WHAT WE DON'T KNOW ABOUT THE 3-TO-1 GRAPH (PLUS A FEW THINGS WE DO KNOW)

ILYA GRIGORIEV

Abstract. This paper presents a method of constructing graphs with m signed conductivities corresponding to a single response matrix. An example 3-to-one graph is studied in detail, and numerous open questions are presented.

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1. How to Construct m-to-one graphs

The first example of a 2-to-one graph is the triangle-in-triangle graph, found by Ernie Esser [4] (Fig. 2 on page 3) . Jennifer French and Shen Pan were able to analyze it thoroughly in [2]. They considered the graph as a union of three 4-stars, and applied the K-star method (best explained in in [3]) to it. They also exhibited numerous other examples of 2-to-one and 2^m-to-one graphs, all based on the triangle-in-triangle graph. This analysis motivated us to look for ways to use n-stars as building blocks for constructing even more complicated graphs.

To understand the construction of the m-to-one graphs, we first mimic the exploration of the triangle-in-triangle graph using somewhat different terminology.

1.1. The four-star.

The four-star. The four-star (Fig. 1) is the simplest example of a star that is of interest to us. We follow [2] in naming the conductance between the interior vertex and the i'th boundary vertex γ_i, and the i,j entry of its 4×4 response matrix Λ — λ_{ij}.

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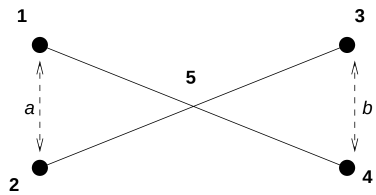


$$\Lambda = \begin{bmatrix} \bullet & a & \lambda_{13} & \lambda_{14} \\ a & \bullet & \lambda_{23} & \lambda_{24} \\ \lambda_{13} & \lambda_{23} & \bullet & b \\ \lambda_{14} & \lambda_{24} & b & \bullet \end{bmatrix}$$

FIGURE 1. The Four-Star as a 2-multiplexer

Any n-star has very simple relations in its response matrix:

Lemma 1.1. *Λ is a response matrix of an n-star if and only if it is $n \times n$, symmetric, its rows sum to 0, and its off-diagonal entries satisfy: $\lambda_{ij}\lambda_{kl} = \lambda_{ik}\lambda_{kl}$, for any distinct i, j, k, l.*

Proof. See [5]. □

In other words, all of the two-by-two determinants of the response matrix that do not involve the diagonal have to be 0.

Since the diagonal entries of the matrix can be found from the others, and do not play a role in determining whether the matrix is a response matrix of a four-star, we replace them with dots. Furthermore, we replace λ_{12} with a and λ_{34} with b to put Λ in the form it appears in Fig. 1.

Now, we can observe several important properties.

Properties 1.1. *With our choice of a and b, the four-star graph satisfies these properties:*

1. *The four-star is recoverable.*
2. *If we know all the entries of the response matrix except for a or b, we cannot recover either of them from the relations we have. Graphically, there is no determinant that contains only a and known, off-diagonal values.*
3. *Once we know either a or b, we know the other from the relation $ab = \lambda_{13}\lambda_{14}$.*

Definition 1.2. We call a graph with some choice of m variables in its response matrix that satisfy the above properties a *multiplexer*, or an *m-multiplexer.*

1.2. **The Triangle-in-Triangle Graph.** The following analysis follows that of [2], but uses slightly different terminology.

We would like to understand the behavior of the triangle-in-triangle graph (Fig. 2) as a union of three four-stars.

If we apply apply the $\bigstar - K$ transformation on the triangle-in-triangle graph, we obtain the multi-graph on the right of Fig. 2. In that graph, the i, j element of the response matrix Λ represents the sum of conductances of the edges between vertices i and j. For instance, since there are two edges connecting vertices 1 and 2, the sum of their conductances is λ_{12}. Similarly, λ_{34} and λ_{67} represent a sum of conductances of two edges.

For all the other pairs of vertices, however, there is at most one edge connecting them. So, the conductance of that edge is either 0 or the same as one of the entries of Λ.

