HARMONIC EXTENSION ON NETWORKS

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Abstract. We study the implication of geometric properties of the graph of a network in the extendibility of all γ-harmonic germs at an interior node. We prove that all γ-harmonic germs at an interior node of a circular planar graph is extendible if the graph is critical (Theorem 4.7). It served as a motivation for many results in this paper. A more general geometric property ensuring γ-harmonic extendibility is given as (Theorem 4.3). Finally, we point out a connection between γ-harmonic extension and the recoverability of a random walk network as Theorem 5.2. A discussion for possible futher research is given in Question 4.8.

1. Introduction

Let $G = (\partial V, \operatorname{int} V, E)$ be a graph with boundary and $\Gamma = (G, \gamma)$ a resistor network. Let $f : W \to \mathbb{R}$ be a function defined on a subset $W \subseteq V$ of its vertices. Then for each vertex $v \in W$ such that its neighbors $\mathcal{N}(v) \in W$, we define the germ of f at v to be the equivalence class of all functions equal to f at $v \cup \mathcal{N}(v)$ [7]. We would like to decide when all γ-harmonic germs can be extended to a γ-harmonic function on the whole graph. An algebraic criterion can be easily given (see Section 2) using the solution to Dirichlet problem. Here, we are concerned with associating the geometry of a graph with the extendibility of all γ-harmonic germs at an interior node.

Most of the results in this paper are for general graphs, but the motivation was the followings. For a circular planar graph G, let p be a strongly interior node, meaning $\mathcal{N}(p) \subseteq \operatorname{int} V(G)$ and let M be the minimum cutset between $\mathcal{N}(p)$ and $\partial V(G)$. Justin Robertson [7] proved that if $|\mathcal{N}(p)| > |M|$, then there exists at least one γ-harmonic germ at p that cannot be globally extended; a counterexample to the converse is given in [7] as the Chinese Star graph (Interestingly, the same graph served as a counterexample to Card Conjecture [5]). Moreover, if $|\mathcal{N}(p)| > |M|$, G will not be critical. But whether criticality is a sufficient condition for global extension was a conjecture, and we give a proof of it, formally stated as Theorem 4.7.

2. Extension Using Kirchhoff's Matrix

Definition 2.1. A graph G' is a subgraph ([2]) of a graph G if the following conditions are met: i) $V' \subseteq V$ with a chosen decomposition into $\partial V(G') \cup \operatorname{int} V(G')$ such that $\operatorname{int} V(G') \subseteq \operatorname{int} V(G)$; ii) $E(G') \subseteq E(G)$; iii) each edge in $E(G)$ that has at least one endpoint as a vertex in $\operatorname{int} V(G')$ must be in $E(G')$. A strongly interior subgraph G' of G is a subgraph of G with $V' \subseteq \operatorname{int} V$. A subgraph G' of G is a weakly interior subgraph if it is not strongly interior.

In this section, Lemma 2.2 gives a necessary and sufficient algebraic condition for the extendibility of all γ-harmonic functions on a strongly interior subgraph G' to the graph G (similar to Section 3.1 of [7]). Remark 2.3 shows a trick for weakly interior subgraphs.

As a note on notation, given a matrix M, we denote $M(I; J)$ as the submatrix of rows I and columns J. Write Kirchhoff's matrix for a resistor network $\Gamma = (G, \gamma)$ as

$$K = \begin{array}{c c} & \begin{array}{cc} \partial V & \operatorname{int} V \end{array} \\ \begin{array}{c} \partial V \\ \operatorname{int} V \end{array} & \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \end{array} .$$

Date: August 8, 2006.

Then the interior potential is $\vec{u}(\operatorname{int} V) = -C^{-1}B^\top \vec{u}(\partial V)$ where $\vec{u}(\partial V)$ is the boundary potential [3]. We are immediately lead to Lemma 2.2.

