PARAMETERIZING RESPONSE MATRICES

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ABSTRACT. In this paper we look at the relationships of entries in the response matrix and determine which entries can be written in terms of the others. This information can be used to recover graphs with only partial information in the response. We have added to the results of Curtis and Morrow in [1], identifying parameter patterns for two more classes of graphs and investigating alternate ways of determining parameter placements.

1. INTRODUCTION

The entries of a Dirichlet to Neumann map for a resistor network have various relationships which depend on the structure of the network. Analysis of these relationships allows us to determine which entries are necessary for recoverability. Knowing this we can recover graphs with only partial information in the response matrix. The Calderon problem with partial data has been studied in the continuous case [2] and in the discrete case [1]. Edward B. Curtis and James A. Morrow generalized the parameterization of square lattice networks in *The Dirichlet to Neumann Map For A Resistor Network*[1]. The Dirichlet to Neumann map for a square lattice is represented by a $4n \times 4n$ matrix with nodes numbered as in figure 1; 1 through n on the north side, $n+1$ through $2n$ on the west side, $2n+1$ through $3n$ on the south side, and $3n+1$ through $4n$ on the east side. The block structure

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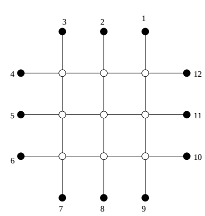

FIGURE 1. Numbering of nodes for a square lattice with $n = 3$

G	A	B	C
A^T	H	E	D
B^T	E^T	I	F
C^T	D^T	F^T	J

FIGURE 2. Block Structure of A Square Lattice Response Matrix

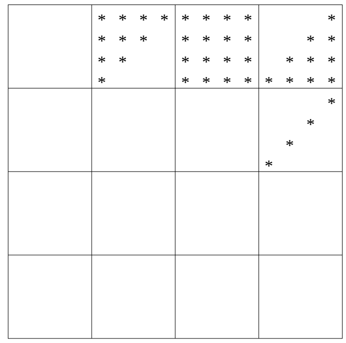

FIGURE 3. Parameter Positioning for Square Lattices

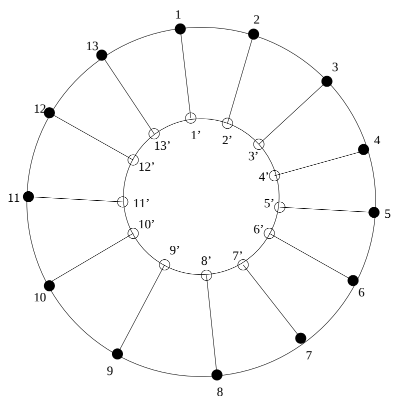

FIGURE 4. A $4n + 1$ spoked wheel with $n = 3$

for square lattices is shown in figure 2–the blocks correspond to the four sides of the graph. One possible parameterization of the response matrix is the following:

1. All the entries of B.
2. All the entries of A on or above the main antidiagonal.
3. All the entries of C on or below the main antidiagonal.
4. All the entries of D on the main antidiagonal.

Figure 3 shows the general positions of these parameters in the response matrix for any square lattice. This characterization of parameter positioning was determined by Curtis and Morrow using determinental relationships of response entries. The proof for this parameterization can be found in [1]. Similar methods can be applied to the parameterization of other graphs. We begin with graphs which have symmetrical properties like the square lattice and attempt to identify symmetrical parameter placements for these graphs.

