Parameterizing Response Matrices

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Abstract

This paper looks at relationships of entries in the response matrix and determines which entries must be written in terms of the others. Information about these parameterizations can be used to recover graphs with only partial information in the response matirx. We analyzed these relationships for the n-gon in n-gon networks, and for the annular network with two circles and three rays.

Contents

1 Introduction

Analysis of the relationships among the entries of a Dirichlet to Neumann map for a resistor network allows us to determine which entries are necessry for recoverability. If we generalize these relationships to call networks of a specific structure, then we can recover graphs with only partial information in the response matrix. Edward B. Curtis and James A. Morrow generalized the parameterization of square lattice networks in [2]. The Dirichlet to Neumann map for a square lattice is represented by a $4n$ x $4n$ matrix with nodes numbered as follows, 1 through n on the north side, $n+1$ through $2n$ on the west side, $2n+1$ through $3n$ on the south side, and $3n+1$ through $4n$ on the east side as in figure 1.

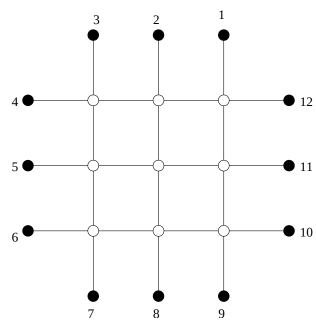

Figure 1: Numbering of nodes for a square lattice with $n = 3$

In figure 2 we show the block structure for square lattices, where the blocks correspond to the sides of the graph.

G	A	B	C
A^T	H	E	D
B^T	E^T	I	F
C^T	D^T	F^T	J

Figure 2: Block Structure of Response Matrix for Square Lattice Networks

One possible parameterization of the response matrix is the following as defined by Curtis and Morrow [2]:

- All the entries of B.
- All the entries of A on or above the main antidiagonal.
- All the entries of C on or below the main antidiagonal.
- All the entries of D on the main antidiagonal.

Figure 3 gives a visual representation of the parameters in the response matrix for any square lattice.

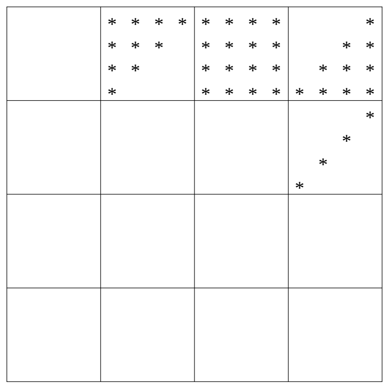

Figure 3: Parameter Positioning for Square Lattices

2 The n-gon in n-gon Networks

2.1 General Definition for *Kirchhoff* Matrix

A *network* is a pair $\Gamma = (G, \gamma)$ where G is a graph and γ is a positive conductivity function defined on all cables in G. If G is a simple graph, then γ is defined on G's edges. If G is not simple, then γ assigns a conductivity to each *cable* of G, not to individual edges [6]. We will adopt Jenny French and Shen Pan's definition for the *Kirchhoff* matrix, so we can use their Characterization for the response matrix. [4] The *Kirchhoff matrix* for Γ, denoted K, is defined such that

$$K_{ij} = \begin{cases} \gamma_{ij} & i \sim j \\ -\sum_{k \neq i} K_{ik} & i = j \\ 0 & i \nsim j \text{ and } i \neq j \end{cases} \tag{1}$$

It's useful to write the Kirchhoff matrix in the following block form:

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

If all interior nodes are numbered such that they appear in the C block and all boundary nodes are in the A block, then the response matrix is just the Schur Compliment of K in C.

$$\Lambda = A - BC^{-1}B^T$$

2.2 Characterization of the Response Matrix for n-gon in n-gon Networks

The following is copied verbatim from Jenny French and Shen Pan's paper [4]. For further explanation, including a construction of the matrix P_k see [4] sections 2, 4, and 6 (particularly Theorems 4.1 and 6.1).

Theorem 2.1 *If $\Lambda = (\lambda_{ij})$ is the response matrix of an n-gon in n-gon network with conventionally numbered nodes and positive real conductivities, and the matrix P_k with $k = n + 1$ is formed from the entries in Λ, then for all $j \leq k$, the determinants p_j and q_j are strictly positive.* [3]

In addition to satisfying the sign conditions for the response matrix, we must also take into account the relations defined on all response matrices for n-gon in n-gon networks.

