SUBGRAPH RECOVERABILITY

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Abstract. The idea behind this paper is to provide a way to quickly identify the non-recoverability of a graph simply by examing its subgraphs. My ideas first came from Ryan Card and Brandon Muranaka's paper on amalgamating networks. In their paper, they state very similar ideas to what I propose about subgraph and graph non-recoverability. In Jeff Russell's paper, he also defines many types of subgraphs and states some similar propositions about his specific types of subgraphs. I show how non-recoverable, non-circular planar graphs can be partially recovered introducing parameters and using medial graphs due to methods found in Jeff Giansiricusa's paper. I also examined medial graphs of non-circular planar graphs, looking for patterns to recognize any non-recoverable subgraphs. My intention is to show how subgraphs can be very useful and quick to determine recoverability of circular and non-circular planar graphs.

Contents

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1. Introduction

A *graph with boundary* is a graph consisting of a set of nodes V, and edges E connecting those nodes, with certain of those nodes designated as boundary nodes ∂V, and the other nodes representing interior nodes $intV$. A *circular planar graph with boundary* , G, is a graph with boundary embedded in a disc so that the boundary nodes lie on the circle, C, which bounds the disc, and the rest of the graph is interior to the disc.

Definition 1.1. A graph G'=(V',E') is a *subgraph* of G =(V,E) if the following conditions are met:
(1) $V' \subset V$ with a chosen decomposition into $\partial V' \cup intV'$ such that $intV' \subset intV$,
(2) $E' \subset E$,
(3) each edge in E that has atleast one endpoint as a vertex in $intV'$ must be in E'.
(Note that $\partial - \partial$ vertices are not required in G').

An electrical network Γ = (G,γ) is a graph with a boundary G together with a function $\gamma > 0$ defined on the edges of G which specifies the conductivity of each edge. Γ is considered to be recoverable when the values of the conductors are uniquely determined by, and can be calculated from a corresponding response matrix, Λ. A graph, G = (V,E), is said to be recoverable if for any conductivity γ, γ can be computed from the corresponding response matrix ,Λ.

2. Subgraph Recoverability

From the previous definitions, the following theorem is derived for both circular planar and non-circular planar graphs.

Theorem 2.1. *If G is recoverable, then G' is recoverable.*

Proof. Let Λ' be the response matrix for G'. Λ' is the Kirchhoff matrix for a graph on $\partial V'$. Let S be the set of boundary vertices of G' that are also boundary vertices

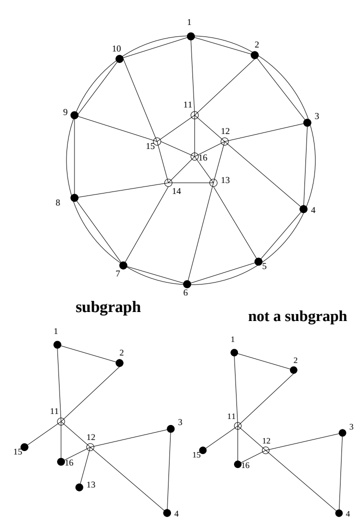

Figure 1. Example of a subgraph

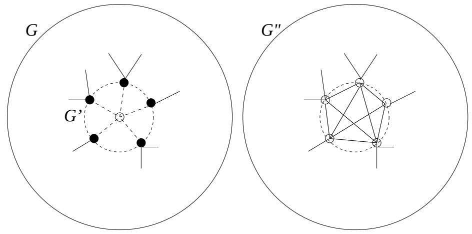

FIGURE 2. Sketch of G, G', G''

of G and T be the set of boundary vertices of G' that are interior vertices of G. From this point on, let $P = S \cup T = \partial V'$ so that P is the set of all boundary vertices of G' and $J = intV'$ so that J is the set of all interior vertices of G'.
Construct $\Gamma'' = (G'', \gamma'')$ as follows:

$V'' = V - intV'$,

$\partial V'' = \partial V$,

$intV'' = intV - intV'$,

$E'' = E - E' \cup E(\Lambda')$ where the conductivities of all edges not in G' are defined to be 1. Figure 2 gives a rough sketch of G, G', and G'' where $\partial V'$ may or may not be in ∂V.

Now, known entries in the Kirchhoff matrix of G with some conductivity function γ, call it K, and Λ' will be used to construct a response matrix Λ for G.