2. THE $4n + 1$ SPOKED WHEEL

Consider the graph with $4n + 1$ rays between two circles with boundary nodes on the outer circle and interior nodes on the inner circle shown in figure 4. There are $3(4n+1)$ edges and thus $3(4n+1)$ conductivities to recover, and there are $(4n+1)^2$ entries in the response matrix. We should be able to find $(4n + 1)^2 - 3(4n + 1)$

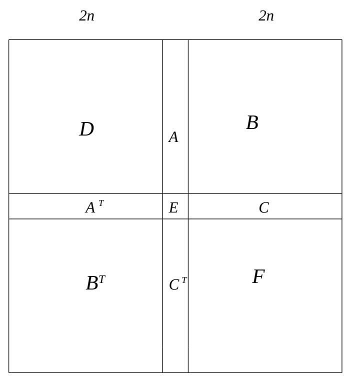

FIGURE 5. Block Structure for the $4n + 1$ Spoked Wheel

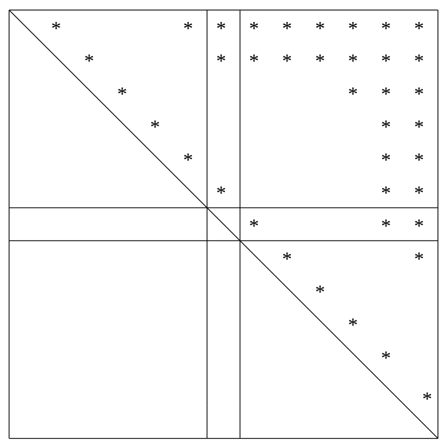

FIGURE 6. Parameter Positioning for Recoverable Wheels

entries which are determined by the rest of the entries in the matrix. The block structure for the $4n + 1$ spoked wheel is shown in figure 5 where D, B, F and B^T are $2n \times 2n$ matrices, A and C^T are column vectors, A^T and C are row vectors, and E is a single entry of the response.

2.1. **Parameters for the response matrix of the $4n + 1$ spoked wheel.**

1. The first two rows and last two columns of B and the entry in the third row and the $(2n - 2)^{th}$ column of B.
2. The first two entries and the last entry of the column vector A.
3. The first entry and the last two entries of the row vector C.
4. The entries directly above the main diagonal of D and the last entry of the first row of D.
5. The entries directly above the main diagonal of F and the last entry of the first row of F.

The positioning of these parameters is illustrated in figure 6.

Claim 2.1. The values of the $3(4n+1)$ parameters of the response matrix for a $4n+1$ spoked wheel determine uniquely the remaining entries of the response matrix.

Lemma 2.2. *There is a two-connection between circular pairs on all wheel graphs.*

Proof. Let the nodes of any $4n+1$ spoked wheel be numbered as in figure 4 with 1 through $4n+1$ boundary nodes and $1'$ through $(4n+1)'$ the corresponding interior nodes. Consider the two connection between the sets of boundary nodes $i, j \in P$ and $k, l \in Q$. Assume $(P;Q)$ is a circular pair, and without loss of generality, let the ordering of the nodes be (i, j, k, l). Any path from the nodes i, j, k and l must travel through their corresponding interior nodes i', j', k' and l' respectively. Consider the path exiting node i through i'. There are two directions available on the graph at the node i'; toward l' and toward j'. Similarly, there are two directions available on the graph at the node j'; toward k' and toward i'. If the path from i' travels toward l' and the path from j' travels toward k', the two paths don't intersect. So there exists a two-connection between P and Q. □

Theorem 2.3. *Suppose* $\Gamma = (G, \gamma)$ *is a circular planar resistor network and* $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ *is a circular pair of sequences of boundary nodes.*
(a) If (P;Q) are not connected through G, then $det\Lambda(P;Q) = 0$.
(b) If (P;Q) are connected through G, then $(-1)^k det\Lambda(P;Q) > 0$.

The proof of this theorem is found in [3]. By Lemma 2.2 and Theorem 2.3, any determinant of $\Lambda(P;Q)$ where $(P;Q)$ is a circular pair with exactly two boundary nodes on the graph, is not equal to zero.