Theorem 2.2 *The relations defined on any response matrix for an n-gon in n-gon network are modeled as follows:*
For $i \leq n$,
$\lambda_{i,(i \ mod \ n)+1}\lambda_{n+i,n+(i \ mod \ n)+1} = \lambda_{(i \ mod \ n)+1,n+i}\lambda_{i,n+(i \ mod \ n)+1}$

The sign conditions coupled with the relations provide a characterization for all response matrices of the n-gon in n-gon networks.

2.3 Triangle-in-Triangle

Consider the Triangle-in-Triangle Network with boundary nodes numbered as in Figure 4. There are 12 edges and thus 12 conductivities to recover, and there are 15 entries in the response matrix. We should be able to find 15-12, or 3 entries which are determined by the rest of the entries in the matrix.

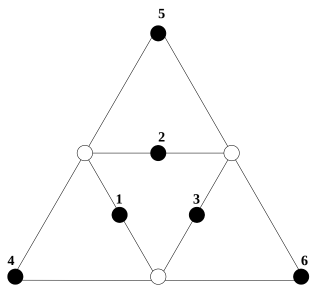

Figure 4: Triangle-in-Triangle Network

The relations for the Triangle-in-Triangle are as follows:

$$\lambda_{1,2}\lambda_{4,5} = \lambda_{2,4}\lambda_{1,5} \tag{2}$$

$$\lambda_{2,3}\lambda_{5,6} = \lambda_{3,5}\lambda_{2,6} \tag{3}$$

$$\lambda_{3,1}\lambda_{6,4} = \lambda_{1,6}\lambda_{3,4} \tag{4}$$

Now consider the order for picking parameters and uniquely determining entries in the response matrix:

1-11. pick parameters > 0
12. pick $\lambda_{2,5} > 0$ and $\lambda_{2,5} > \frac{\lambda_{2,3}\lambda_{5,6}}{\lambda_{3,6}}$ and $\lambda_{2,5} > \frac{\lambda_{1,2}\lambda_{4,5}}{\lambda_{1,4}}$ and $\lambda_{2,5} > \frac{\lambda_{1,2}\lambda_{4,5}\lambda_{3,6}+\lambda_{1,4}\lambda_{2,3}\lambda_{5,6}}{\lambda_{1,4}\lambda_{2,3}}$
13. determine $\lambda_{1,5} = \frac{\lambda_{1,2}\lambda_{4,5}}{\lambda_{2,4}}$
14. determine $\lambda_{2,6} = \frac{\lambda_{2,3}\lambda_{5,6}}{\lambda_{3,5}}$
15. determine $\lambda_{1,6} = \frac{\lambda_{3,1}\lambda_{6,4}}{\lambda_{3,4}}$

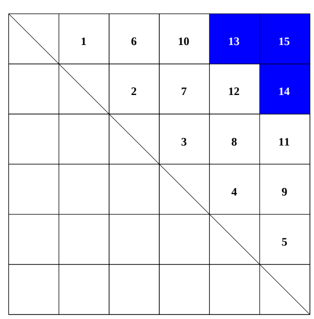

Figure 5: Order of Parameter Placement for the Triangle-in-Triangle Network

2.4 Square-in-Square

Consider the Square-in-Square Network with boundary nodes numbered as in Figure 6. There are 16 edges and thus 16 conductivities to recover, and 28 entries in the response matrix. Unlike the Triangle-in-Triangle the response matrix for this network has a series of zero entries. We will place a zero in the response matrix at $\lambda_{i,j}$, when there is no path connecting boundary nodes i and j in the graph. We should be able to uniquely determine 4 entries in the response.

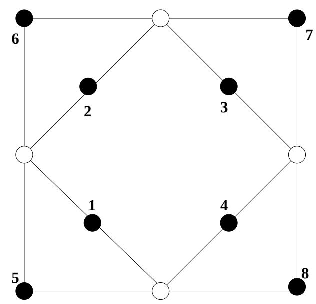

Figure 6: Square-in-Square Network

The relations for the Square-in-Square are as follows:

$$\lambda_{1,2}\lambda_{5,6} = \lambda_{2,5}\lambda_{1,6} \tag{5}$$

$$\lambda_{2,3}\lambda_{6,7} = \lambda_{3,6}\lambda_{2,7} \tag{6}$$

$$\lambda_{3,4}\lambda_{7,8} = \lambda_{4,7}\lambda_{3,8} \tag{7}$$

$$\lambda_{4,1}\lambda_{8,5} = \lambda_{1,8}\lambda_{4,5} \tag{8}$$

The order for picking parameters and uniquley determining entries in the response matrix is very similar to that of the Triangle-in-Triangle.