To find Λ, consider the block structure of K.

$$K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \tag{1}$$

Let S precede all other vertices in the A block, T precede all other vertices in the B and C blocks, and J procede all other vertices in the B and C blocks as in Figure 2. The *'s refer to the unknown data that correspond to all edges in G'. The 0's can automatically be filled in from the definition of subgraph. Any edge appearing in G but not in G' was assigned a conductivity of 1 and hence the remainder of K can be filled in.

Let

$$\tilde{K} = K(P+J; P+J) = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{B}^T & \tilde{C} \end{pmatrix}. \tag{2}$$

$\tilde{A} = \tilde{A}' + \tilde{A}''$ where $\tilde{A}'$ is the sum of the conductivities of edges not in G', which is known, and $\tilde{A}''$ is the sum of the conductivities of edges in G', which is unknown. $\tilde{B}$ and $\tilde{B}^T$ represent the boundary to interior conductivities and $\tilde{C}$ represents all interior to interior conductivities of G'.

If only J is interiorized, $K/K(J;J)$ only changes entries in K where the *'s in figure 2 are located. This is due to the placement of the 0's in the rows and columns of J. Thus, those entries have been placed in $\tilde{K}$ for simplicity of computation. $\tilde{K}/\tilde{K}(J;J) = \tilde{A} - \tilde{B}\tilde{C}^{-1}\tilde{B}^T = \tilde{A}' + \tilde{A}'' - \tilde{B}\tilde{C}^{-1}\tilde{B}^T$. However, $\tilde{A}'' - \tilde{B}\tilde{C}^{-1}\tilde{B}^T$ corresponds to Λ', which was given. Thus, $\tilde{K}/\tilde{K}(J;J) = \tilde{A}' + \Lambda'$ is known.

Now, place the values of $\tilde{K}/\tilde{K}(J;J)$ into $K/K(J;J)$. Enough information is known to complete the Schur complement of K by computing $\frac{K/K(J;J)}{K(I;I)/K(J;J)}$ which

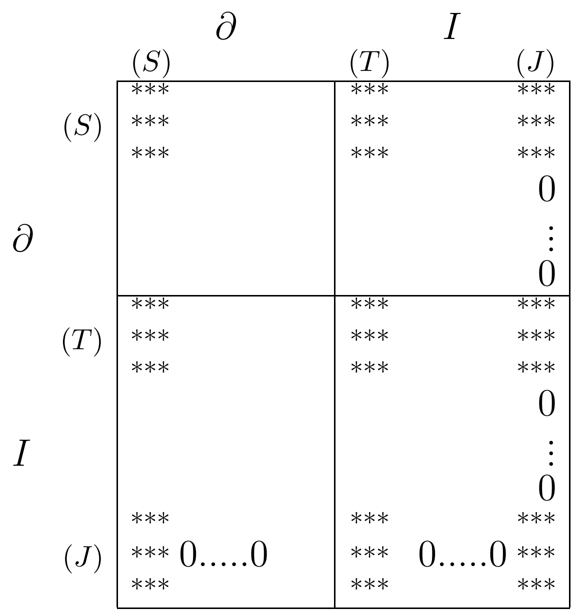

FIGURE 3. Detailed Kirchhoff matrix of G

gives Λ. Since G is recoverable, the conductivities of all edges in G can be found, including the edges in G'. Therefore, G' is recoverable. □

3. Graph Recoverability

The theorem just stated may seem very obvious and quite trivial. However, the useful conclusion from this theorem is the contrapositive.

Corollary 3.1. *If a subgraph G' of a graph G is not recoverable, then G is not recoverable.*

Thus, when studying the recoverability of graphs, if a non-recoverable subgraph is detected right away, much valuable time spent on tedious calculations can be saved.

4. Non-Critical Circular Planar Graphs

With a clear understanding of subgraphs, they can now be used to identify the non-recoverability of a non-critical circular planar graph, G, and minimize the steps needed to construct an electrically-equivalent critical graph, G'. The new graph can be used to partially recover the conductivities of edges in G. First, several concepts must be introduced and defined.

Definition 4.1. A graph G is called a *critical graph* if the removal of any edge breaks some connection through G.