Lemma 2.4. *Let* $(P;Q) = (p_1, p_2, p_3; q_1, q_2, q_3)$ *be a circular pair on a spoked wheel graph. If* $det(\Lambda(p_1, p_2, p_3; q_1, q_2, q_3)) = 0$, *then one of the entries of this* 3×3 *submatrix can be determined in terms of the other eight.*

Proof. Let $i, j, k \in P$ and $l, m, n \in Q$ be two sets of boundary nodes on a spoked wheel graph and let the determinant of $\Lambda(P;Q)$ be equal to zero. Then

$$det\Lambda(P;Q) = \lambda_{il}(\lambda_{jm}\lambda_{kn} - \lambda_{km}\lambda_{jn}) - \lambda_{jl}(\lambda_{im}\lambda_{kn} - \lambda_{km}\lambda_{in}) + \lambda_{kl}(\lambda_{im}\lambda_{jn} - \lambda_{jm}\lambda_{in}) = 0 \tag{1}$$

Now we can solve this equation for any of the entries in $\Lambda(P;Q)$, for example:

$$\lambda_{jm} = \frac{\lambda_{jl}(\lambda_{im}\lambda_{kn} - \lambda_{km}\lambda_{in}) + \lambda_{il}\lambda_{km}\lambda_{jn}}{\lambda_{il}\lambda_{kn} - \lambda_{kl}\lambda_{in}} \tag{2}$$

The denominator in this example is the determinant of the submatrix $\Lambda(i, k; l, n)$. Since $(P;Q)$ is a circular pair, $(p_i, p_k; q_l, q_n)$ is a circular pair for any i, j, k and l. It can be seen that solving for any entry will always give a fraction with the determinant of a 2×2 submatrix in the denominator. By Lemma 2.2 and Theorem 2.3, this denominator can never be equal to zero and so the equation is always defined. □

Definition 2.5. Let i and j be boundary nodes on a $4n+1$ spoked wheel graph with vertices numbered as in figure 4, and without loss of generality, let $i < j$. i is a *neighbor* of j if $i + 1 = j$ or if $i = 1$ and $j = (4n+1)$.

Lemma 2.6. *Let* $(P;Q) = (p_1, p_2, p_3; q_1, q_2, q_3)$ *be a circular pair of sequences of three boundary nodes of a* $4n+1$ *spoked wheel graph. If none of the nodes in* P *is a neighbor of any of the nodes in* Q *then there exists no three connection between* P *and* Q.

	1	2	3	4	5	6	7	8	9
1		29	27	23	20	16	1	2	3
2	29		30	25	21	17	4	5	6
3	27	30		31	22	18	7	8	9
4	23	25	31		32	19	12	10	11
5	20	21	22	32		33	15	13	14
6	16	17	18	19	33		34	26	24
7	1	4	7	12	15	34		35	28
8	2	5	8	10	13	26	35		36
9	3	6	9	11	14	24	28	36	

FIGURE 7. Order of parameter placement for the response of a 9-spoked wheel

Proof. Consider the following ordering for P and Q on the wheel graph: $(p_1, p_2, p_3, q_1, q_2, q_3)$. Looking at the structure of the graph as shown in figure 4, we can see that connections from any of these boundary nodes p_i, q_i must go through their corresponding interior nodes, p_i' or q_i' respectively. Any path beginning at p_2 must go through either p_1' or p_3' before reaching any node in the set Q, blocking any paths from either p_1 or p_3. Therefore, there is no three connection from P to Q. $\square$

By Theorem 2.3, these particular 3×3 submatrices of Λ have zero determinant. Now, given the parameterization in figure 6, we can determine the remaining entries of Λ by using eight parameters or a combination of parameters and determined entries to calculate the ninth entry of the 3×3 submatrix which satisfies the conditions of Lemma 2.6. Since Λ is symmetric ($\lambda_{ij} = \lambda_{ji}$) and the column sums and row sums are equal to zero, the remaining entries of Λ can be calculated, proving Claim 2.1.

2.2. **Determining Parameters.** The following example shows how the parameters for the $4n + 1$ spoked wheel were determined. The entries in the response matrix for the 9-spoked wheel are numbered in figure 7, and these numberings correspond to the order in which the parameters were chosen and in which the other entries were determined. Whenever a parameter is added, there are no more entries that can be determined in terms of the others, showing that each parameter is necessary.

