \leq N\|BA_k - I\| \to 0$$

Therefore $\|B - B_k\|$ vanishes.

Next we show that $\|B_k\|$ is bounded. The set of invertible operators is an open set; since A_k converges to A, it follows that for large enough k, A_k is invertible. Furthermore,

$$\|B_k - A_k^{-1}\| = \|A_k^{-1}(A_kB_k - I)\| \leq \|A_k^{-1}\|\|A_kB_k - I\|$$

so if $\|A_k^{-1}\|$ is bounded then B_k approaches A_k^{-1}; in particular, $\|B_k\|$ is bounded. It remains only to show that A_k^{-1} is bounded.

Since $A_k \to A$ we can write A as a sum of A_k and some matrix E_k, whose norm approaches zero; so say $\|E_k\| \leq \varepsilon$.

$$\begin{aligned} A_k &= A - E_k \\ &= A(I - A^{-1}E_k) \\ &= A(I - F_k) \end{aligned}$$

And $\|F\| \leq \|A^{-1}\|\|E_k\| \leq \varepsilon' < 1$, since the norm of A^{-1} is some fixed number. Then we invert A_k:

$$\begin{aligned} A_k^{-1} &= (I - F_k)^{-1}A^{-1} \\ &= (I + F_k + F_k^2 + F_k^3 + ...)A^{-1} \end{aligned}$$

And, since $\sum \varepsilon'^k = (1 - \varepsilon')^{-1}$, we conclude that

$$\|A_k^{-1}\| \leq \|A^{-1}\|(1 - \varepsilon')^{-1}$$

That is, $\|A_k^{-1}\|$ is bounded, and therefore converges to A^{-1}. □

Since $\widetilde{L}_k\widetilde{L_k^{-1}}$ approaches the identity operator, the lemma implies that $\widetilde{L_k^{-1}}$ converges to $\widetilde{L}^{-1}$.

Consider any Riemann-integrable function $\phi : \partial\Omega \to \mathbf{R}$. There exists some sequence of functions $\phi_k : \partial G_k \to \mathbf{R}$ whose extensions converge to ϕ. Let $\psi_k = \Lambda_k \phi_k$ and $\psi = \Lambda\phi$. Since Λ_k converges to Λ, ψ_k approaches ψ. L_k^{-1} converges to L^{-1}, thus,

$$v_k = L_k^{-1} \begin{bmatrix} \psi_k \\ 0 \end{bmatrix} \longrightarrow L^{-1} \begin{bmatrix} \psi \\ 0 \end{bmatrix} = v \tag{6}$$

Now pick some point $p \in \partial\Omega$. Let c_k be the constant function $\tilde{\phi}_k(p) - \widetilde{v_k}(p)$. Clearly c_k converges to the constant function $c(x) = \phi(p) - v(p)$. Then $u_k = v_k + c_k$ converges to $u = v + c$. Since u_k and v_k differ by a constant,

$$Ku_k = Kv_k = Lv_k = \begin{bmatrix} \psi_k \\ 0 \end{bmatrix} \tag{7}$$

So u_k is a γ-harmonic function that satisfies the Neumann data ψ_k. Any two functions with this property differ by a constant; in particular, if two such functions are equal at one point, they are equal everywhere. Since $\Lambda_k \phi_k = \psi_k$, there is a γ-harmonic function equal to ϕ_k on ∂G that satisfies the Neumann condition. u_k equals ϕ_k on $cell(p)$, so u_k must be that function; i.e., $u_k = \phi_k$ on the boundary.

Finally, since u_k converges to u and ϕ_k converges to ϕ, $u = \phi$ on $\partial\Omega$. u and v differ by a constant, so $Ku = Kv = Lv = \begin{bmatrix} \psi \\ 0 \end{bmatrix}$. That is to say, u is a γ-harmonic function that satisfies both the Dirichlet data ϕ and the Neumann data $\Lambda\phi$. So u is exactly the function we needed to find. Ergo, γ is the solution of the inverse problem (Ω, Λ).

□

3 Incidentally,

the norm of $\widetilde{L^{-1}}$ is actually connected to the discrete and continuous Dirichlet norms.

$$\begin{aligned} \|\widetilde{L_k^{-1}}\| &= \|L_k^{-1}\|_\omega \\ &= \left(\text{maximum eigenvalue of } (L_k^{-1})^\top (L_k^{-1})\right)^{1/2} \\ &= \left(\text{minimum eigenvalue of } ((L_k^{-1})^\top (L_k^{-1}))^{-1}\right)^{-1/2} \\ &= \left(\text{minimum eigenvalue of } L_k L_k^\top\right)^{-1/2} \end{aligned}$$

Consider the quadratic form $\langle v, L_k L_k^\top v\rangle_\omega$. Since $L_k L_k^\top$ is symmetric, it has an orthonormal eigenbasis. Any vector can be decomposed as $v = \sum c_i e_i$ where each e_i is a unit λ_i-eigenvector of $L_k L_k^\top$. Then $\langle v, L_k L_k^\top v\rangle_\omega = \sum \langle c_i e_i, \lambda_i c_i e_i\rangle_\omega = \sum c_i^2 \lambda_i$, which is minimal when v lies on an eigenvector of smallest eigenvalue. Thus the square root of the smallest eigenvalue of $L_k L_k^\top$ is equal to

$$\min_{\|v\|_\omega = 1} \left(\langle v, L_k L_k^\top v\rangle_\omega\right)^{1/2} \tag{8}$$

Since $L_k^\top$ approaches L_k, this will approach

$$\min_{\|v\|_\omega=1} \langle v, L_k v\rangle_\omega = \min_{\substack{\|v\|_\omega=1 \\ v\in V_k}} \langle v, K_k v\rangle_\omega$$

Moreover, we can easily demonstrate that the quadratic form $\langle v, K_k v\rangle_\omega$ limits to the Dirichlet norm $\langle v, Kv\rangle = W_\gamma(v) = \int_\Omega \gamma(\nabla v)^2$. It follows that

$$\min_{\substack{\|v\|_\omega=1 \\ v\in V_k}} \langle v, K_k v\rangle_\omega$$

converges to

$$\inf_{\substack{\int_\Omega v^2=1 \\ v\in V}} W_\gamma(v) \tag{9}$$

So the boundedness of $\|\widetilde{L_k^{-1}}\|$ is equivalent to the fact that (9) is positive.