Definition 4.2. Suppose $\mathcal{A}$ is a family of arcs in the disc D, which intersect at two distinct points p and q. A subgraph L of the graph formed by the family $\mathcal{A}$ is called a lens if the following two conditions are satisfied:

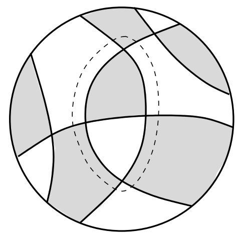

FIGURE 4. Defining G_0

(1) there are two arc fragments $a_0 a_1 ... a_l b_0$ and $b_0 b_1 ... b_m a_0$. The sequence of arc segments
$$\mathcal{P} = a_0 a_1 ... a_l b_0 b_1 ... b_m a_0$$
is a simple closed path in the interior of D

(2) L consists of the vertices and arc segments of $\mathcal{P}$ together with all vertices and arc segments of $\mathcal{A}$ in the interior of the bounded component of the complement of $\mathcal{P}$.

Observation 4.1. The only repeated vertices are a_0 and b_0, called the *poles* of the lens. If $b_0 \neq a_0$, the path $\mathcal{P}$ will be called a *simple lens*. It can happen that $b_0 = a_0$ (that is, there are no points b_i for $i > 0$). In this case, there is only one pole a_0, and the path $\mathcal{P}$ will be called a *loop* in $\mathcal{A}$. This degenerate case of a lens is handled in much the same way as a true lens which has two poles. It can also happen that the lens has no pole, in which case it is called a *bubble*.

Theorem 4.3. *Suppose that $\mathcal{A}$ is a family of arcs that has one or more lenses. Then by a finite sequence of switches and uncrossings of arcs that form empty lenses, $\mathcal{A}$ can be reduced to a family that is lensless.*

Theorem 4.4. *A circular planar graph G is critical if and only if its medial graph $\mathcal{M}$ is lensless.*

Theorem 4.5. *A graph is recoverable if and only if it is critical circular planar.*

Theorem 4.6. *Let e be an edge in a circular planar graph G, and let v_e be the vertex corresponding to e in the medial graph $\mathcal{M}$ associated to G. If v_e is not contained in any lens then $\gamma(e)$ is recoverable.*

Details of the previous theorems and definitions can be found in [1] (except for Theorem 4.6 which came from [2]). The goal here is to find the smallest subgraphs associated with each lens of the medial graph. Then, the process of emptying and uncrossing the lenses through a finite series of $Y - \Delta$ transformations and replacement of two edges in series or parallel by a single edge, constructs a new electrically-equivalent graph which is then used to partially recover the original edges of G.

4.1. **Defining G_0.** Let G be a non-critical circular planar graph. To find the non-recoverable subgraphs of G, construct the medial graph $\mathcal{M}$ of G. (For detailed directions on how to draw medial graphs, see [1]).

For *every* lens, there is a corresponding non-critical circular planar subgraph, $G_0 = (V_0, E_0)$ that can be constructed as follows. Draw a circle around the lens

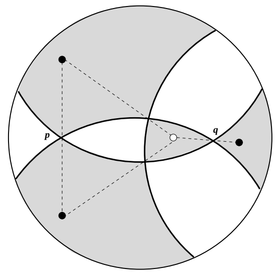

FIGURE 5. Defining edges of G_0

that includes only the regions interior and adjacent to the entire lens in G as in Figure 4. The interior of this circle represents G_0. Every node interior to the lens is defined as $intV_0$ and every edge in G connected to $intV_0$ must be in E_0 by definition of subgraph. Also, all edges that are associated with the poles must be included as in Figure 5. All vertices in V_0 not enclosed in the lens are labeled ∂V_0. Therefore, under this construction, G_0 corresponds to a smaller non-recoverable subgraph of G.

4.2. **Emptying and Uncrossing the Lenses.** The next steps are to empty the lens and then uncross it. To do so, consider simple lenses first. By a series of $Y - \Delta$ or $\Delta - Y$ transformations, the lens can be emptied. Figure 7(a) shows the correspondence between the medial graph and the graph of this transformation. Each $Y - \Delta$ or $\Delta - Y$ transformation produces an electrically-equivalent graph to the previous graph.