1-11. parameters
12. $\Lambda(2, 3, 4; 7, 8, 9)$
13, 14. parameters
15. $\Lambda(3, 4, 5; 7, 8, 9)$
16, 17. parameters
18. $\Lambda(1, 2, 3; 6, 7, 8)$
19. $\Lambda(2, 3, 4; 6, 7, 8)$
20, 21. parameters
22.$\Lambda(1, 2, 3; 5, 6, 7)$
23, 24. parameters
25. $\Lambda(4, 5, 6; 1, 2, 9)$
26. $\Lambda(4, 5, 6; 1, 8, 9)$
27. $\Lambda(3, 4, 5; 1, 8, 9)$
28. $\Lambda(5, 6, 7; 1, 8, 9)$
29-36. parameters

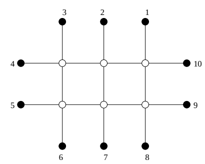

FIGURE 8. $n \times (n+1)$ lattice with $n = 2$

	$n+1$	n	$n+1$	n
$n+1$	G	A	B	C
n	A^T	H	F	D
$n+1$	B^T	F^T	I	E
n	C^T	D^T	E^T	J

FIGURE 9. Block structure for the $n \times (n+1)$ lattice

3. THE $n \times (n+1)$ RECTANGULAR LATTICE

Rectangular lattices, as shown in figure 8 have less symmetry than square lattices or spoked wheels and so the patterns in the determinental relations of the response matrix are harder to identify. There are $2(n+1)^2 - 1$ edges and thus $2(n+1)^2 - 1$ conductivities to recover, and there are $[2(2n+1)]^2$ entries in the response matrix. We should be able to find $[2(2n+1)]^2 - 2(n+1)^2 + 1$ entries which are determined by the rest of the entries in the matrix. The block structure for the $n \times (n+1)$ lattice is shown in figure 9; the boundary nodes are grouped according to the four sides of the lattice as with the square lattice.

3.1. **Parameters for the response matrix of the $n \times (n+1)$ lattice.**

1. All of the entries in B.
2. The first $n-1$ entries of the last column of G.
3. The last $n-1$ entries of the first column of D.
4. In the top $n \times n$ submatrix of A, all of the entries on and above the main antidiagonal except for the middle $n-2$ entries in the first column.
5. In the top $n \times n$ submatrix of C, all of the entries on and above the main diagonal except for the last entry in the first row, plus the last entry in the last column of C.

The positioning of these parameters is illustrated in figure 10.

Claim 3.1. The values of the $2(n+1)^2 - 1$ parameters of the response matrix for an $n \times (n+1)$ lattice determine uniquely the remaining entries of the response matrix.

Lemma 3.2. *Let $(P;Q) = (p_1, p_2; q_1, q_2)$ be a circular pair of boundary nodes of an $n \times (n+1)$ lattice and assume $n > 1$. Then there is no two connection between P and Q if and only if the nodes in P or Q form a boundary antenna.*

Proof. First assume that p_1 and p_2 form a boundary antenna. There is clearly no two connection between P and Q because paths from p_1 and p_2 must always travel