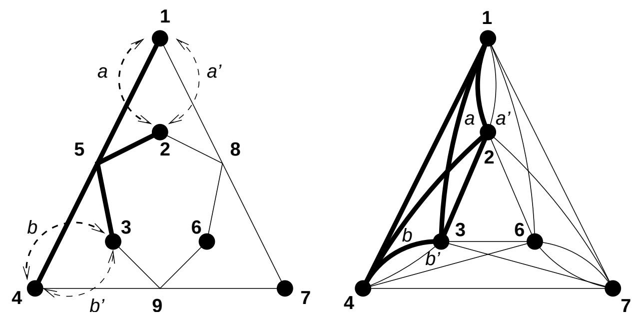


FIGURE 2. The triangle-in-triangle graph (left) and its $\bigstar - K$ transformation.

Since the four-star is recoverable (Property 1.1.1), we can recover the conductances of the bold edges of the triangle-in-triangle graph from the bold edges of the $\bigstar - K$-ed graph. In fact, the relations between the Kirchoff matrix of a four-star and its response matrix are rather simple (from [2]):

$$\lambda_{ij} = \frac{\gamma_i \gamma_j}{\sum_{k=1}^{4} \gamma_k} \quad \text{and} \quad \gamma_i = -\lambda_{ii} + \frac{\lambda_{ij}\lambda_{ik}}{\lambda_{jk}} = \sum_{i \neq j} \lambda_{ij} + \frac{\lambda_{ij}\lambda_{ik}}{\lambda_{jk}}$$

However, we do not know the bold conductances between vertices 1 and 2, and between vertices 3 and 4 - the conductances that correspond to a and b from previous section. By Property 1.1.2, we cannot recover this information from the other conductances.

So, we leave the bold conductance between vertices 1 and 2 as a. Then, the bold conductance between 3 and 4 becomes, by Property 1.1.3, $b = \frac{\lambda_{13}\lambda_{24}}{a}$. The sum of the two conductances between 3 and 4 has to be λ_{34}. So, the conductance b' between 3 and 4 is $b' = \lambda_{34} - b = \lambda_{34} - \frac{\lambda_{13}\lambda_{24}}{a}$.

We can continue this process with the other two four-stars to obtain a continued fraction:

$$a' = \frac{\lambda_{26}\lambda_{17}}{\lambda_{67} - \frac{\lambda_{36}\lambda_{47}}{b'}} = \frac{\lambda_{26}\lambda_{17}}{\lambda_{67} - \frac{\lambda_{36}\lambda_{47}}{\lambda_{34} - \frac{\lambda_{13}\lambda_{24}}{a}}}$$

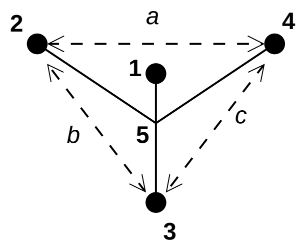
Note that this fraction can be simplified to a linear fractional of a (a fraction of two linear polynomials of a), as on every step of the way only the operations of taking inverses, multiplication by constants, and subtraction from constants were used.

The sum of the two conductances between 1 and 2 has to be equal to λ_{12}. So, if we denote $a' = l(a)$, we obtain the equation:

$$a + l(a) = \lambda_{12}$$

If we multiply both sides by the denominator of $l(a)$, we see that it is a quadratic. Once we pick a to be one of its two roots, we know every conductance in the $\bigstar - K$-ed graph, so we can recover the conductances in the original triangle-in-triangle graph.

For further study of this equation and its solutions, see [2].



$$\Lambda = \begin{bmatrix} \bullet & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{12} & \bullet & b & a \\ \lambda_{13} & b & \bullet & c \\ \lambda_{14} & a & c & \bullet \end{bmatrix}$$

FIGURE 3. A three-multiplexer.

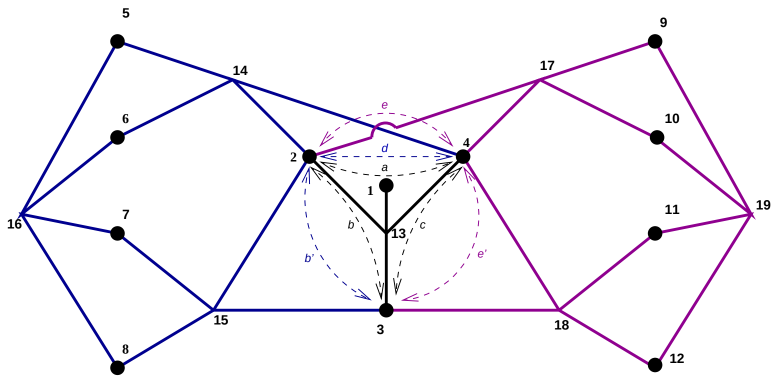


FIGURE 4. Three-to-one Graph

1.3. **Three-to-One Graphs.** As we used it before, the four-star relates an unknown response on one end to an unknown response on the other end. This was sufficient to impose the condition of $a + l(a) = const$ on the graph.