Once the interior of the lens has been cleared of all arcs, the lens must be uncrossed to represent a recoverable graph. The empty simple lens will either correspond to two edges in parallel connection or two edges in series connection as in Figure 7(b). In both cases, the two edges can be contracted into one and the resulting graph, G_0', is electrically equivalent to G_0. However, G_0' corresponds to a lensless medial graph, and thus, by Theorem 4.4, is recoverable.

For lenses within lenses, the above process can be used on G_0 by starting with an interior lens with the smallest number of interior regions. Follow the above steps to empty and uncross the lens. If the lens represents either a loop or a bubble and is crossed by at least one arc, that in turn forms a simple lens. These lenses must be emptied and uncrossed first, in the same manner as above. During this process, it is possible that uncrossing an interior lens may also uncross a larger lens. That is

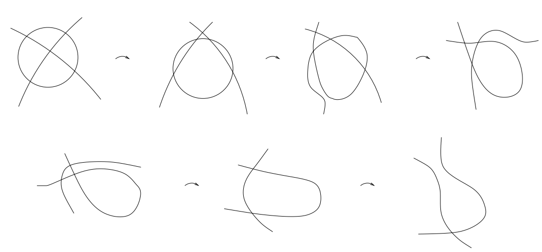

FIGURE 6. Clearing a bubble transforms it into a loop

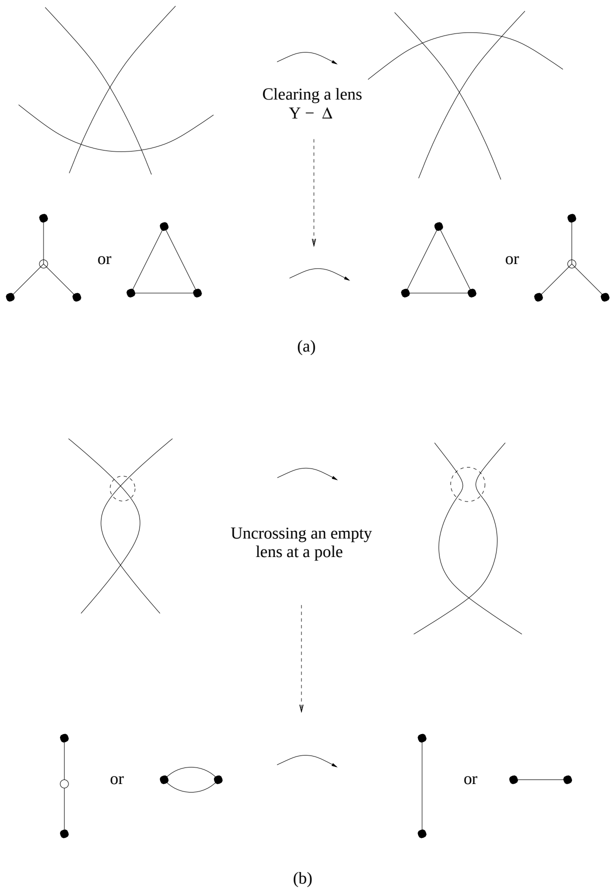

FIGURE 7. Graph and medial graph correspondence when a lens is emptied (a) and uncrossed (b)

why it is important to start with the smaller interior lenses. An empty bubble can be disregarded because no edges are associated with it. In the process of emptying and recovering a non-empty bubble, it becomes a loop as in Figure 6. A loop, once it has been emptied, is uncrossed at the pole which is similar to untwisting the lens. Once this has been done, a check must be made for any final lenses. Thus, G_0' is found for all G_0.

4.3. **Replacing G_0 with G_0'.** After each lens has been cleared and uncrossed, the new graph, G_0' replaces G_0 in G. This is possible because each process described above in constructing G_0' preserves the number of boundary nodes of G_0. Thus, once this process is completed for all lenses in $\mathcal{M}$ of G, the newly constructed, electrically-equivalent graph G' is now lensless and hence, recoverable by Theorems 4.4 and 4.5.