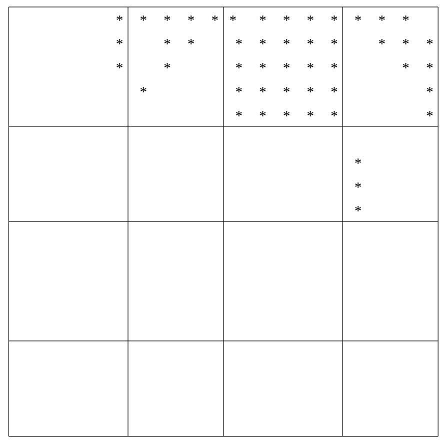

FIGURE 10. Parameter Positioning for $n \times (n+1)$ lattices

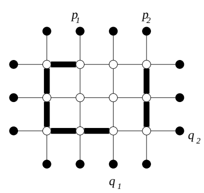

FIGURE 11. Connection travelling along the interior rectangle

through the same interior node. Now we wish to show that if there is no two connection between P and Q then the nodes in either P or Q form a boundary antenna. It suffices to prove the converse. Assume that the nodes of P and Q do not form boundary antennas. We wish to show that there is a two-connection between P and Q. Consider the interior nodes which are connected to boundary nodes in the graph. Since $n > 1$, this set of nodes forms a closed rectangle within the graph. So we contract the boundary spikes and consider only this interior rectangle. Each of the nodes in P is connected to a distinct interior node and each of the nodes in Q is connected to a distinct interior node. Since $(P;Q)$ is a circular pair, we can connect the corresponding interior nodes of P and Q along the boundary of this interior rectangle (see figure 11) and the two paths will not intersect. Also, none of the edges of these paths is used more than once since $n > 1$. Therefore when $n > 1$ and neither P nor Q contains a boundary antenna, there is a two connection between P and Q. □

We now have a class of 2×2 submatrices of Λ which have determinant zero. These can be used to calculate one entry from three known entries. We also have a class of 2×2 submatrices of Λ which have determinant not equal to zero, and so any 3×3 determinant which equals zero and has no 2×2 submatrix with determinant zero can be used to calculate one entry in terms of eight others as with the spoked wheel.

	1	2	3	4	5	6	7	8	9	10
1		15	11	9	5	6	33	1	2	45
2	15		12	10	8	7	39	3	4	13
3	11	12		40	26	25	35	21	22	14
4	9	10	40		28	27	36	23	24	16
5	5	8	26	28		41	37	30	29	17
6	6	7	25	27	41		38	32	31	18
7	33	39	35	36	37	38		42	43	34
8	1	3	21	23	30	32	42		44	19
9	2	4	22	24	29	31	43	44		20
10	45	13	14	16	17	18	34	19	20	

FIGURE 12. Order of parameter placement for the response of the 2×3 lattice

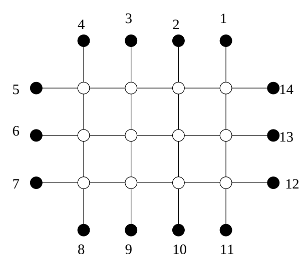

FIGURE 13. The 3×4 lattice

3.2. **The 2×3 lattice.** Using only the information from Lemma 3.2 and the consequence of Theorem 2.3, we can come up with placements for the $2(n+1)^2 - 1$ parameters in the response matrix for the 2×3 lattice. The numbering of the entries of Λ as shown in figure 12 corresponds to the following placement of parameters and determined entries. As with the 9-spoked wheel, each parameter is added only when no more entries can be determined.

1-3. parameters
4. $\Lambda(1, 2; 8, 9)$
5-7. parameters
8. $\Lambda(1, 2; 5, 6)$
9-11. parameters
12. $\Lambda(1, 2; 3, 4)$
13,14. parameters
15. $\Lambda(2, 3; 1, 10)$
16. $\Lambda(3, 4; 1, 10)$
17. $\Lambda(4, 5; 1, 10)$
18. $\Lambda(5, 6; 1, 10)$
19. $\Lambda(6, 8; 1, 10)$
20. $\Lambda(8, 9; 1, 10)$
21. parameter
22. $\Lambda(2, 3; 8, 9)$
23. $\Lambda(3, 4; 1, 8)$
24. $\Lambda(3, 4; 8, 9)$
25. parameter
26. $\Lambda(2, 3; 5, 6)$
27. $\Lambda(3, 4; 1, 6)$
28. $\Lambda(3, 4; 5, 6)$
29. parameter
30. $\Lambda(3, 5; 8, 9)$
31. $\Lambda(5, 6; 9, 10)$
32. $\Lambda(5, 6; 8, 9)$
33. parameter
34. $\Lambda(7, 8; 1, 10)$
35. parameter
36. $\Lambda(3, 4; 6, 7)$
37. $\Lambda(1, 2, 5; 7, 9, 10)$
38. $\Lambda(5, 6; 7, 8)$
39. parameter
40. $\Lambda(1, 2, 3; 4, 5, 7)$
41. $\Lambda(6, 7, 8; 2, 4, 5)$
42. $\Lambda(4, 6, 7; 8, 10, 2)$
43. $\Lambda(6, 7; 8, 9)$
44. $\Lambda(6, 7, 8; 9, 10, 2)$
45. $\Lambda(1, 2, 3; 7, 9, 10)$

