# Extending Harmonic Germs

Justin Robertson

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**Abstract**

In this paper we will consider when $\gamma$-harmonic functions that are defined locally can be extended to $\gamma$-harmonic functions that are defined on the whole graph.

## 1 Introduction

The notation used in this paper is consistent with the notation used in [1]. We consider resistor networks $\Gamma = (G, \gamma)$ with postive conductivities. A $\gamma$-harmonic function is a function $u$ that satisfies the following equation:

$$\sum_{q \in \mathcal{N}(p)} \gamma(p,q)[u(p) - u(q)] = 0,$$

for each interior node $p$. This means that the current flowing out of each interior node is zero.

Since we will be considering locally defined $\gamma$-harmonic functions in this paper it is useful to have the following definition:

**Definition 1.1.** Let $G = (V, E)$ be a graph, and let $f : W \to \mathbb{R}$ be a function defined on a subset $W \subset V$ of its vertices. Then for each vertex $v \in W$ such that $\mathcal{N}(v) \subset W$, we may define the *germ* of $f$ at $v$ to be the equivalence class of all functions equal to $f$ at $v$ and at all vertices in the neighborhood $\mathcal{N}(v)$. [2]

The following observation will also be useful in this paper.

**Observation 1.1.** A $\gamma$-harmonic germ, $v$, at $p$ with $n$ neighbors can be completely specified by giving the values of the potentials at the neighbors of $p$. The value at $p$ itself is determined by the averaging principle.

It is not the case that every $\gamma$-harmonic germ can be extended to a $\gamma$-harmonic function on the whole graph. To make this clear, let us consider a simple case. Let $G$ be the graph pictured in Figure 1. It has been shown in [1] that if $u$ is a $\gamma$-harmonic function, then the maximum and minimum values of $u$ occur on the boundary of $G$. Since this graph contains only one boundary node, any $\gamma$-harmonic function on this graph must be constant. But we can

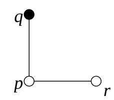

Figure 1: This graph does not extend to a global $\gamma$-harmonic function.

easily choose a $\gamma$-harmonic germ, $v$, at $p$ so that $v$ is not constant on the graph. For example, if the conductivities on the two edges in the graph are 1, then we can choose the potential at $q$ to be 1 and the potential at $r$ to be 3. Since $v$ is $\gamma$-harmonic, the potential at $p$ is 2 by the averaging principal. Thus there is a $\gamma$-harmonic germ on this graph that cannot be extended to a globally $\gamma$-harmonic function.

In this paper we will be interested in the following question. Given an arbitrary graph $G$, when can all $\gamma$-harmonic germs at an interior node $p$ be extended to a $\gamma$-harmonic function on the whole graph? When all germs at an interior node $p$ can be extended in this way, we will say that all $\gamma$-harmonic germ at $p$ can be *globally extended*.

# 2 Determining when $\gamma$-harmonic Germs Cannot be Globally Extended

To determine when $\gamma$-harmonic germs cannot be globally extended, the following terminology is useful.

**Definition 2.1.** Given a graph $G = (V, E)$ (where $V$ is the set of vertices and $E$ is the set of edges) and two disjoint sets of vertices $P, Q \subset V$ a *cutset* of the pair $P, Q$ is a set $W \subset V$ such that all paths from a vertex $p \in P$ to a vertex $q \in Q$ intersect $W$. A *minimal cutset* of the pair $P, Q$ is a cutset of $P, Q$ of minimal cardinality among all such cutsets. [3]

**Definition 2.2.** The *valence* of a germ at $p$ is the number of neighbors of $p$.

It can be shown that given any graph $G$, we can determine at which interior nodes there exists at least one $\gamma$-harmonic germ that cannot be globally extended by looking at a minimum cutset of a specific sets of vertices.

**Theorem 2.1.** *Let $G$ be a connected circular planar graph, let $P$ be the neighbors of an interior node $p$, and let $Q$ be the boundary nodes. If the cardinality of a minimum cutset from $P$ to $Q$ is less than the valence of $p$, then there exists at least one $\gamma$-harmonic germ at $p$ that cannot be globally extended.*

*Proof.* Let $G$ be a connected circular planar graph. Let $P$ be the neighbors of $p$ and let $Q$ be the boundary nodes. Let $W$ be a minimum cutset from $P$ to $Q$. Let $G'$ be the graph created by deleting all nodes that you have to go through $W$ to reach when starting at a node in $P$, and by deleting all edges connected to the deleted nodes. Define $W$ to be the boundary nodes of $G'$. Let the cardinality of $W$ be equal to $m$, let the valence of $p$ be equal to $n$, and let the total number of nodes in $G$ be equal to $N$. Assume $m < n$.