4.4. **Recovering Conductivities.** Now that G' is recoverable, the resulting edge conductivities can be used to partially recover the edges corresponding to the lenses. The conductivities of edges that were not associated with any lens in G are taken from G'. For those edges that were modified, the following three diffeomorphisms, taken from [2], can be used to partially recover the remaining edge conductivities. For the joining of two edges in parallel, it is known there is a diffeomorphism ϕ_{par} that sends the conductivity of the single edge, a, to the two edges in parallel by introducing a parameter, t, given by:

$$\phi_{par} : (a, t) \mapsto (at, a(t-1)). \tag{3}$$

Similarly, for edges in series there is also a diffeomorphism ϕ_{ser} that sends the conductivity of the single edge, a, to the two edges in series by introducing a parameter t, given by:

$$\phi_{ser} : (a, t) \mapsto (\frac{a}{t}, \frac{a}{1-t}). \tag{4}$$

For any $Y - \Delta$ or $\Delta - Y$ transformation, the conductivities can be recovered as follows. From [2], it is also known there is a diffeomorphism $\phi_{Y\Delta}$ that sends conductivities on a Y network to their equivalent conductivities on a Δ given by:

$$\phi_{Y\Delta} : (a, b, c) \mapsto (\frac{bc}{a+b+c}, \frac{ac}{a+b+c}, \frac{ab}{a+b+c}). \tag{5}$$

Figure 8 shows this relationship where $\alpha = \frac{bc}{a+b+c}$, $\beta = \frac{ac}{a+b+c}$, and $\gamma = \frac{ab}{a+b+c}$. Similarly, conductivities on a Δ can be found for a Y where $a = \frac{\beta\gamma}{\alpha} + \beta + \gamma$, $b = \frac{\alpha\gamma}{\beta} + \alpha + \gamma$, and $c = \frac{\alpha\beta}{\gamma} + \alpha + \beta$. Therefore, a non-critical circular planar graph can be partially recovered up to a finite set of parameters.

4.5. **Special Cases that Minimize the Parameters.** Every time an edge is removed and ϕ_{par} or ϕ_{ser} is used in recovery, a new parameter is introduced. Therefore, the number of parameters is equal to the total number of edges in the original graph G minus the total number of edges in the new graph G' as was stated in [2]. At first, one might think that the number of parameters would correspond to the original number of lenses in G. This is not the case, however. In some instances, uncrossing one lens may uncross multiple lenses which decreases the number of parameters. There is a special case when this happens. Thus, the following definitions are introduced.

Definition 4.7. If a simple lens in a medial graph contains a smaller simple lens that has both poles on one arc of the bigger lens, define the larger lens to be

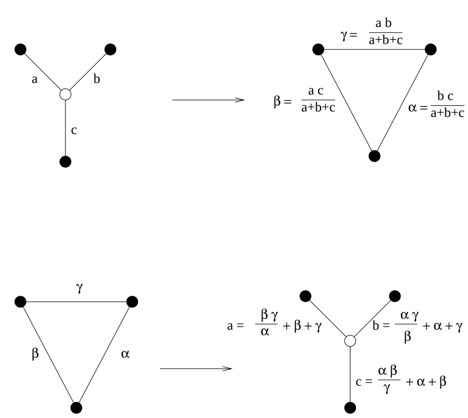

FIGURE 8. Conductivities of $Y - \Delta$ and $\Delta - Y$ Transformations

the *boundary lens* and the smaller lens to be the *interior lens*. Define the shared boundary arc segment to be $\boldsymbol{X}$ and the other arc segment of the boundary lens to be $\boldsymbol{Y}$. It can happen that an interior lens has more than one boundary lens. See Figure 9.

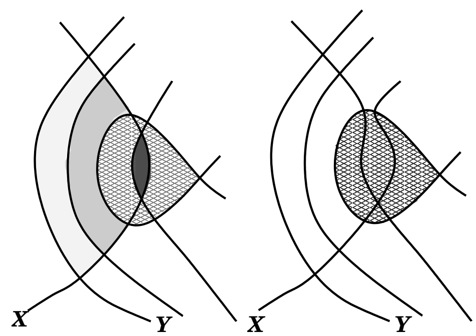

FIGURE 9. Uncrossing the interior lens uncrosses all boundary lenses but not loops or bubbles reducing the number of parameters

If it happens that the arc segment of the interior lens on the opposite of the boundary does not intersect the arc of $\mathbf{Y}$ or form a loop or bubble with itself at any other location on $\mathcal{M}$, then the uncrossing of the innermost interior lens uncrosses all the boundary lenses of that lens. This makes the edges of all the boundary lenses minus the smaller lens recoverable. Therefore, this situation reduces the number of parameters needed to solve for the edges associated with the lenses. It is important to note that this does not reduce the parameters with any other type of lenses.