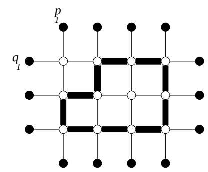

FIGURE 14.

3.3. **The 3×4 lattice.** As n increases, the information from Lemma 3.2 is no longer enough to calculate all of the entries in the response of $n \times (n+1)$ lattices because higher connections come into play, and when $n > 2$ there are more 3×3 connections possible.

Lemma 3.3. *Let $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ be sets of boundary nodes which do not include a boundary antenna on an $n \times (n+1)$ lattice and let $(P; Q)$ be a circular pair. If $n > 2$ and p_1 and q_1 form a boundary antenna or p_3 and q_3 form a boundary antenna, then there exists a three connection between P and Q.*

Proof. Assume p_1 and q_1 form a boundary antenna. Connecting these two nodes, we use only one interior node. Recall the interior rectangle described in Lemma 3.2. The interior node used to connect p_1 and q_1 is on one corner of this interior rectangle. So we alter the interior rectangle so that its boundary travels around this corner node as in figure 14, forming another closed path. By the same reasoning as in Lemma 3.2, the nodes p_2, p_3, q_2 and q_3 can be connected through disjoint paths around this closed path. None of the edges of these paths is travelled more than once since $n > 2$ and since the closed path does not involve the interior node travelled by the connection from p_1 to q_1, there exists a three connection between P and Q. □

As a consequence of this lemma, we cannot determine the entries $\lambda_{4,5}, \lambda_{7,8}, \lambda_{11,12}$ or $\lambda_{1,14}$ using 3×3 submatrices. We must look at four-connections to determine these entries. And to make use of any 4×4 submatrix with zero determinant, we must be sure that there is no 3×3 or 2×2 submatrix with zero determinant. The following process calculates the entries of Λ for a 4×3 lattice (refer to figure 15 for the ordering of calculation):

1.$\Lambda(1,2;11,12)$
2.$\Lambda(2,3;11,12)$
3.$\Lambda(3,4;11,12)$
4.$\Lambda(1,2;7,8)$
5.$\Lambda(2,3;7,8)$
6.$\Lambda(3,4;7,8)$
7.$\Lambda(1,2;4,5)$
8.$\Lambda(2,3;4,5)$
9.$\Lambda(3,4;1,14)$
10.$\Lambda(2,3;1,14)$
11.$\Lambda(1,2,4;10,12,13)$
12.$\Lambda(1,2,3;10,12,13)$
13.$\Lambda(1,2,3;6,8,9)$
14.$\Lambda(2,3,4;6,7,9)$
15.$\Lambda(6,7,9;12,1,2)$
16.$\Lambda(6,8,9;12,1,2)$
17.$\Lambda(4,5;1,14)$
18.$\Lambda(5,6;1,14)$
19.$\Lambda(6,7;1,14)$
20.$\Lambda(7,8;1,14)$
21.$\Lambda(8,9;1,14)$
22.$\Lambda(9,10;1,14)$
23.$\Lambda(10,11;1,14)$
24.$\Lambda(11,12;1,14)$
25.$\Lambda(12,13;1,14)$
26.$\Lambda(12,14,2;3,4,6)$
27.$\Lambda(1,2,3,4;5,7,9,10)$
28.$\Lambda(1,2,3,4;9,10,12,14)$
29.$\Lambda(3,5,6;12,14,2)$
30.$\Lambda(4,5;11,12)$
31.$\Lambda(5,6;11,12)$
32.$\Lambda(6,7;11,12)$
33.$\Lambda(7,8;11,12)$
34.$\Lambda(8,9;11,12)$
35.$\Lambda(4,5;2,6)$
36.$\Lambda(4,5;6,7)$
37.$\Lambda(4,5;7,8)$
38.$\Lambda(4,5;8,9)$
39.$\Lambda(4,5;9,10)$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1		10	9	*	*	*	*	*	*	*	*	*	*	28
2	10		26	*	7	*	4	*	*	*	*	1	*	*
3	9	26		8	*	13	5	*	*	*	*	2	12	*
4	*	*	8		27	14	6	*	*	*	*	3	11	*
5	*	7	*	27		35	36	37	38	39	30	29	40	17
6	*	*	13	14	35		45	44	43	42	31	*	41	18
7	*	4	5	6	36	45		59	57	49	32	*	46	19
8	*	*	*	*	37	44	59		56	50	33	16	47	20
9	*	*	*	*	38	43	57	56		58	34	15	48	21
10	*	*	*	*	39	42	49	50	58		54	55	51	22
11	*	*	*	*	30	31	32	33	34	54		60	52	23
12	*	1	2	3	29	*	*	16	15	55	60		53	24
13	*	*	12	11	40	41	46	47	48	51	52	53		25
14	28	*	*	*	17	18	19	20	21	22	23	24	25	