1-14. pick parameters > 0
15. pick $\lambda_{4,8} > 0$ and $\lambda_{4,8} > \lambda_{7,8}$
16. pick $\lambda_{2,6} > 0$ and $\lambda_{2,6} > \frac{\lambda_{2,3}\lambda_{6,7}}{\lambda_{3,7}}$ and $\lambda_{2,6} > \frac{\lambda_{2,3}\lambda_{6,7}\lambda_{4,8}}{\lambda_{3,7}\lambda_{4,8}-\lambda_{3,7}\lambda_{7,8}}$ and $\lambda_{2,6} > \frac{\lambda_{1,2}\lambda_{5,6}\lambda_{3,7}+\lambda_{1,5}\lambda_{2,3}\lambda_{6,7}}{\lambda_{3,7}\lambda_{1,5}}$ and $\lambda_{2,6} > \frac{\lambda_{1,2}\lambda_{5,6}\lambda_{3,7}\lambda_{4,8}+\lambda_{1,5}\lambda_{2,3}\lambda_{6,7}\lambda_{4,8}-\lambda_{1,2}\lambda_{5,6}\lambda_{3,4}\lambda_{7,8}}{\lambda_{3,7}\lambda_{1,5}\lambda_{4,8}-\lambda_{3,7}\lambda_{1,5}\lambda_{7,8}}$
17. determine $\lambda_{1,6} = \frac{\lambda_{1,2}\lambda_{5,6}}{\lambda_{2,5}}$
18. determine $\lambda_{2,7} = \frac{\lambda_{2,3}\lambda_{6,7}}{\lambda_{3,6}}$
19. determine $\lambda_{3,8} = \frac{\lambda_{3,4}\lambda_{7,8}}{\lambda_{4,7}}$
20. determine $\lambda_{1,8} = \frac{\lambda_{1,4}\lambda_{5,8}}{\lambda_{4,5}}$

	1	0	8	13	17	0	20
		2	0	9	16	18	0
			3	0	10	14	19
				4	0	11	15
					5	0	12
						6	0
							7

Figure 7: Order of Parameter Placement for the Square-in-Square Network

2.5 Pentagon-in-Pentagon

Now we will look at the Pentagon-in-Pentagon network with boundary nodes ordered as in Figure 8. There are 20 edges and thus 20 conductivities to recover. There are 45 total entries in the response matrix. As in the Square-in-Square network there are a series of zeros corresponding to nodes in the graph which have no connecting path. There are 20 total zeros. So, we should be able to uniquely determing 5 entries in the response.

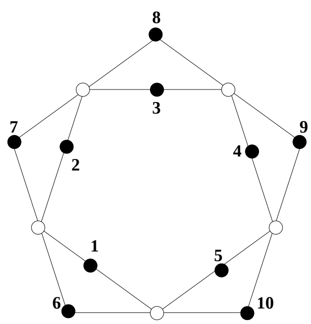

Figure 8: Pentagon-in-Pentagon Network

The relations for the Pentagon-in-Pentagon are as follows:

$$\lambda_{1,2}\lambda_{6,7} = \lambda_{2,6}\lambda_{1,7} \tag{9}$$

$$\lambda_{2,3}\lambda_{7,8} = \lambda_{3,7}\lambda_{2,8} \tag{10}$$

$$\lambda_{3,4}\lambda_{8,9} = \lambda_{4,8}\lambda_{3,9} \tag{11}$$

$$\lambda_{4,5}\lambda_{9,10} = \lambda_{5,9}\lambda_{4,10} \tag{12}$$

$$\lambda_{5,1}\lambda_{10,6} = \lambda_{1,10}\lambda_{5,6} \tag{13}$$

The order for picking parameters follows the same patterns set by both the Triangle-in-Triangle and Square-in-Square networks. The order is as follows:

1-17. pick parameters > 0
18. pick $\lambda_{4,9} > 0$ and $\lambda_{4,9} > \frac{\lambda_{8,9}\lambda_{3,4}}{\lambda_{3,8}}$
19. pick $\lambda_{5,10} > 0$ and $\lambda_{5,10} > \frac{\lambda_{9,10}\lambda_{4,5}\lambda_{3,8}}{\lambda_{3,8}\lambda_{4,9}-\lambda_{8,9}\lambda_{3,4}}$
20. pick $\lambda_{2,7} > 0$ and $\lambda_{2,7} > \frac{\lambda_{4,9}\lambda_{2,3}\lambda_{7,8}}{\lambda_{4,9}\lambda_{3,8}-\lambda_{8,9}\lambda_{3,4}}$
and $\lambda_{2,7} > \frac{\lambda_{7,8}\lambda_{2,3}\lambda_{4,9}\lambda_{5,10}-\lambda_{7,8}\lambda_{2,3}\lambda_{4,5}\lambda_{9,10}}{\lambda_{3,8}\lambda_{4,9}\lambda_{5,10}-\lambda_{8,9}\lambda_{3,4}\lambda_{5,10}}$ and $\lambda_{2,7} > \frac{\lambda_{1,6}\lambda_{4,9}\lambda_{2,3}\lambda_{7,8}+\lambda_{4,9}\lambda_{3,8}\lambda_{1,2}\lambda_{6,7}-\lambda_{8,9}\lambda_{3,4}\lambda_{1,2}\lambda_{6,7}}{\lambda_{1,6}\lambda_{4,9}\lambda_{3,8}-\lambda_{1,6}\lambda_{8,9}\lambda_{3,4}}$
and $\lambda_{2,7} > \frac{\lambda_{1,6}\lambda_{7,8}\lambda_{5,10}\lambda_{4,9}\lambda_{2,3}+\lambda_{5,10}\lambda_{4,9}\lambda_{3,8}\lambda_{1,2}\lambda_{6,7}-\lambda_{1,6}\lambda_{7,8}\lambda_{9,10}\lambda_{4,5}\lambda_{2,3}-\lambda_{9,10}\lambda_{4,5}\lambda_{3,8}\lambda_{1,2}\lambda_{6,7}-\lambda_{8,9}\lambda_{5,10}\lambda_{3,4}\lambda_{1,2}\lambda_{6,7}}{\lambda_{1,6}\lambda_{5,10}\lambda_{4,9}\lambda_{3,8}-\lambda_{1,6}\lambda_{9,10}\lambda_{4,5}\lambda_{3,8}-\lambda_{8,9}\lambda_{5,10}\lambda_{3,4}\lambda_{1,6}}$
21. determine $\lambda_{1,7} = \frac{\lambda_{1,2}\lambda_{6,7}}{\lambda_{2,6}}$
22. determine $\lambda_{2,8} = \frac{\lambda_{2,3}\lambda_{7,8}}{\lambda_{3,7}}$
23. determine $\lambda_{3,9} = \frac{\lambda_{3,4}\lambda_{8,9}}{\lambda_{4,8}}$
24. determine $\lambda_{4,10} = \frac{\lambda_{4,5}\lambda_{9,10}}{\lambda_{5,9}}$
25. determine $\lambda_{1,10} = \frac{\lambda_{1,5}\lambda_{6,10}}{\lambda_{5,6}}$

	1	0	0	10	16	21	0	0	25
		2	0	0	11	20	22	0	0
			3	0	0	12	17	23	0
				4	0	0	13	18	24
					5	0	0	14	19
						6	0	0	15
							7	0	0
								8	0
									9

Figure 9: Order of Parameter Placement for the Pentagon-in-Pentagon Network

2.6 Parameters for the Response Matrix of n-gon in n-gon Networks

For any n-gon in n-gon network, the response matrix will always be $2n$ x $2n$. It will always have $n + (n - 1)$ diagonals above the main diagonal, which we will refer to as *upper diagonals*. They are numbered in increasing numerical order, with the 1^{st} upper diagonal directly above the main diagonal and the $n + (n - 1)^{st}$ upper diagonal consisting only of the entry $\lambda_{1,2n}$.

There will always be a total of $2(n-3)$ upper diagonals consisting only of zeros:

- One block of $(n-3)$ diagonals with all entries equaling zero will start at the 2^{nd} upper diagonal and end at the $(n-2)^{nd}$ upper diagonal.
- Another block of $(n-3)$ diagonals with all entries equaling zero will start at the $(n+2)^{nd}$ upper diagonal and end at the $n+(n-2)^{nd}$ upper diagonal.