A way of constructing a 3-to-1 graph would be to impose the condition $a + l_1(a) + l_2(a) = const$ on it. (This is clearly a cubic - just multiply both sides by the denominators of l_1 and l_2.) To do that, however, we would need to use a structure that would relate an unknown response a on one end to *two* other unknown responses, b, and c. In other words, we need to construct a 3-multiplexer. This is possible using a star such as that in Fig. 3:

Note that this graph satisfies Properties 1.1:

(1) It is recoverable. (In fact, it is also a four-star)
(2) If we do not know a, b, or c, we cannot recover any of them from the relations we have. There is no determinant that relates any of the variables to three known values.
(3) Once we know one of a, b, or c, we can recover the rest using the relations:

$$a\lambda_{13} = b\lambda_{14} \quad \text{and} \quad a\lambda_{13} = c\lambda_{12}$$

Now, all that is left to us is to create a graph that would impose a second relation between a and b, and between a and c, so that an equation is produced. We can do this using two chains of three four-stars each, to obtain the graph in Fig. 4.

To recover it, assume we know a. From it, we know $b = \frac{a\lambda_{1,3}}{\lambda_{1,4}}$ and $c = \frac{a\lambda_{1,3}}{\lambda_{1,2}}$. In the same manner as with the triangle-in-triangle graph, we use the chains of four-stars to express d and e from b and c:

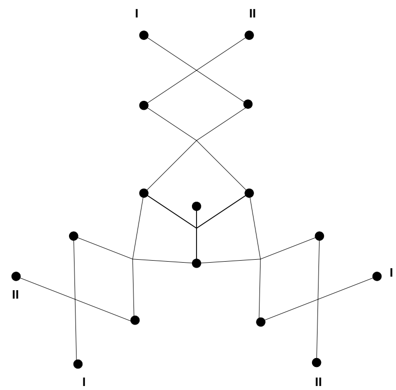


FIGURE 5. Another three-to-one graph. Identify all the vertices marked by I as one vertex. Similarly, the vertices marked by II are the same vertex.

$$d = \frac{\lambda_{2,6}\lambda_{4,5}}{\lambda_{5,6} - \frac{\lambda_{5,8}\lambda_{6,7}}{-\frac{\lambda_{2,7}\lambda_{3,8}}{-b+\lambda_{2,3}}+\lambda_{7,8}}} \quad \text{and} \quad e = \frac{\lambda_{2,9}\lambda_{4,10}}{\lambda_{9,10} - \frac{\lambda_{9,12}\lambda_{10,11}}{-\frac{\lambda_{3,12}\lambda_{4,11}}{-c+\lambda_{3,4}}+\lambda_{11,12}}}$$

You can verify that d and e are linear fractionals. Finally, since we know the response $\lambda_{1,2}$, we obtain a cubic equation,

$$a + d + e = a + l_1(a) + l_2(a) = \lambda_{1,2} \tag{1}$$

The equation has at most three complex solutions. Once we pick an a, we know the conductance of every edge in the $\bigstar - K$-ed graph (you can think of this as knowing the response matrix for each individual four-star that composes the graph), so we can recover all the conductances of the graph.

Figure 7 on page 9 presents a typical example of a response matrix and three corresponding Kirchoff matrices . The values there are approximate only for viewing convenience, the exact values are available. For instance, the three possible values for the $(1, 13)$ entry of the Kirchoff matrix with that response matrix are:

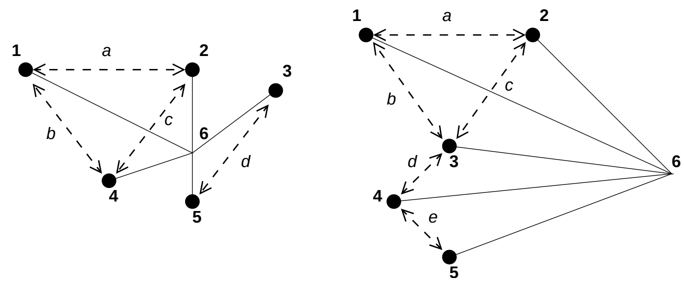
$$92, \ \frac{92\left(4749777544859660751646598533831 - 1643948\sqrt{220826732286257802787039228087101081465856067059}\right)}{5097651132204254186871790467297 5},$$

$$\text{and } \frac{92\left(4749777544859660751646598533831 + 1643948\sqrt{220826732286257802787039228087101081465856067059}\right)}{50976511322042541868717904672975}$$