Apply Kirchoff's law at all interior nodes of $G'$, and consider the system of equations we get. We know these equations will not use any nodes outside of $G'$ because of the definition of a cutset. We also know that any interior node of $G'$ is connected to either another interior node of $G'$ or to a node in $W$. Suppose that we did not use a node in $W$, then we get a contradiction since that node would not be in $W$ by the definition of a cutset.

Let $\widetilde{K} = K(W, int'; W, int')$. Then $\widetilde{K}$ can be partitioned as follows:

$$\begin{array}{cc} & \begin{array}{cc} W & int' \end{array} \\ \begin{array}{c} W \\ int' \end{array} & \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{B}^T & \widetilde{C} \end{pmatrix} \end{array}$$

Therefore

$$\widetilde{K}u' = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{B}^T & \widetilde{C} \end{pmatrix} \begin{pmatrix} u_W \\ u_{int'} \end{pmatrix} = \begin{pmatrix} * \\ 0 \end{pmatrix}, \tag{1}$$

where $u_W$ is the column vector of the potentials of $W$ and $u_{int'}$ is the column vector of the potentials of $int'$. By the definition of $\widetilde{K}$, $C = \widetilde{C}$ or $\widetilde{C}$ is a proper principal submatrix of $C$. Since $C$ is symmetric positive definite, then $\widetilde{C}$ is symmetric positive definite. Therefore $\widetilde{C}$ is invertible. From the previous matrix equation we get

$$\widetilde{B}^T u_W + \widetilde{C} u_{int'} = 0 \tag{2}$$

Since $\widetilde{C}$ is invertible we can solve the equation to get

$$u_{int'} = -\widetilde{C}^{-1}\widetilde{B}^T u_W \tag{3}$$

Therefore, the potentials of the nodes in $W$ determine the potentials of the interior nodes of $G'$. Define $D := [-\widetilde{C}^{-1}\widetilde{B}^T](P; W)$. So $Du_W = u_P$, where $u_P$ is the column vector of the potentials of the neighbors of $p$.

Define $\phi$ on the boundary nodes of $G$. Since the Dirichlet problem has a unique solution, there exists a unique $\gamma$-harmonic function $u$ defined on $G$ with $u(q) = \phi(q)$, for $q \in \partial V$. Define $E$ to be the $m \times N$ restriction matrix such that $E\mathbf{u} = u_W$, where $\mathbf{u} = [u_1, u_2, ..., u_N]^T$. Define $M := DE$. Therefore,

$$M\mathbf{u} = DE\mathbf{u} = D(E\mathbf{u}) = D(u_W) = Du_W = u_P \tag{4}$$

So $M$ is the matrix from $\mathbb{R}^N$ to $\mathbb{R}^n$ that maps globally defined $\gamma$-harmonic functions to the values of $u$ at the neighbors of $p$. Since $M = DE$, we know

$$rankM \leq min\{rankD, rankE\}.$$

Since $D$ is an $n \times m$ matrix, rank$D \leq m$. Since $E$ is an $m \times N$ matrix, rank$E$ $\leq m$. Thus,

$$rankM \leq min\{rankD, rankE\} \leq min\{m, m\} = m \tag{5}$$

Since the rank of the $n \times N$ matrix $M$ is at most $m$, $M$ is not onto. Therefore, there exists at least one $\gamma$-harmonic germ at $p$ that does not extend globally. $\square$

# 3 Extending Germs that are Strongly Interior

In this section we will look at two different methods of extending $\gamma$-harmonic germs globally for certain germs in a graph. To talk about these specific types of germs the following definition is useful.

**Definition 3.1.** Let $G = (V, \partial V, E)$ be a graph with boundary nodes. We say that a germ at $p$ is *strongly interior* if all of the neighbors of $p$ are not boundary nodes in $G$.

Let us now look at the first method of extending $\gamma$-harmonic germs globally for germs which are strongly interior.