Lemma 2.2. *Let $\Gamma = (G, \gamma)$ be a resistor network and G' a strongly interior subgraph of G. Let $f : V(G) \to \mathbb{R}$ be a γ-harmonic function on the set $V(G')$, then there exist a global γ-harmonic function $g : V(G) \to \mathbb{R}$ such that $g|_{V(G')} = f|_{V(G')}$ if and only if the matrix*

$$D := -C^{-1}(\partial V(G'); \operatorname{int} V(G))B^\top$$

has full rank.

Lemma 2.2 can be applied to decide the extendibility γ-harmonic germs at a strongly interior node p of G by taking $G' = p \cup \mathcal{N}(p)$.

Remark 2.3. To extend γ-harmonic germs at a weakly interior node p, we can adjoin two boundary spikes (with adjustment to the Kirchhoff's matrix) to each $v \in \mathcal{N}(p) \cap \partial V$ and thus making p to be a strongly interior node. It is important to note that adjoining boundary spikes has no effect on harmonic extendibility. Also, if the graph is critical circular planar, then adjoining two boundary spikes to one node will preserve criticality, and this can be easily seen from the medial graph and we will use this fact later.

3. A Layered Structure

Let p be an interior node of a graph G, we can construct a layered graph $\mathcal{O}_p(G)$ with center p. We give the construction inductively. Let $\mathcal{N}^{(0)}(p) = p$, then $\mathcal{N}^{(k)}(p) = \mathcal{N}(\mathcal{N}^{(k-1)}(p)) \setminus \bigcup_{j \in \{0,\ldots,k-1\}} \mathcal{N}^{(j)}(p)$. Moreover, if $\Omega = \mathcal{N}^{(k-1)}(p) \cap \partial V \neq \emptyset$ and $\mathcal{N}^{(k)}(p) \neq \emptyset$, then for all $v \in \Omega$, we interiorize v by adjoining to it two boundary spikes. The resulting graph will be $\mathcal{O}_p(G)$. Referring to Remark 2.3, we know that all γ-harmonic germs at p is extendible to G if and only if they are extendible to $\mathcal{O}_p(G)$. If G is circular planar, we have $\mathcal{O}_p(G)$ is critical if G is critical and we can embed $\mathcal{O}_p(G)$ in the layered plane with p as the center and $\mathcal{N}^{(i)}(p)$ on the i-th layer (Figure 1).

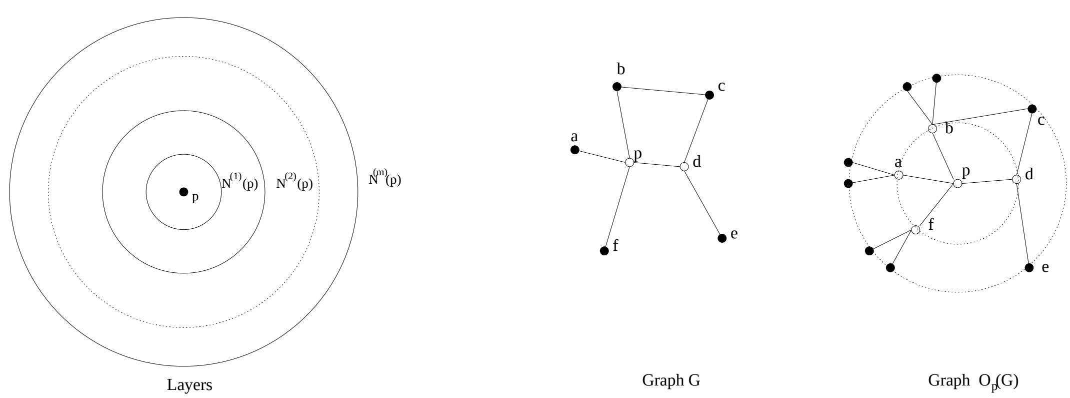

Figure 1. Layered Embedding

4. Geometry of γ-harmomic Extendible Networks

Lemma 4.1. *For a resistor network $\Gamma = (G, \gamma)$, let p be an interior node of G. Define $J := \mathcal{O}_p(G)$ and suppose $\partial V(J) = \mathcal{N}^{(m)}(p)$ with $m \geq 2$, then the matrix $D := -C^{-1}(\mathcal{N}^{(m-1)}(p); \operatorname{int} V(J))B^\top$ has full rank if and only if the matrix $E := B^\top(\mathcal{N}^{(m-1)}(p); \partial V(J))$ has full rank.*