1.4. **More Graphs.** We can construct another three-to-one graph by connecting four-stars to the multiplexer differently (see Fig. 5). The resulting graph has properties very similar to those of the first one.

To construct a four-to-one graph, a five-to-one graph, or an even worse graph, we only need an appropriate multiplexer. Four-stars are insufficient for these purposes, but five-stars such as those in Fig. 6 work fine. It's easy to verify that the three properties we need are satisfied.

Unfortunately, the only method I know of so far for coming up with m-multiplexers is drawing matrix frames and trying to creatively place m letters in them to insure that properties 1.1 are satisfied.



$$\Lambda = \begin{bmatrix} \bullet & a & \lambda_{13} & d & \lambda_{15} \\ a & \bullet & \lambda_{23} & b & \lambda_{25} \\ \lambda_{13} & \lambda_{23} & \bullet & \lambda_{34} & c \\ d & b & \lambda_{34} & \bullet & \lambda_{45} \\ \lambda_{15} & \lambda_{25} & c & \lambda_{45} & \bullet \end{bmatrix} \quad \Lambda = \begin{bmatrix} \bullet & a & b & \lambda_{14} & \lambda_{15} \\ a & \bullet & c & \lambda_{24} & \lambda_{25} \\ b & c & \bullet & d & \lambda_{35} \\ \lambda_{14} & \lambda_{24} & d & \bullet & e \\ \lambda_{15} & \lambda_{25} & \lambda_{35} & e & \bullet \end{bmatrix}$$

Figure 6. Examples of five-stars serving as a 4-multiplexer and a 5-multiplexer.

Then, we can connect chains of four-stars to the multiplexers in either of the two ways we know to obtain a 4-to-one or a five-to-one graph.

(*Note:* I haven't actually verified this by hand yet. So, if you like searching for mistakes in papers, this is a good place to look).

Furthermore, the only reason we have used stars in the previous discussion is because stars are easy to understand. Any other graphs that satisfy the properties of a 2-multiplexer will serve in the place of the four-star, and there is no reason for the 3-, 4-, and 5-multiplexers to be stars. It is possible that there exist multiplexers with other, more convenient, properties.

The only thing we would have to be careful about here is that we rely on the relation between different variables in a multiplexer being a linear fractions. This insures that the equations we get are sums of as many linear fractionals as we want, and therefore have the correct degree. However, the properties of non-linear-fractional multiplexers and graphs based on them may also be very interesting.

2. Oh, No! I've Created a Monster!

So far, my investigation of the three-to-one graph from Fig. 4 on page 4 was motivated by the desire to find an example of three real Kirchoff matrices *with positive conductances* that have the same response matrix. So, the procedure was to take a random Kirchoff matrix (with real, positive conductances between 100 and 300), compute its response matrix, and then recover three Kirchoff matrices - the original matrix and two new "sister" ones - from it.

There are several very interesting patterns that appeared in this investigation, though the proof of any of them has eluded me so far.

Note that the following conjectures are limited by the quality of the random number generator. For instance, some of them may be true only in the case when the numbers in the matrix have similar orders of magnitude, since this is the kind of matrices that were tested.

2.1. Roots are real. First of all, for all the examples I've tried, all of the entries of the sister Kirchoff matrices were real. This is very strange, because most arbitrary cubics have complex solutions, and there is no obvious reason for the cubic (1) to have all real solutions.

Conjecture 2.1. *For any real, positive matrix K for our graph, if its two sisters will have all-finite conductances, they will both have real conductances for all of their edges.*

It seems clear that this should fail if we allow negative conductances in the original matrix. (I don't have an example yet, though).

In fact, the problem is simplified slightly because we started with a real, positive Kirchoff matrix, and therefore we know that one of the roots of the cubic is real and positive. It is possible to factor out this solution we know in advance, and we are left with a (very, very ugly) quadratic equation. Still, its discriminant is a long an ugly expression, and it is not obvious at all how to find where it is positive.