## 3.1 Extension Using the Kirchhoff Matrix of $G$

Let $G = (V, \partial V, E)$ be a connected circular planar graph with given conductivities, but with no potentials imposed on the nodes. We know that we can store the information of this graph in a Kirchhoff matrix, $K$. Now let us consider the germ, $v$, at an interior node $p$ which has valence $n$, and let us assume it is strongly interior.

Next, let us number the nodes of $G$ in the conventional way: boundary nodes first then interior nodes. We can now partition the Kirchhoff matrix as below.

$$\begin{array}{cc} & \begin{array}{cc} \partial & int \end{array} \\ \begin{array}{c} \partial \\ int \end{array} & \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \end{array} \tag{6}$$

In this matrix $A$ represents all of the conductivities between boundary nodes, $B$ represents all of the conductivities between boundary and interior nodes, and $C$ represents all of the conductivities between interior nodes. It was shown in [1] that if we are given a Kirchhoff matrix then we can determine the interior potentials in the following way:

**Observation 3.1.** The values of the potential $u$ at the interior nodes $s$ are:

$$u(s) = g(s) = [-C^{-1}B^T f](s), \tag{7}$$

where $f$ is a function imposed at the boundary nodes and $g$ is the resulting potential at the interior nodes.

Let us say that the graph $G$ has $r$ boundary nodes and $m$ interior nodes. Then the matrix $-C^{-1}B^T$ is an $m \times r$ matrix. Let $\psi = [u_1, u_2, ..., u_r]^T$ be the column vector in $\mathbb{R}^r$ where $u_1, u_2, ..., u_r$ are the potentials at boundary nodes $1, 2, ..., r$, respectively. Then

$$[-C^{-1}B^T]\psi = [w_1, w_2, ..., w_m]^T := \mathbf{w},$$

where $w_1, w_2, ..., w_m$ are the potentials at the $m$ interior nodes. Let us label the $n$ neighbors of $p$: $i$, $i+1$, ..., $j-1$, $j$. Therefore

$$\{[-C^{-1}B^T](i, i+1, ..., j-1, j; 1, ..., r)\}\psi = [w_i, w_{i+1}, ..., w_{j-1}, w_j]^T := \mathbf{w}'$$

So the vector $\mathbf{w}'$ holds the potentials at the $n$ neighbors of $p$. For convenience let us define $\{[-C^{-1}B^T](i, i+1, ..., j-1, j; 1, ..., r)\} := D$. We can now state the following lemma.

**Lemma 3.1.** Let $D$ be the matrix defined above. If $D$ is onto, then all $\gamma$-harmonic germs at node $p$ can be globally extended.

*Proof.* Let $D$ be the matrix defined above. Assume $D$ is onto. Then given any vector $\mathbf{w}' \in \mathbb{R}^n$ there exists at least one vector $\psi \in \mathbb{R}^r$ so that $D\psi = \mathbf{w}'$. Hence all $\gamma$-harmonic germs at $p$ can be globally extended. $\square$

## 3.2 Extension using a new graph

We now consider circular planar graphs that looks like the graph $G_1$ in Figure 2 or like the graph $G_2$ in Figure 3, where the black nodes are boundary nodes and the white nodes are interior nodes.

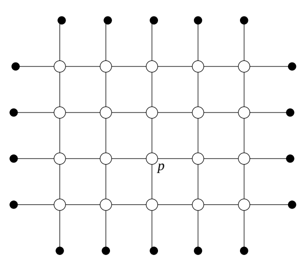


Figure 2: This is the graph $G_1$.

We want to determine if all $\gamma$-harmonic germs at an interior node $p$ which is strongly interior can be globally extended. We begin by creating a new graph in the following manner. First, delete the node $p$ and all of the edges that are connected to $p$. Then promote each of the interior nodes that were adjacent to $p$ to boundary nodes in the new graph. We will then be left with a new graph. The new graph $G_1'$ is shown in Figure 4 and the new graph $G_2'$ is shown in Figure 5.

In the new graph $G'$, let $V'$ denote the nodes in $G'$, let $\partial V'$ denote the boundary nodes in $G'$ which are the same as in $G$, let $\partial^* V'$ denote the boundary

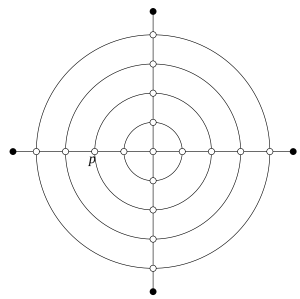


Figure 3: This is the graph $G_2$.