Proof. Denote $\Phi = \mathcal{N}^{(m-1)}(p)$ and $\Psi = \bigcup_{j \in \{0,\ldots,m-2\}} \mathcal{N}^{(j)}(p)$, and define $\tilde{C} := -C^{-1}(\Phi; \Phi)$. Because all nodes in Ψ are strongly interior, the submatrix $B^\top(\Psi; \partial V(J))$ of $B^\top$ is a zero matrix. Hence,

$$-C^{-1}(\Phi; \operatorname{int} V)B^\top = -\left(\begin{array}{cc} & \begin{array}{cc} \Phi & \Psi \end{array} \\ \Phi & \begin{bmatrix} & C^{-1} & \end{bmatrix} \end{array} \right) \cdot \left(\begin{array}{cc} & \partial V \\ \begin{array}{c} \Phi \\ \Psi \end{array} & \begin{bmatrix} E \\ 0 \end{bmatrix} \end{array} \right) = \tilde{C}E.$$

Because C is positive definite ([3]), so is C^{-1}; we have $\tilde{C}$ is invertible. Thus, $rank(D) = rank(\tilde{C}E) = rank(E)$.

□

Lemma 4.2. *Let $\Gamma = (G, \gamma)$ be a resistor network and p an interior node. Define $J := \mathcal{O}_p(G)$ and suppose $\partial V(J) = \mathcal{N}^{(m)}(p)$. If there are no disjoint sets A, $B \in \mathcal{N}^{(m-1)}(p)$ such that $\mathcal{N}(A) \cap \partial V(J) = \mathcal{N}(B) \cap \partial V(J)$, then the matrix*

$$E := B^\top(\mathcal{N}^{(m-1)}(p); \partial V(J))$$

has full rank.

Proof. All arguments below are with respect to the graph J. If E does not have full rank, then there are rows $P_{i_1}, ..., P_{i_k}; P_{j_1}, ..., P_{j_l}$, corresponding to disjoint sets of nodes $A = \{p_{i_1}, ..., p_{i_k}\}$ and $B = \{p_{j_1}, ..., p_{j_l}\}$ such that $(c_{i_1}P_{i_1} + \cdots + c_{i_k}P_{i_k}) - (c_{j_1}P_{j_1} + \cdots + c_{j_l}P_{j_l}) = 0$ with $c_m > 0$. We can also write it in the form

$$c_{i_1}P_{i_1} + \cdots + c_{i_k}P_{i_k} = c_{j_1}P_{j_1} + \cdots + c_{j_l}P_{j_l}. \tag{1}$$

Equation 1 says that A and B share a common set of boundary nodes, that is $\mathcal{N}(A) \cap \partial V(J) = \mathcal{N}(B) \cap \partial V(J)$. It is because all entries in $B^\top$ have the same sign, so nothing will cancel off. This completes the proof.

□

The following theorem gives a sufficient condition for determining γ-harmonic extendibility from the geometry of the graph.

Theorem 4.3. *Let $\Gamma = (G, \gamma)$ be a resistor network and let p be an interior node. Define $H := \mathcal{O}_p(G)$. If for all layers $\mathcal{N}^{(j)}(p)$ of H with $j \geq 2$, there are no disjoint sets of A, $B \in \mathcal{N}^{(j-1)}(p)$ such that $\mathcal{N}(A) \cap \mathcal{N}^{(j)}(p) = \mathcal{N}(B) \cap \mathcal{N}^{(j)}(p)$, then all γ-harmonic germs at p can be globally extended.*

Proof. First consider the case when $j = 2$. By Lemma 4.2 with $J = \bigcup_{i=\{0,1,2\}} \mathcal{N}^{(i)}(p)$ and thus $\mathcal{N}^{(j-1)}(p) = \mathcal{N}(p)$, we can conclude that the matrix $E = B^\top(\mathcal{N}(p); \mathcal{N}^{(2)}(p))$ has full rank. Thus $D = -C^{-1}(\mathcal{N}(p); \operatorname{int} V(J))B^\top$ has full rank by Lemma 4.1. Then by Lemma 2.2, we know all γ-harmonic germs can be extended to J. Hence we can determine the values on $\mathcal{N}^{(2)}(p)$.