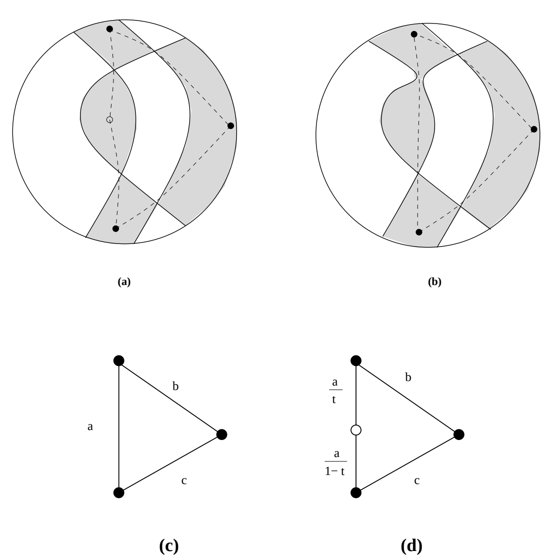

FIGURE 10. Example of uncrossing an interior lens and partially recovering the edges of the original graph

4.6. **Example.** Figure 10 illustrates an example of how uncrossing the interior lens also uncrosses the boundary lens, introducing only one parameter. This process corresponds to joining the two edges in series as shown in (a) and (b). Since the new medial graph in (b) is now lensless, the corresponding graph in (c) can now be recovered. Then, using diffeomorphism (4), the edges of the original graph can be partially recovered, as shown in (d).

5. Non-Circular Planar Graphs

The use of medial graphs is very important in determining the non-recoverable subgraphs of a circular planar graph. However, medial graphs of non-circular planar graphs are not well-defined and much more complicated. Nonetheless, if properly defined and constructed, they can be used to find patterns of non-recoverability of a non-circular planar graph.

5.1. **Annular Graphs.** An annular graph is defined by the number of rays and circles. Specifically, annular graphs will be denoted as $G(r, c)$ where $r = \#$ of rays and $c = \#$ of circles. Figure 11 is an example of $G(3, 2)$ where $r = 3$ and $c = 2$.

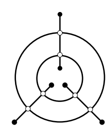

Figure 11. $G(3, 2)$

Definition 5.1. An *annular subgraph* of an annular graph, $G(r, c)$, is a subgraph that contains at least one circle with r interior nodes.

In [5], the following theorem was stated and proved.

Theorem 5.2. *All* $G(2n, n)$ *networks are recoverable.*

For example, by this theorem, it is guaranteed that $G(6, 3)$ is recoverable. Then by Theorem 2.1, all annular subgraphs of $G(6, 3)$ are recoverable. That is $G(6, 2)$ and $G(6, 1)$ must be recoverable. Thus, the following corollary can be stated.

Corollary 5.3. *For every* $G(2n, n)$ *recoverable annular graph, all annular subgraphs* $G'(2n, x)$*, for* $1 \leq x \leq n$*, are recoverable.*

It has been shown in [5] and [6] that $G(3, 2)$ and $G(5, 3)$ are not recoverable. By applying Corollary 3.1, these graphs are annular subgraphs of an infinite collection of non-recoverable graphs of the form $G(3, y)$ for $y \geq 2$ and $G(5, z)$ for $z \geq 3$.

5.2. **Annular Medial Graphs.** In [4], the medial graphs of annular networks are drawn on a torus as in Figure 12. Another interpretation of these graphs can be found by embedding them on a cylinder. These medial graphs are found by drawing two boundary circles on the original graph connecting the two sets of boundary nodes and then using the same convention as a circular planar graph to construct the corresponding medial graph. Using what has already been defined and proven with annular graphs, patterns can be found in the recoverable and non-recoverable annular medial graphs.