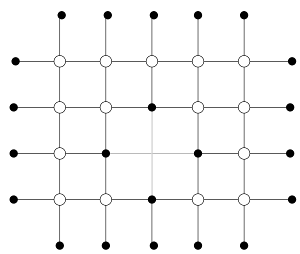

Figure 4: This is the graph $G'_1$.

nodes in $G'$ which are the newly promoted boundary nodes, and let $E'$ denote the edges of $G'$.

With this method we are given different information than in the previous method. We are given the potentials at the four nodes in $\partial^* V'$ and the current flowing out of each edge that leaves one of the four nodes in $\partial^* V'$. These currents that we know come from the equation:

$$\sum_{q \in \mathcal{N}(r_i)} \gamma(r_i, q)[u(r_i) - u(q)] = I_{r_i} = \gamma(p, r_i)[u(p) - u(r_i)], \tag{8}$$

for $1 \leq i \leq n$, where each $r_i$ is a neighbor of $p$, and $n$ is the valence of $p$.

We can partition the Kirchhoff matrix, $K'$, of $G'$ in the following way:

$$\begin{array}{c} \\ \partial V' \\ int' \\ \partial^* V' \end{array} \begin{array}{c} \begin{array}{ccc} \partial V' & int' & \partial^* V' \end{array} \\ \begin{pmatrix} A & B & C \\ B^T & D & E \\ C^T & E^T & F \end{pmatrix} \end{array} \tag{9}$$

We then have the equation:

$$\begin{pmatrix} A & B & C \\ B^T & D & E \\ C^T & E^T & F \end{pmatrix} \begin{pmatrix} X \\ Y \\ v \end{pmatrix} = \begin{pmatrix} * \\ 0 \\ J \end{pmatrix}, \tag{10}$$

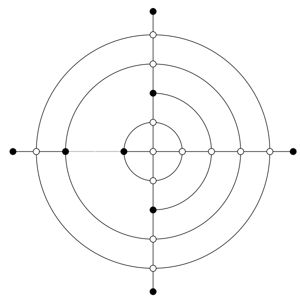

Figure 5: This is the graph $G'_2$.

where $X$ is the column vector of the potentials of $\partial V'$, $Y$ is the column vector of the potentials of the interior of $G'$, $v$ is the column vector of the potentials of $\partial^* V'$ (which are known), and $J$ is the column vector of the currents flowing out of the nodes in $\partial^* V'$ (which are known). From the previous matrix equation we get the following equations:

$$\begin{aligned} B^T X + DY + Ev &= 0 \\ C^T X + E^T Y + Fv &= J \end{aligned} \tag{11}$$

We can rewrite the these equations to get:

$$\begin{aligned} B^T X + DY &= -Ev \\ C^T X + E^T Y &= J - Fv \end{aligned} \tag{12}$$

Define $M$ to be the following submatrix of $K'$

$$M := \begin{pmatrix} B^T & D \\ C^T & E^T \end{pmatrix} \tag{13}$$

**Lemma 3.2.** Let $M$ be the matrix defined above. If $M$ is onto then all $\gamma$-harmonic germs at an interior node $p$ of $G$ that is strongly interior can be globally extended.

*Proof.* Let $M$ be the matrix defined above. Assume that $M$ is onto. Thus, given any $\gamma$-harmonic germ, $v$, at $p$ there exists values at all others nodes in $G$ so that Kirchhoff's law is satisfied at all interior nodes. Therefore all $\gamma$-harmonics germs at $p$ can be globally extended. $\square$

# 4 Circular Planar Graphs and Minimum Cutsets

In this section we will consider some properties of circular planar graphs and their minimum cutsets. It can be shown that if we look at a particular subgraph of a connected circular planar graph, then the subgraph is also circular planar.