Proving by induction, we assume that all γ-harmonic germs can be extended to the m-th layer, that is we know the values of the extended potential function on $\mathcal{N}^{(m)}(p)$ and note that the values on the set of nodes $\bigcup_{k \in \{0,...,m-1\}} \mathcal{N}^{(k)}(p)$ are uniquely determined as the solution to the Dirichlet problem with boundary nodes $\mathcal{N}^{(m)}(p)$. By Lemma 4.2 with $J = \bigcup_{i \in \{0,...,m+1\}} \mathcal{N}^{(i)}(p)$, we have that the matrix $E = B^\top(\mathcal{N}^{(m)}(p); \mathcal{N}^{(m+1)}(p))$ has full rank, and so by Lemma 4.1, the matrix $D = -C^{-1}(\mathcal{N}^{(m)}(p); \operatorname{int} V(J))B^\top$ has full rank. By Lemma 2.2, all γ-harmonic germs at p can be extended to the $(m+1)$-th layer. Therefore, all γ-harmonic germs at p can be extended to the entire graph H, and so to the entire graph G. □

Corollary 4.4. *If $|\mathcal{N}^{(j)}(p)| = k$ for all $j \geq 1$ and k a constant, then the statement of Theorem 4.3 is a if and only if statement.*

Proof. In this case, $D_j = -C^{-1}(\mathcal{N}^{(j)}(p); \operatorname{int} V(H))B^\top$ is always a square matrix. Since all γ-harmonic germs can be globally extended, by Lemma 2.2, $D_1 = -C^{-1}(\mathcal{N}(p); \operatorname{int} V(J))B^\top$ has full rank, meaning D_1 is invertible. Define the graph $J := \bigcup_{i=\{0,1,2\}} \mathcal{N}^{(i)}(p)$, we have for $\vec{u}$ a potential function, $\vec{u}(\mathcal{N}^{(2)}(p)) = D\vec{u}(\mathcal{N}(p))$ and $\vec{u}(\mathcal{N}(p)) = D^{-1}\vec{u}(\mathcal{N}^{(2)}(p))$. Thus, let U_j be the subspace spanned by all vectors $\vec{u}(\mathcal{N}^{(j)}(p))$, then $\dim(U_2) = \dim(U_1) = k$. Continue inductively, we see that $\dim(U_j) = k$ for all $j \geq 1$, which requires D_j to have full rank for all $j \geq 1$, not just for $j = 1$. Thus, it is necessary that for all layers $\mathcal{N}^{(j)}(p))$ of H with $j \geq 2$, sets $A, B \in \mathcal{N}^{(j-1)}(p)$ described in Theorem 4.3 not exist.

□

It is a natural to ask when a germ has an unique global γ-harmonic continuation.

Theorem 4.5. *Let $\Gamma = (G, \gamma)$ be a resistor network and p be a strongly interior node. If all γ-harmonic germs at p are extendible, then the extension is unique if and only if $|\mathcal{N}(p)| = |\partial V|$.*

Proof. Referring to Lemma 2.2 and the discussion before it. Let $D := -(C^{-1}B^{\top})(\mathcal{N}(p); \partial V)$, then we have $\vec{u}(\mathcal{N}(p)) = D\vec{u}(\partial V)$. By the hypothesis of the theorem, all germs are extendible, and so D has full rank. Hence, the continuation is unique if and only if D is a square matrix (i.e. $|\mathcal{N}(p)| = |\partial V|$). □

All contents above have no assumption about G being circular planar. We now show a result, Theorem 4.7, that links criticality of cirular planar graphs to γ-harmonic extendibility.