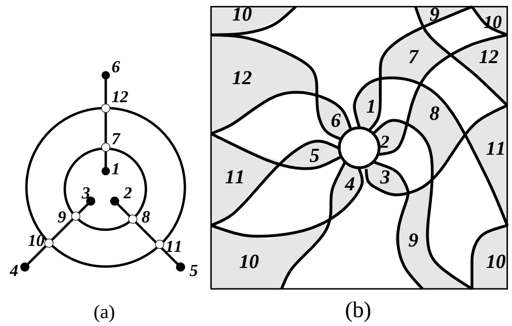

FIGURE 12. $G(3,2)$ with its medial graph drawn on a torus

Figure 13(a) shows the underlying graph, $G(3,2)$, in dotted lines, the corresponding geodesics that form the medial graph in solid lines, and the two boundary circles in dashed lines. The regions are still two-colorable, however, this step is omitted at this point. The four highlighted geodesics intersect in such a way to form the two shaded diamond regions in Figure 13(a).

Definition 5.4. Define the four geodesics that form the diamond region as in Figure 13(c) as the *diamond geodesics* .

Definition 5.5. The region bounded by the intersection of the diamond geodesics as in Figure 13(c) is defined as an *inner diamond* if it is on the innermost circle of the graph. If the diamond geodesics intersect to form another bounded region in the medial graph, that region is defined as an *outer diamond* . In some cases, an outer diamond is also bounded by the outer boundary circle. An outer diamond does not have to correspond to a vertex of the graph.

After examing all combinations of subgraphs of $G(3,2)$, it has been found that $G(3,2)$ is itself the smallest non-recoverable subgraph. Therefore, $G(3,n)$, for $n \geq 2$, is not recoverable as was previously discussed. Figure 13(b) shows the same type of medial graph for $G(3,4)$. Notice the two diamonds. The inner diamond remains the same, however, the outer diamond is now fully bounded by the four geodesics. From this, $G(3,2)$ can be detected as a smaller non-recoverable subgraph of $G(3,4)$. This can be generalized by keeping the same r and letting c be the number of circles interior to the innermost-outer diamond. Thus, the following conjecture is made.

Conjecture 5.6. If the diamond geodesics in the medial graph of an annular graph, $G(r,c)$, form an outer diamond, then the annular graph is not recoverable. Furthermore, if the outer diamond is bounded by the outer boundary circle, then the annular graph is itself the smallest non-recoverable subgraph.

Conjecture 5.7. In contrast, if the graph is recoverable, the diamond geodesics never intersect again with each other or the boundary circle to form an outer diamond.

Figure 13(c) shows $G(4,2)$ which is recoverable by Theorem 5.2. This is a common pattern found in all $G(2n,n)$.

5.3. **Flowers.** A flower is a graph with no boundary spikes and no boundary-to-boundary connections. Consider the flower and its corresponding medial graph

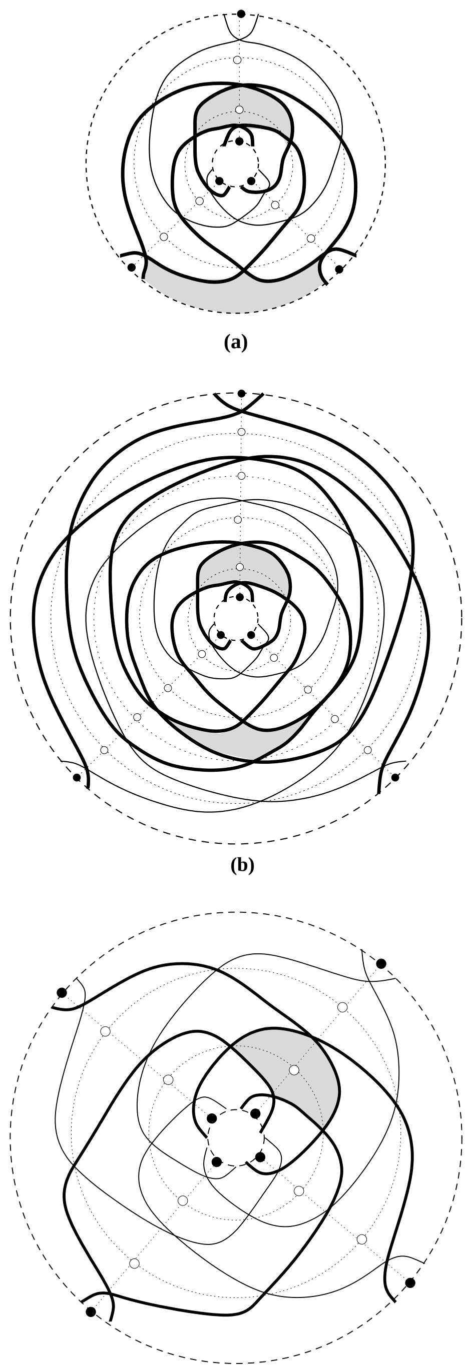

FIGURE 13. $G(3,2)$ and its medial graph (a), $G(3,4)$ and its medial graph (b), and $G(4,2)$ and its medial graph (c)