**Theorem 4.1.** *Let $G$ be a connected circular planar graph. Let $p$ be an interior node of $G$ which is strongly interior. Let $P$ be the neighbors of $p$ and let $Q$ be the boundary nodes of $G$. Let $W$ be a minimum cutset from $P$ to $Q$. Consider all paths that start at $p$ and stop when they reach a node in $W$. Define $G'$ to be the graph with all the nodes and edges that were used in these paths and with $\partial G' = W$. Then $G'$ is circular planar.*

*Proof.* Since $G$ is circular planar we can draw a medial graph for $G$. Let $U$ be a subset of the medial graph of $G$ defined as follows. Let $U$ be the closure of the union of all black cells corresponding to nodes in $G'$ and all edge adjacent white cells. We know that all nodes in $G'$ are connected to each other through $p$, so $G'$ is connected. Since the interior of $U$ is path-connected, open, and bounded then $U$ is topologically equivalent to a circular region with a boundary, and perhaps some small holes in the larger circle, as demonstrated in Figure 6.

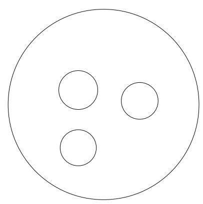

Figure 6: Large circle with small holes.

We know than all nodes in $W$ are on $\partial U$ because if there exists some $r \in W$ not on $\partial U$, then all of its neighbors would still be in $G'$ and it would not be in the minimum cutset, a contradiction. Moreover, since $W$ is a *minimum* cutset, none of the small holes in the big circle exist, since they could be deleted and we would still have a cutset. Lastly, all $q \in G' - W$ are not on $\partial U$, because if there exists an $s \in G' - W$ with $s$ on $\partial U$, then we can get to the boundary of $G$ without going through the cutset (this is demonstrated in Figure 7, where the thicker line to the right of the cell representing node $s$ is $\partial U$), a contradiction. Therefore $G'$ is circular planar.

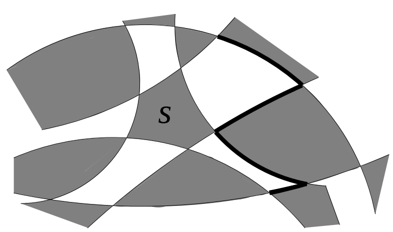


Figure 7: *Part* of the medial graph of $G$.

□

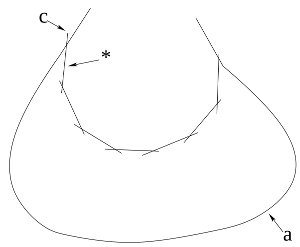


Figure 8: Part of $\mathcal{M}'$.

We can now state a result about criticality with circular planar graphs. (Note: Thanks to Owen Biesel who suggested using the medial graph to prove the following theorem.)

**Theorem 4.2.** *Let $G$ be a circular planar graph. Let $p$ be an interior node of $G$ which is strongly interior. Let $P$ be the neighbors of $p$ and let $Q$ be the boundary nodes of $G$. Let $W$ be a minimum cutset from $P$ to $Q$. Consider all paths that start at $p$ and stop when they reach a node in $W$. Define $G'$ to be the graph with all the nodes and edges that were used in these paths and with $\partial G' = W$. If the cardinality of $W$ is less than the valence of $p$, then $G'$, and hence $G$, is not critical.*

*Proof.* Since $G$ is circular planar, $G'$ is circular planar by Theorem 4.1. So we can draw a medial graph for $G'$, $\mathcal{M}'$. Denote the valence of $p$ by $n$ and the cardinality of $W$ by $m$. We assumed $m < n$. So there is an n-star at $p$, and the black cell in $\mathcal{M}'$ representing $p$ is an n-gon. Since only $m$ geodesics touch the boundary of $\mathcal{M}'$ (because $\partial G' = W$), either at least one of the edges of the n-gon loops back to itself or at least two of the edges are part of the same geodesic. So in the first case, if one of the edges loops back to itself, then we get a lens in $\mathcal{M}'$. Now let us consider the second case. If the two edges that are part of the same geodesic are adjacent then we get an immediately obvious lens. So let us assume the two edges that are part of the same geodesic are not adjacent. Then the part of $\mathcal{M}'$ that we are interested in looks something like Figure 8. Let us now look at one of the two edges, the one marked with a $*$ in Figure 8, that are adjacent to the geodesic labeled $a$. Since all intersections of the n-gon have the edges going *into* the region bounded by $a$, since geodesics must cross traversally at intersections, and since $*$ has to end at the boundary of $\mathcal{M}'$ or loop around to $c$, it must eventually cross the geodesic $a$, by the Jordan Curve Theorem, making a lens. Thus in all cases we get a lens in $\mathcal{M}'$. So $G'$, and hence $G$, is not critical.