2.2. **Roots are never positive.** I have not succeeded in finding a Kirchoff matrix such that its two sisters have all positive entries. In fact, a very interesting pattern emerges:

Conjecture 2.2. *For any real, positive matrix K, one of its sisters will have exactly two negative conductances, and the other sister will have exactly four negative conductances.*

This may fail only if the sisters have 0 conductance along some of their edges.

The position of the negative conductances in the matrix varies, but their number does not.

It is possible to concentrate our attention of a given pair of entries, and vary the Kirchoff matrix continuously between a point where the entries are negative to a point where they are positive. What seems to happen is that, at some point, four conductances become 0. Then, our pair of conductances becomes positive, and another pair - negative.

Another conjecture that may, perhaps, shed light one the previous one is:

Conjecture 2.3. *In the three-to-one graph, whenever a conductance of an edge in a four-star is negative, the sum of the conductances of the edges in the four-star is also negative.*

Equivalently, the corresponding value on the diagonal of the Kirchoff matrix is positive.

2.3. **The Singular Cases.** A very interesting case occurs if we set all the conductances in our graph to 1. In this case, the two sister Kirchhoff matrices turn out to have infinite conductances on some of their edges. It is possible to evaluate the sister matrices for Kirchhoff matrices that approach all ones. In this case, several entries of the sister entries go to ∞, and several others go to $-\infty$, all with different speed.

Jim proposed that we can think about this as the matrix approaching projective infinity from some direction. Then, it is natural to imagine the map Λ as a map from complex projective space to complex projective space. For some specific infinity, we can define Λ on it as the limit of $\Lambda(K)$ as K approaches that infinity.

The caveat here is that Λ may be a multi-valued function: $\Lambda(all\ ones)$ or $\Lambda(some\, infinity)$ may depend on which direction we approach the point from.

Question 2.1. *Is Λ as a function on projective space a single-valued function? Is there an example where it isn't?*

2.4. **Rank of the Differential.** Ernie has run his "rank of the differential" program on the graph. The program computed the differential of Λ as a function of K symbolically, and then performed Gaussian elimination on the result. He then observed, that the denominators in the results involve only addition and multiplication, and are therefore positive as long as K has positive conductances. So, the differential will have full rank.

This implies that the map Λ will be locally one-to-one at K. In particular, the root of the cubic (1) that corresponds to K cannot have multiplicity more than 1.

Note that this is invalid if K does not have positive conductances, or if some of its conductances are infinite.

Question 2.2. *Is it possible that the differential of Λ at K has full rank, but does not have full rank at the sisters of K?*

What would a "yes" or a "no" answer to this mean?

3. Conclusion: Help me Scare the Big Bad Graph Away.

The three-to-one graph has numerous properties that seem to be very abiding and easy to define, but hard to prove. It is likely that all of these distinct properties can be connected by a single pretty idea. So, any help figuring out what's lurking in the shadows would be greatly appreciated.

Help, I need somebody,
Help, not just anybody,
Help, you know I need someone, help.

Appendix A. Tools

I've used quite a few tools in this project that may be useful to others. So far, they are not very user-friendly, but if you shame me enough, I'll try to make them more decent.

The conversion from one Kirchoff matrix to three sister matrices was performed by:

(1) Using a modified Nick Addington's recovery program to create instructions for *Mathematica* on how to recover the graph (once per graph).
(2) Computing the response matrix of the graph, and telling *Mathematica* to recover it. The result is dependent on parameter (that is closely related to our a).
(3) Computing the Schur complement of that, and telling *Mathematica* to equate it to the response matrix we started from, and solve the equation to obtain three solutions.

The advantages of this method are that it is exact, fast (for the three-to-one graph, it takes about 20 seconds to go from one Kirchoff matrix to three, and that's without any attempt on my part to optimize it), and able to cope with symbolic variables. In fact, it can easily recover small graphs symbolically.