A <u>flower</u> is a circular planar graph with at least one edge but no $\partial - \partial$ edges or ∂ spikes.

Lemma 4.6. *Given G a flower, and let v be an interior node of G. Let G_n be the graph with n boundary spikes added to v and still be circular planar. Then G_n is not critical for any n.*

Proof. By Lemma 8.6 of [3], every critical circular planar graph must have at least three $\partial - \partial$ edges or ∂ spikes. Thus G is not critical and so are G_1 and G_2. Now consider G_2 and name the two new ∂ nodes v_1 and v_2 and the two ∂ spikes $e(v_1 v)$ and $e(v_2 v)$. we can recover one of the spikes by contracting it to v, without loss of generality, we contract $e(v_1 v)$. We can do so because contracting $e(v_1 v)$ will break the connection between v_2 and other ∂ nodes except for the node v since v is the only ∂ node connected to v_2 after contracting $e(v_1 v)$. Now, we can recover the $e(v_2 v)$ because it is a $\partial - \partial$ edge. Since G_2 is not recoverable, there is some other edge $e(ij) \in E(G_2)$ such that deleting or contracting $e(ij)$ does not break any connection in G_2.

Now consider G_n, we want to show that deleting or contracting edge $e(ij)$ does not break any connection in G_n. Certainly, it does not break any connection which existed in G_2. Now let $(P; Q)$ be any connection on G_n which does not exist on G_2. If this connection exists, there can be at most one node $v_p \in \{v_3, ..., v_n\}$ have $v_p \in P$ and at most one node $v_q \in \{v_3, ..., v_n\}$ have $v_q \in Q$. Since all of the ∂ spikes are attached to the node v, they are connected to the rest of the graph in the same way. Without loss of generality, we may replace v_p and v_q with v_1 and v_2. The result is that $(P; Q)$ is a connection existing in G_2, which will not break by deleting or contracting $e(ij)$. Hence, no connection will break by deleting or contracting $e(ij)$, and this shows that G_n is not critical. □

Theorem 4.7. *Let G be a* critical *circular planar graph and $\Gamma = (G, \gamma)$ a resistor network. Let p be an interior node. Then all γ-harmonic germs at p can be globally extended.*

Proof. Define $H := \mathcal{O}_p(G)$, and by Remark 2.3, H is critical. If there is at least one layer $\mathcal{N}^{(j)}(p)$ of H with $j \geq 2$ such that there are disjoint sets of A, $B \in \mathcal{N}^{(j-1)}(p)$ with $\mathcal{N}(A) \cap \mathcal{N}^{(j)}(p) = \mathcal{N}(B) \cap \mathcal{N}^{(j)}(p)$, then take the smallest j for which this happens and define a graph $J := \bigcup_{k \in \{0,...,j\}} \mathcal{N}^{(k)}(p)$. So, $\partial V(J) = \mathcal{N}^{(j)}(p)$. The graph J is critical since it is a subgraph (Definition 2.1) of a critical graph H [2].

If $A \cup B = \mathcal{N}^{(j)}(p)$, then J does not have any ∂ spikes since every ∂ nodes is shared by at least one node in A and one node in B. Note that deleting $\partial - \partial$ edges or contracting ∂ spikes does not affect criticality. After deleting all $\partial - \partial$ edges, J is reduced to a flower and so it is not critical, which is a contradiction. Otherwise, let $L := \mathcal{N}^{(j)}(p) \backslash (A \cup B)$, we use the following algorithm to reduce J to a simpler form while preserving criticality (note that the graph J changes constantly with the algorithm):

1. Delete all $\partial - \partial$ edges.
2. Contract all ∂ spikes, but do not contract a boundary spike to the node p.
3. Repeat steps 1 and 2 until they can no longer be performed.