□

The following conjecture has been refuted with a counterexample.

**Conjecture 4.1.** Let $G$ be a circular planar graph. Let $p$ be an interior node which is strongly interior. Let $P$ be the neighbors of $p$ and let $Q$ be the boundary

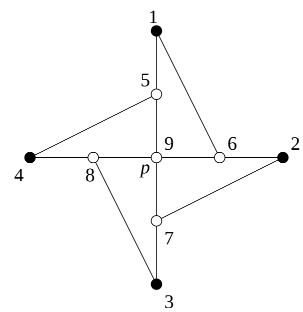


Figure 9: The Chinese star graph.

nodes of $G$. Let the cardinality of a minimum cutset from $P$ to $Q$ be equal to the valence of $p$. Then all $\gamma$-harmonic germs at $p$ can be globally extended.

Due to its shape, we call the counterexample graph the Chinese star. The Chinese star graph is shown in Figure 9.

Let all of the conductivities of the Chinese star be equal to 1. Let the potentials at nodes 5, 6, 7, and 8 be 7, -6, 5, and 6, respectively. Then taking the $\gamma$-average of the neighbors of node 9 (which is the same as node $p$), the value at node 9 is

$$\frac{7-6+5+6}{4} = \frac{12}{4} = 3 \tag{14}$$

We now have a $\gamma$-harmonic germ at node 9. Next, let the potentials at nodes 1, 2, 3, and 4 be $x$, $y$, $z$, and $w$, respectively. We then need the following system of equations to have a solution in order for this germ to be extended to a $\gamma$-harmonic function on the whole graph.

$$x + y + 3 = -6 \times 3 = -18 \tag{15}$$

$$y + z + 3 = 5 \times 3 = 15 \tag{16}$$

$$z + w + 3 = 6 \times 3 = 18 \tag{17}$$

$$w + x + 3 = 7 \times 3 = 21 \tag{18}$$

These equations can be rewritten as

$$x + y = -21 \tag{19}$$

$$y + z = 12 \tag{20}$$

$$z + w = 15 \tag{21}$$

$$w + x = 18 \tag{22}$$

Substituting the value of $y$ from equation 20 into equation 19 gives $z = x + 33$. Then substituting this value of $z$ into equation 21 gives $w + x = -18$. But equation 22 gives $w + x = 18$, so this system of equations has no solution. Thus, not all $\gamma$-harmonic germs at the strongly interior node $p$ of the Chinese star can be globally extended even though the cardinality of a minimum cutset from $P$ to $Q$ is equal to the valence of $p$. Thus, we have a counterexample to Conjecture 4.1.

The previous counterexample was not found arbitrarily, and it is instructive to look at how it was found.

Let $G$ be a circular planar graph with $r$ boundary nodes and $m$ interior nodes. Let $p$ be an interior node which is strongly interior and with valence $n$. Let $P$ be the neighbors of $p$ and let $Q$ be the boundary nodes of $G$. Let $W$ be a minimum cutset from $P$ to $Q$ with cardinality equal to $n$. Now let us partition the Kirchhoff matrix, $K$, for the network $\Gamma = (G, \gamma)$ as below:

$$\begin{array}{cc} & \begin{array}{cc} \partial & int \end{array} \\ \begin{array}{c} \partial \\ int \end{array} & \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \end{array} \tag{23}$$

We know from Observation 3.1 that $-C^{-1}B^T$ is the $r \times m$ matrix that takes boundary potentials to interior potentials. Define $D := -C^{-1}(P; 1, ..., m)$. So the matrix $DB^T$ is the $n \times r$ matrix that takes boundary potentials to the potentials at the neighbors of $p$. So if $DB^T$ is onto, then all $\gamma$-harmonic germs at $p$ can be globally extended. Since $-C^{-1}$ has full rank and $D$ is a submatrix of $-C^{-1}$, then $D$ has full rank. So if the rank of $B^T$ is less than $n$ we will have a counterexample for Conjecture 4.1.

Next, let us consider the following theorem, Menger's Theorem [3].

**Theorem 4.3.** *Let $G = (V, E)$ be a graph and $A,B \subseteq V$. Then the cardinality of the minimum cutset of the pair $A$,$B$ in $G$ is equal to the maximum number of disjoint paths from $A$ to $B$ in $G$.*

Therefore since the cardinality of $W$ is equal to $n$, the valence of $p$, we have $n$ disjoint paths from $p$ to $G$.