This paper was typeset with LyX (`http://www.lyx.org`), which makes LaTeX far easier on the eyes. It also helps if you don't want to care whether your pictures are in EPS, PDF, or JPG.

$$
\Lambda = \begin{pmatrix}
-\frac{51612}{653} & \frac{9844}{653} & \frac{30728}{653} & \frac{11040}{653} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{9844}{653} & -\frac{2610886402973}{4281053634} & \frac{59292970}{554397} & \frac{85238311}{840411} & \frac{55040}{891} & \frac{15910}{297} & \frac{73168}{849} & \frac{39440}{849} & \frac{8365}{143} & \frac{68593}{858} & 0 & 0 \\
\frac{30728}{653} & \frac{59292970}{554397} & -\frac{79726485908}{170199879} & \frac{28666128}{200471} & 0 & 0 & \frac{43847}{849} & \frac{23635}{849} & 0 & 0 & \frac{44776}{921} & \frac{39904}{921} \\
\frac{11040}{653} & \frac{85238311}{840411} & \frac{28666128}{200471} & -\frac{48730500194}{86002059} & \frac{512}{9} & \frac{148}{3} & 0 & 0 & \frac{4270}{143} & \frac{17507}{429} & \frac{20844}{307} & \frac{18576}{307} \\
0 & \frac{55040}{891} & 0 & \frac{512}{9} & -\frac{34266878}{91773} & \frac{3522461}{30591} & \frac{5649}{103} & \frac{78548}{927} & 0 & 0 & 0 & 0 \\
0 & \frac{15910}{297} & 0 & \frac{148}{3} & \frac{3522461}{30591} & -\frac{3159956}{10197} & \frac{3717}{103} & \frac{17228}{309} & 0 & 0 & 0 & 0 \\
0 & \frac{73168}{849} & \frac{43847}{849} & 0 & \frac{5649}{103} & \frac{3717}{103} & -\frac{29227862}{87447} & \frac{9223583}{87447} & 0 & 0 & 0 & 0 \\
0 & \frac{39440}{849} & \frac{23635}{849} & 0 & \frac{78548}{927} & \frac{17228}{309} & \frac{9223583}{87447} & -\frac{84016580}{262341} & 0 & 0 & 0 & 0 \\
0 & \frac{8365}{143} & 0 & \frac{4270}{143} & 0 & 0 & 0 & 0 & -\frac{5351402}{19591} & \frac{2188548}{19591} & \frac{4902}{137} & \frac{5111}{137} \\
0 & \frac{68593}{858} & 0 & \frac{17507}{429} & 0 & 0 & 0 & 0 & \frac{2188548}{19591} & -\frac{326475763}{822822} & \frac{77142}{959} & \frac{80431}{959} \\
0 & 0 & \frac{44776}{921} & \frac{20844}{307} & 0 & 0 & 0 & 0 & \frac{4902}{137} & \frac{77142}{959} & -\frac{301313554}{883239} & \frac{95754206}{883239} \\
0 & 0 & \frac{39904}{921} & \frac{18576}{307} & 0 & 0 & 0 & 0 & \frac{5111}{137} & \frac{80431}{959} & \frac{95754206}{883239} & -\frac{294492862}{883239}
\end{pmatrix}
$$

$$
K_1 = \begin{pmatrix}
-92.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 92.0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -833. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 107. & 215. & 272. & 0 & 239. & 0 & 0 \\
0 & 0 & -729. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 334. & 0 & 163. & 0 & 0 & 232. & 0 \\
0 & 0 & 0 & -764. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 120. & 198. & 0 & 0 & 122. & 324. & 0 \\
0 & 0 & 0 & 0 & -525. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 256. & 0 & 269. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -399. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 222. & 0 & 177. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -458. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 269. & 189. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -437. & 0 & 0 & 0 & 0 & 0 & 0 & 145. & 292. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -343. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 210. & 0 & 133. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -586. & 0 & 0 & 0 & 0 & 0 & 0 & 287. & 0 & 299. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -451. & 0 & 0 & 0 & 0 & 0 & 0 & 193. & 258. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -441. & 0 & 0 & 0 & 0 & 0 & 172. & 269. \\
92.0 & 107. & 334. & 120. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -653. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 215. & 0 & 198. & 256. & 222. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -891. & 0 & 0 & 0 & 0 & 0 \\
0 & 272. & 163. & 0 & 0 & 0 & 269. & 145. & 0 & 0 & 0 & 0 & 0 & 0 & -849. & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 269. & 177. & 189. & 292. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -927. & 0 & 0 & 0 \\
0 & 239. & 0 & 122. & 0 & 0 & 0 & 0 & 210. & 287. & 0 & 0 & 0 & 0 & 0 & 0 & -858. & 0 & 0 \\
0 & 0 & 232. & 324. & 0 & 0 & 0 & 0 & 0 & 0 & 193. & 172. & 0 & 0 & 0 & 0 & 0 & -921. & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 133. & 299. & 258. & 269. & 0 & 0 & 0 & 0 & 0 & 0 & -959.
\end{pmatrix}
$$