FIGURE 15. Order of calculation of Λ for the 3×4 lattice

40.$\Lambda(4, 5; 12, 13)$
41.$\Lambda(3, 5, 6; 12, 13, 14)$
42.$\Lambda(3, 5, 6; 10, 13, 14)$
43.$\Lambda(3, 5, 6; 9, 13, 14)$
44.$\Lambda(3, 5, 6; 8, 13, 14)$
45.$\Lambda(5, 6; 7, 8)$
46.$\Lambda(3, 6, 7; 13, 14, 2)$
47.$\Lambda(7, 8; 13, 14)$
48.$\Lambda(6, 7, 9; 12, 13, 14)$
49.$\Lambda(3, 5, 7; 10, 11, 13)$
50.$\Lambda(7, 8; 10, 11)$
51$\Lambda(6, 7, 10; 13, 14, 2)$
52.$\Lambda(9, 10, 11; 13, 1, 2)$
53.$\Lambda(11, 12; 13, 14)$
54.$\Lambda(5, 6, 8, 10; 11, 13, 14, 3)$
55.$\Lambda(9, 10; 11, 12)$
56.$\Lambda(2, 5, 6, 8; 9, 10, 11, 14)$
57.$\Lambda(7, 8; 9, 10)$
58.$\Lambda(6, 8, 9; 10, 12, 14)$
59.$\Lambda(2, 3, 5, 7; 8, 9, 10, 11)$
60.$\Lambda(8, 9, 10, 11; 12, 14, 2, 3)$

We used 4×4 determinants in steps 27, 28, 54, 56, 59, and 60 to determine the entries $\lambda_{4,5}, \lambda_{1,14}, \lambda_{10,11}, \lambda_{8,9}, \lambda_{7,8}$ respectively. The specific submatrices chosen to recover these entries have no 3×3 or 2×2 submatrix with determinant zero. As n increases, it may be harder to show that these larger matrices with zero determinant have no submatrix with zero determinant.

3.4. **Remark on parameter placement techniques.** Notice that in determining the parameters of the response for the rectangular lattice we first made use of connections which did not exist, namely those involving boundary antennas. A similar technique was used in [1] to parameterize square lattices. But for the spoked wheel, we first made use of existing connections, namely 3-connections involving neighboring boundary vertices. For the lattice graphs we might have first noticed that there is always a connection from all boundary nodes on the north side and those on the south side of the lattice, placing parameters on the whole submatrix B. Of course, we are not required to parameterize all of the entries of B in the response for lattices or the upper 3×3 submatrix of B in the response for wheels, but this may be a good way to order the parameters and identify placement patterns.