To simplify things, let us suppose $n = 4$ and the conductivity for each edge is 1. So we have 4 disjoint paths to the boundary. Let us label the 4 boundary nodes these paths end at 1, 2, 3, and 4. So our situation looks something like Figure 10.

In Figure 10 nodes $i$, $i+1$, $i+2$, and $i+3$ are the interior nodes on the 4 disjoint paths that are adjacent to nodes 1, 2, 3, and 4, respectively. Consider the submatrix of $B^T$, $E := B^T(i, i+1, i+2, i+3; 1, 2, 3, 4)$. Let us see if it has rank 4. So far we know $E =$

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} i \\ i+1 \\ i+2 \\ i+3 \end{array} & \begin{pmatrix} -1 & ? & 0 & ? \\ ? & -1 & ? & 0 \\ 0 & ? & -1 & ? \\ ? & 0 & ? & -1 \end{pmatrix} \end{array}, \tag{24}$$

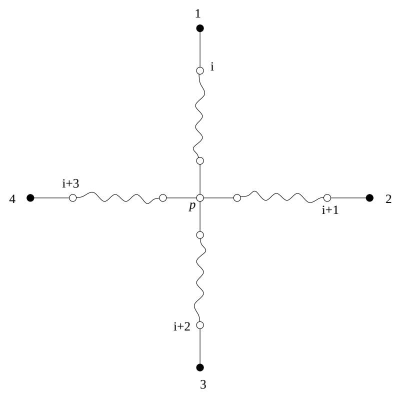


Figure 10: The $n = 4$ case.

where the 4 zeros appear because $G$ must be circular planar, so these edges cannot exist.

Now suppose we have an edge from 1 to $i+1$. Since the graph is planar there can be no edge from 2 to $i$. So we now have $E =$

$$\begin{array}{c} \\ i \\ i+1 \\ i+2 \\ i+3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & ? \\ -1 & -1 & ? & 0 \\ 0 & ? & -1 & ? \\ ? & 0 & ? & -1 \end{pmatrix} \end{array} \tag{25}$$

Next suppose we have an edge from 2 to $i+2$. Since the graph is planar there can be no edge from 3 to $i+1$. So we now have $E =$

$$\begin{array}{c} \\ i \\ i+1 \\ i+2 \\ i+3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & ? \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & ? \\ ? & 0 & ? & -1 \end{pmatrix} \end{array} \tag{26}$$

Now suppose we have an edge from 3 to $i+3$. Since the graph is planar there can be no edge from 4 to $i+2$. So we now have $E =$

$$\begin{array}{c} \\ i \\ i+1 \\ i+2 \\ i+3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & ? \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ ? & 0 & -1 & -1 \end{pmatrix} \end{array} \tag{27}$$

Lastly, suppose there is an edge from 4 to $i$. Since the graph is planar there can be no edge from 1 to $i+3$. So we now have $E =$

$$\begin{array}{c} \\ i \\ i+1 \\ i+2 \\ i+3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix} \end{array} \tag{28}$$

If we put $E$ into row reduced echelon form we have $E =$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{29}$$

Thus $E$ has rank 3. So if we assume the graph from Figure 10 has only 1, 2, 3, and 4 as boundary nodes and only $i$, $i+1$, $i+2$, $i+3$, and $p$ as interior nodes (which implies the graph is the Chinese star), then $B^T =$

$$\begin{array}{c} \\ i \\ i+1 \\ i+2 \\ i+3 \\ p \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{30}$$

So the rank of $B^T$ is 3. Thus $DB^T$ is not onto. Therefore not all $\gamma$-harmonic germs at the strongly interior node $p$ can be globally extended.

Even though Conjecture 4.1 has been refuted, it may still be true with one condition added, namely letting $G$ be a *critical* circular planar graph. But it has not been shown whether this would make Conjecture 4.1 true or false.

# 5 Future Research

There are still several questions relating to extending $\gamma$-harmonic germs that have not been answered.

## 5.1 Germs and Criticality

In at the end of section 4, it was asked if the following conjecture is true.