$$
K_2 = \begin{pmatrix}
-84.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 84.3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -685. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 240.3 & 179. & 60.1 & 0 & 205.2 & 0 & 0 \\
0 & 0 & -865. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 750. & 0 & 36.04 & 0 & 0 & 79.2 & 0 \\
0 & 0 & 0 & -650. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 269.5 & 165. & 0 & 0 & 104.8 & 110.6 & 0 \\
0 & 0 & 0 & 0 & -512. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 333. & 0 & 179.3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -406. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 288. & 0 & 118.0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -295.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -114.3 & 410. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -572. & 0 & 0 & 0 & 0 & 0 & 0 & -61.6 & 634. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -353. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 272. & 0 & 81.7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -555. & 0 & 0 & 0 & 0 & 0 & 0 & 371. & 0 & 183.6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -380. & 0 & 0 & 0 & 0 & 0 & 0 & -725. & 1105. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -507. & 0 & 0 & 0 & 0 & 0 & -646. & 1153. \\
84.3 & 240. & 750. & 270. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1344. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 179. & 0 & 165. & 333. & 288. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -966. & 0 & 0 & 0 & 0 & 0 \\
0 & 60.1 & 36.0 & 0 & 0 & 0 & -114. & -61.6 & 0 & 0 & 0 & 0 & 0 & 0 & 79.8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 179. & 118. & 410. & 634. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.34\times 10^{3} & 0 & 0 & 0 \\
0 & 205. & 0 & 105. & 0 & 0 & 0 & 0 & 272. & 371. & 0 & 0 & 0 & 0 & 0 & 0 & -953. & 0 & 0 \\
0 & 0 & 79.2 & 111. & 0 & 0 & 0 & 0 & 0 & 0 & -725. & -646. & 0 & 0 & 0 & 0 & 0 & 1.18\times 10^{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 81.7 & 184. & 1.11\times 10^{3} & 1.15\times 10^{3} & 0 & 0 & 0 & 0 & 0 & 0 & -2.52\times 10^{3}
\end{pmatrix}
$$

$$
K_3 = \begin{pmatrix}
-87.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 87.1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -797. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 162.6 & 139. & 183.7 & 0 & 312. & 0 & 0 \\
0 & 0 & -786. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 507. & 0 & 110.1 & 0 & 0 & 168.3 & 0 \\
0 & 0 & 0 & -704. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 182.3 & 128. & 0 & 0 & 159.1 & 235.1 & 0 \\
0 & 0 & 0 & 0 & -450. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 697. & 0 & -247. & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -442. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 604. & 0 & -162.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -554. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 496. & 58.1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -357. & 0 & 0 & 0 & 0 & 0 & 0 & 267. & 89.8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -335. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 159. & 0 & 175.6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -612. & 0 & 0 & 0 & 0 & 0 & 0 & 217. & 0 & 395. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -456. & 0 & 0 & 0 & 0 & 0 & 0 & 257. & 199.1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -436. & 0 & 0 & 0 & 0 & 0 & 229. & 207.6 \\
87.1 & 163. & 507. & 182. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -940. & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 139. & 0 & 128. & 697. & 604. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1568. & 0 & 0 & 0 & 0 & 0 \\
0 & 184. & 110. & 0 & 0 & 0 & 496. & 267. & 0 & 0 & 0 & 0 & 0 & 0 & -1057. & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -247. & -162. & 58.1 & 89.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 261.6 & 0 & 0 & 0 \\
0 & 312. & 0 & 159. & 0 & 0 & 0 & 0 & 159. & 217. & 0 & 0 & 0 & 0 & 0 & 0 & -847. & 0 & 0 \\
0 & 0 & 168. & 235. & 0 & 0 & 0 & 0 & 0 & 0 & 257. & 229. & 0 & 0 & 0 & 0 & 0 & -889. & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 176. & 395. & 199. & 208. & 0 & 0 & 0 & 0 & 0 & 0 & -977.
\end{pmatrix}
$$

FIGURE 7. Sample response matrix with three corresponding Kirchoff matrices.

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E-mail address: ilyag@uchicago.edu