4. 2 CIRCLES, 4 RAYS

In the case of wheels and lattices, we were able to identify specific determinants in the response matrix which were equal to zero because some k-connections did not exist, but for some graphs, all k-connections do exist. In such cases, and when we take into consideration non-circular planar graphs, it is not always possible to find single determinental properties within the response. Consider the case of the

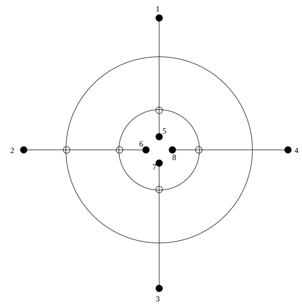

FIGURE 16.

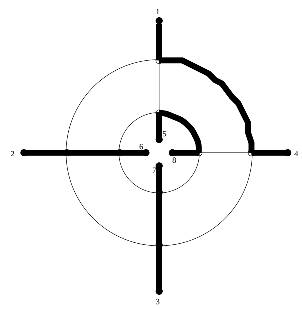

FIGURE 17.

2 circles 4 rays graph shown in figure 16. Any two, three, or four-connection exists in this graph, so there are not any necessary submatrices with determinant equal to zero in the response. But there are 20 conductivities and 28 entries in the response matrix, so there are probably eight entries which are uniquely determined by the other 20. Though all four-connections exist in the graph, some four-connections have uniquely determined paths. For example, the connection from $P = (1, 5, 6, 7)$ to $Q = (2, 3, 4, 8)$ must take the paths shown in figure 17. The connection from $P = (1, 2, 3, 5)$ to $Q = (4, 6, 7, 8)$ is also required to take this path. There are six relations between two different four connections in the graph which must take the same paths. By the determinental formula in [3], the determinants of these 4×4 submatrices must be equal to each other. So we have,

$$
\begin{aligned}
det(\Lambda(1,5,6,7;2,3,4,8)) &= det(\Lambda(1,2,3,5;4,6,7,8)) \\
det(\Lambda(1,2,6,7;3,4,5,8)) &= det(\Lambda(1,2,5,8;3,4,6,7)) \\
det(\Lambda(2,3,7,8;1,4,5,6)) &= det(\Lambda(2,3,5,6;1,4,7,8)) \\
det(\Lambda(1,2,3,7;4,5,6,8)) &= det(\Lambda(3,5,6,7;1,2,4,8)) \\
det(\Lambda(2,6,7,8;1,3,4,5)) &= det(\Lambda(2,3,4,6;1,5,7,8)) \\
det(\Lambda(2,5,6,8;1,3,4,7)) &= det(\Lambda(1,2,4,6;3,5,7,8))
\end{aligned}
$$

Since there are no submatrices with determinant equal to zero, we can solve each of these equations for one of the sixteen entries in one of the submatrices. There are two other linear relationships between the determinants, and presumably, if we have eight relationships which don't depend on each other, we will be able to determine eight entries in terms of the others.

5. Further research

(1) An interesting question to consider is what parameter placement looks like for non-circular planar graphs like the 2 circles 4 rays graph. For this specific graph, it is certainly possible to determine the other linear relationships between determinants in the response matrix besides the six listed in this paper–we ran out of time this summer. But once we have more linear relationships, the process of parameter placement will be different because in each step we have to consider more than one determinant.
(2) We can also continue to parameterize different classes of circular planar graphs and analyze different patterns of these parameters. The parameters for the $n \times (n+1)$ lattice look very similar to those for the square lattice, so there might be a general parameter placement for all lattice graphs.
(3) Since we relied on connection properties of the graphs in this paper, it might be interesting to consider the relationship between these parameter patterns and the connection properties associated with the graph structures. Connectivity types have been addressed in [3] and investigated in detail for critical circular planar graphs in [4]. Taking into consideration which connections were used in our parameterizations, we may be able to make use of or add to the research done by Miao Xu on connectivity types.

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