**Conjecture 5.1.** Let $G$ be a *critical* circular planar graph. Let $p$ be an interior node which is strongly interior. Let $P$ be the neighbors of $p$ and let $Q$ be the boundary nodes of $G$. Then all $\gamma$-harmonic germs at $p$ can be globally extended.

The counterexample, the Chinese star graph, is not critical; so, perhaps, Conjecture 5.1 which requires that $G$ be critical will make the conjecture true.

## 5.2 Extending Germs at Weakly Interior Nodes

In section 3 we looked at when germs which are strongly interior can be globally extended, but we did not consider the other case.

**Definition 5.1.** Let $G = (V, \partial V, E)$ be a graph with boundary nodes. We say that a germ at $p$ is *weakly interior* if at least one of the neighbors of $p$ is a boundary node.

When a node is weakly interior it is more difficult to determine when all $\gamma$-harmonic germs at that node can be globally extended. Consider the method that we used in section 3.1 where we could see if all germs at an interior node $p$ could be globally extended by checking if the matrix $D$ is onto. When we have a weakly interior node the problem is that at least one of the boundary potentials will already be determined when we choose a particular germ. Hence, we can not freely choose whatever boundary values that we want as we could with a strongly interior node.

Now consider the method that we used in section 3.2 where we looked at the Kirchhoff matrix of the new graph $G'$. The difficultly that we get when we look at a $\gamma$-harmonic germ at a weakly interior node $p$ arises with how we partition the Kirchhoff matrix $K'$. With strongly interior nodes we were able to partition $K'$ into $\partial V'$, $int'$, and $\partial^* V'$, but with weakly interior nodes there is at least on node $s$ with $s \in \partial V' \cap \partial^* V'$. This leaves the question of where to put the row (and column) of $K'$ that correspond to node $s$. Do we put it with just $\partial V'$, with just $\partial^* V'$, or with both $\partial V'$ and $\partial^* V'$; or do we need to use an altogether different method?

## 5.3 The Relationship Between $D$ and $M$

In sections 3.1 and 3.2 we looked at the matrices $D$ and $M$, respectively, to see if all $\gamma$-harmonic germs at a strongly interior node $p$ could be globally extended. If $D$ or $M$ is onto then all $\gamma$-harmonic germs at $p$ can be globally extended. But $D$ and $M$ are different matrices with, in general, different dimensions. One important difference is that all $\gamma$-harmonic germs at $p$ can be globally extended *if and only if* $D$ is onto; but if $M$ is not onto it is still possible that all $\gamma$-harmonic germs at $p$ could be globally extended. It would be interesting to look into how these two matrices are related.

## 5.4 Extending Harmonic Globs

In the paper we considered when we could extend $\gamma$-harmonic germs, which are defined at a node $p$ and its neighbors. But what about considering functions that are defined on a larger neighborhood.

**Definition 5.2.** Let $G = (V, E)$ be a graph, and let $f : W \to \mathbb{R}$ be a function defined on a subset $W \subset V$ of its vertices. Let $S \subset V$ such that each $s \in S$ is adjacent to at least one other $r \in S$. Then for each such $S \subset V$ such that

$\mathcal{N}(S) \subset W$, we may define the *glob* of $f$ on $S$ to be the equivalence class of all functions equal to $f$ on $S$ and at all vertices in the neighborhood $\mathcal{N}(S)$.

It would be interesting to see *how far* $\gamma$-harmonic globs can be $\gamma$-harmonically extended on a graph $G$.

### 5.5 The Green's Function

Let $G$ be a connected circular planar graph. Let $\Gamma = (G, \gamma)$ be a resistor network. Let $p$ be a strongly interior node of $G$. Let $v$ be a germ at $p$ so that the net current flow out of $p$ is 1. Assume that we can extend $v$ to a function which is $\gamma$-harmonic everywhere on $G$ except at node $p$ where the net current flow out is 1. Now, let us define $u$ to be the $\gamma$-harmonic function with $u = v$ on $\partial G$. We know this function $u$ exists because there is a unique solution to the Dirichlet problem.

Next, we define another function, $w$, on $G$ so that $w := v - u$. So $w = 0$ on $\partial G$ and is $\gamma$-harmonic at all interior nodes except at $p$, where the current flow out is 1. Therefore $w$ is the Green's function. So $w$ equals the $p^{th}$ column of $C^{-1}$. So whenever we can extend the germ $v$ to a $\gamma$-harmonic function, we can then find the Green's function.

## References


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