Let $\tilde{J}$ be reduced graph of J by the algorithm above. The graph $\tilde{J}$ is a flower with the addition of n boundary spikes to the interior (may not be strongly interior after the algorithm) node p for some $n \geq 0$. By Lemma 4.6, $\tilde{J}$ is not critical, and this is a contradiction. So, sets A, $B \in \mathcal{N}^{(j-1)}(p)$ described earlier cannot exist for any $j \geq 2$. Therefore, by Theorem 4.3, all γ-harmonic germs at p can be globally extended if G is critical.

□

Question 4.8. What are some other geometric criterions that ensures all γ-harmonic germs at an interior node p have global γ-harmonic continuation? We gave a sufficient condition in Theorem 4.3. But, it is not good enough because, given a graph, it is hard to check whether the hypotheses of the theorem are met. Also, it maybe too sufficient since it excludes a class of extendible graphs such as the one in Figure 2. It may be worth to look at what graphs are in that class. The necessary and sufficient condition given in Corollary 4.4 is too restrictive. Another way to work on this problem may be to study what will amalgamating networks do to γ-harmonic extendibility. There are some obvious amalgamations, but whether we can use this technique to extract more information is unclear to me. It is also to note that all the results above was discovered in trying to proof Theorem 4.7. However, we don't have a nice geometric concept like criticality handy for non-planar graphs, so it is hard to formulate a conjecture. We can only look for things that stand out amongst others. Furthermore, Lemma 2.2 gives a necessary and sufficient algebraic condition for γ-harmonic extendibility. I avoided handling the matrix $D := -C^{-1}(\partial V(G'); \operatorname{int} V(G))B^{\top}$ by using Lemma 4.1. In doing so, I was restricted to consider the geometry of the graph in layers. I have not been able to find more interesting properties by only considering each layer at a time, hence I think in order to obtain a better geometric condition, we may need to look at a bigger part of the matrix D. Or, it maybe possible that one can find some other algebraic criterions involved, but I do not have an idea about this. Finally, we may be able to use some results from the inverse problem for random walk networks. A small connection between random walk networks and γ-harmonic extendibility is established in the next section.

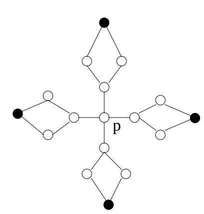

Figure 2. An extendible graph not predicted by Theorem 4.3 (cf. Question 4.8).

5. Link to Random Walk Networks

5.1. Background.

The background information discussed here is more or less taken from [4], [5], and [6]; more detailed discussions can be found in them.

Definition 5.1. (Random Walk Network) Let $G = (\operatorname{int} V, \partial V, E)$ be a directed edge graph-with-boundary which satisfy the following properties:

(1) If two nodes $v_i, v_j \in \operatorname{int} V$, $i \neq j$, are connected in G, they are connected by exactly two edges (one for each direction).

(2) If two nodes $v_i \in \operatorname{int} V$, $v_j \in \partial V$ are connected in G, they are connected by exactly one edge $e_{ij} \in E$ (the edge directed from node v_i to node v_j).
(3) For all $v_i \in \partial V$, the only edge from node v_i is e_{ii} (a loop from node v_i back to itself).
(4) For all $v_i \in \operatorname{int} V$, there exists a set of directed edges linking node v_i to the boundary.

We assign for each directed edge e_{ij} a transition probability p_{ij} and form a *transition matrix*, P, with p_{ij} being an entry of the i-th row and j-th column. So,

$$P = \begin{array}{c c} & \begin{array}{cc} \partial V & \operatorname{int} V \end{array} \\ \begin{array}{c} \partial V \\ \operatorname{int} V \end{array} & \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix} \end{array}.$$

Then the *absorbing matrix*, Λ, is given by

$$\Lambda = \lim_{n\to\infty} P^n = \lim_{n\to\infty} \begin{bmatrix} I & 0 \\ (I + Q + \cdots + Q^{n-1})R & Q^n \end{bmatrix} = \begin{bmatrix} I & 0 \\ (I - Q)^{-1}R & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ N & 0 \end{bmatrix}.$$

A proof of the convergence of $\sum_{n=0}^{\infty} Q^n$ can be found in [4]. Note that an entry n_{ij} of the matrix N represents the probability of a random walk starting from the interior node i and ending (or absorbed) by the boundary node j. The *inverse problem* for a random walk network would be to recover the transition matrix P given the absorbing matrix Λ.