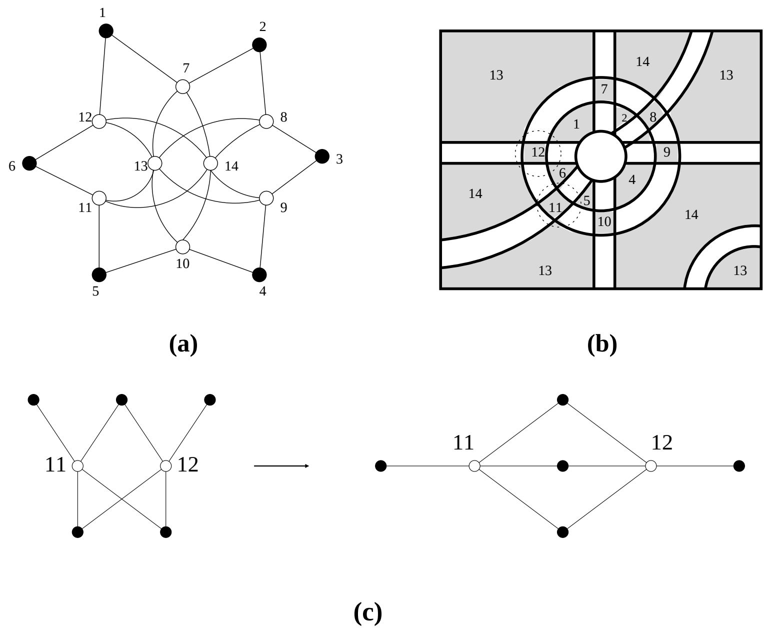

FIGURE 14. Graph of a flower (a), its corresponding medial graph on a torus (b), and a non-recoverable subgraph found from nodes 11 and 12 (c)

drawn on a torus in Figure 14(a) & (b). This graph is actually $\infty - 1$ and hence, not recoverable (see [8]). The non-recoverable subgraph corresponding to vertices 11 and 12 is shown in Figure 14(c), (see [9] for details on this specific graph). However, at this time, no pattern has been detected that makes this subgraph stand out as lenses do in the circular planar case.

6. Smallest Non-Recoverable Subgraphs

The simplest (but not so common) method for non-recoverable detection in both circular planar and non-circular planar graphs is observation. If a small non-recoverable subgraph can be spotted right away, no further computation needs

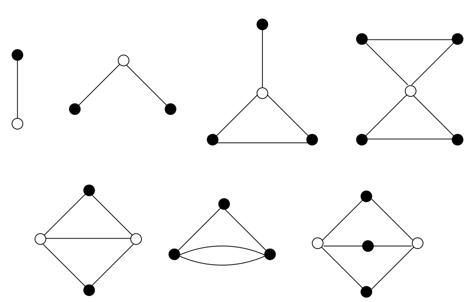

FIGURE 15. Some examples of the smallest non-recoverable subgraphs

to be done. Some examples of the smallest non-recoverable graphs are shown in Figure 15.

Corollary 6.1. *A graph composed of all recoverable subgraphs does not have to be recoverable.*

In such a case, the graph itself is the smallest non-recoverable graph. For example, the last graph in Figure 15 is not recoverable, however, the two Y subgraphs are recoverable. Nyssa Thompson spent much time this year working on the smallest recoverable graphs which is a complement of this section (see [9]).

7. Conclusion

The two sections in this paper dealing with non-circular planar graphs leave much room for further investigation. Medial graphs of this type are often hard to draw and interpret. There is also a lot of new information from Nick Reichert on medial graphs of non-circular planar graphs in his paper written in 2004 (see [8]) that would be useful for future investigation. It would also be nice to have theorems for the diamond arguments in section 5.2. At this point, however, time has run short so these ideas are left up to future researchers.

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