5.2. **An Equivalence.**
Let Kirchhoff's matrix for a possibly non-symmetric electrical network be

$$K = \begin{array}{c c} & \begin{array}{cc} \partial V & \operatorname{int} V \end{array} \\ \begin{array}{c} \partial V \\ \operatorname{int} V \end{array} & \begin{bmatrix} A & B \\ L & C \end{bmatrix} \end{array}.$$

A relation between conductance, γ_{ij}, of an electrical network and transition probability, p_{ij}, of a random walk network is given by

$$p_{ij} = \frac{\gamma_{ij}}{\sum_{i\sim k} \gamma_{ik}} = \frac{\gamma_{ij}}{\gamma_i}. \tag{2}$$

Suppose $\operatorname{int} V = \{i_1, ..., i_n\}$, we define

$$\Omega = \begin{bmatrix} \gamma_{i_1} & & 0 \\ & \ddots & \\ 0 & & \gamma_{i_n} \end{bmatrix}.$$

Then, using the same notation as those in Kirchoff's matrix and the transition matrix, we have by equation 2

$$I - Q = \Omega^{-1}C, \quad (I - Q)^{-1} = C^{-1}\Omega, \quad \text{and} \quad R = \Omega^{-1}L. \tag{3}$$

Theorem 5.2. *Let $\Gamma = (G, \gamma)$ be a directed electrical network and $\Pi = (G, p)$ be the corresponding random walk network. Let i be a strongly interior node of G, then all γ-harmonic germs at i are extendible if and only if probabilities p_{ik} can be recovered for all $k \in \mathcal{N}(i)$.*

Proof. For a strongly interior node i, denote $M := \Lambda(\mathcal{N}(i); \partial V)$, we have

$$\Lambda(i; \partial V) = P(i; \mathcal{N}(i))M \quad \Rightarrow \quad \Lambda(i; \partial V)M^* = P(i; \mathcal{N}(i))MM^*, \tag{4}$$

where M^* is the conjugate of M. We now show MM^* is invertible if and only if M has full rank. There is a singular value decomposition of M as $M = U\Sigma V^*$ where U and V are unitary matrices. The matrix Σ has the same size as M, and it consists of non-negative numbers on the diagonal and zeros off diagonal. We also have

$$MM^* = (U\Sigma V^*)(V\Sigma^* U^*) = U\Sigma\Sigma^* U^*$$

and note that the size of MM^* is the same as the size of $\Sigma\Sigma^*$. Since $|\mathcal{N}(i)| \leq |\partial V|$, the matrix $\Sigma\Sigma^*$ is invertible if and only if Σ has full rank. Because U and V are invertible, $rank(\Sigma) = rank(M)$ and $rank(MM^*) = rank(\Sigma\Sigma^*)$. Hence, MM^* is invertible if and only if M has rull rank.

We then have by Equation 4

$$P(i; \mathcal{N}(i)) = [\Lambda(i; \partial V)M^*][MM^*]^{-1}$$

is recoverable if and only if M has full rank. Then applying Equations 3 gives,

$$M = \Lambda(\mathcal{N}(i); \partial V) = (I - Q)^{-1}(\mathcal{N}(i); \operatorname{int} V)R = [C^{-1}(\mathcal{N}(i); \operatorname{int} V)\Omega][\Omega^{-1}L]. = C^{-1}(\mathcal{N}(i); \operatorname{int} V)L.$$

Hence $rank[\Lambda(\mathcal{N}(i); \partial V)] = rank[C^{-1}(\mathcal{N}(i); \operatorname{int} V)L]$. Since Lemma 2.2 applies to directed network as well, we have proved the theorem. □

6. Acknowledgment

I wish to thank Owen Biesel, Ryan Card, Ernie Esser, Peter Mannisto, and Jim Morrow for their extreme patience and numerous helpful discussions.

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