The placement of parameters in the response matrix:

1. The 1^{st} upper diagonal.
2. The $(n-1)^{st}$ upper diagonal.
3. The n^{th} upper diagonal.

The entries in the response matrix that are involved in the relations do not appear in the equations for the sign conditions. Thus, when picking our last parameter we can ensure that all sign conditions will be satisfied. If we always pick $\lambda_{2,(n+2)}$ last we will be able to fix all the necessary signs with that one choice. This is because the entry $\lambda_{2,(n+2)}$ appears in all equations for sign conditions, and it can always be written as $\lambda_{2,(n+2)} >$ expressions that are in terms of previously picked parameters. Thus, we can always choose $\lambda_{2,(n+2)}$ to be greater that the maximum of these expressions.

3 The Two Circle Three Ray Network

3.1 General Definition for the Kirchhoff Matrix

We need to change our interpretation of the Kirchhoff matrix from that used for the n-gon in n-gon networks, because we will be using Ernie Esser's conjecture about the characterization of the response matrix for the G(3,2) [3]. For a network with n nodes, the Kirchhoff matrix K is an $n \times n$ symmetric matrix formed by taking for $i \neq j$

$$K(i,j) = \left\{ \begin{array}{ll} -\gamma(i,j) & \text{if there is an edge from i to j} \\ 0 & \text{if no such edge exists} \end{array} \right\}$$

Then the diagonal entries are chosen such that each row sums to zero. It's useful to write the Kirchhoff matrix in the following block form:

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

If all interior nodes are numbered such that they appear in the C block and all boundary nodes are in the A block, then the response matrix is just the Schur Compliment of K in C.

$$\Lambda = A - BC^{-1}B^T$$

For simplification purposes we will represent the response matrix Λ as

$$\Lambda = \begin{bmatrix} \Sigma & a & b & c & d & e \\ a & \Sigma & f & g & h & p \\ b & f & \Sigma & q & r & s \\ c & g & q & \Sigma & t & u \\ d & h & r & t & \Sigma & v \\ e & p & s & u & v & \Sigma \end{bmatrix} \tag{14}$$

3.2 Characterization of the Response Matrix for G(3,2)

For our parameterization we must assume Ernie Esser's conjecture about the characterization of the response matrix for G(3,2) is true [3]. The conjecture came from a G(3,2) with boundary nodes numbered as in Figure 10.

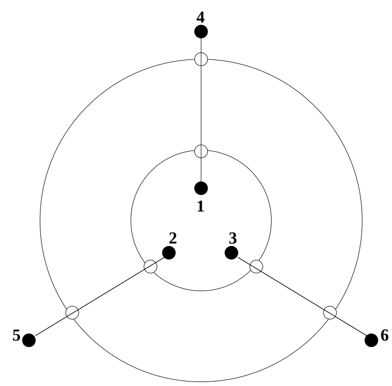

Figure 10: G(3,2)

In [3] it was found that there are 13 sign conditions which come in the form of determinantal inequalities. They are

1. $\det \Lambda(1,2,3;4,5,6) < 0$
2. $\det \Lambda(1,2,5;3,4,6) > 0$
3. $\det \Lambda(2,1,4;3,5,6) > 0$
4. $\det \Lambda(2,3,5;1,4,6) > 0$
5. $\det \Lambda(2,3,6;1,5,4) > 0$
6. $\det \Lambda(1,3,4;2,5,6) > 0$
7. $\det \Lambda(1,3,6;2,4,5) > 0$
8. $\det \Lambda(1,2;4,5) > 0$

9. $\det \Lambda(2,3;5,6) > 0$

10. $\det \Lambda(1,3;4,6) > 0$

11. $\det \Lambda(1,4;2,5) > 0$

12. $\det \Lambda(1,4;3,6) > 0$

13. $\det \Lambda(2,5;3,6) > 0$

These determinants, however, aren't enough to characterize Λ. It was also found in [3] that there is existance of one relation such that for any Λ for G(3,2) labeled as in equation 14, relation R can be expressed as

$$R = aqv + fdu + bpt - tef - vgb - ura - dpq + gre = 0 \tag{15}$$

This is also equivaltent to saying,

$$R = \det(\Lambda(1,2,4;3,5,6)) - \det(\Lambda(1,4,5;2,3,6)) = 0$$

The following is Ernie Esser's conjecture concerning the characterization of the response matrix for G(3,2), which will be used in our parameterization.

Conjecture 3.1 *The relation*

$$R = \det(\Lambda(1,2,4;3,5,6)) - \det(\Lambda(1,4,5;2,3,6)) = 0$$

combined with the 13 previously mentioned sign conditions constitutes a characterization of the response matrix for $G(3,2)$ [3].

3.3 Parameterization for G(3,2)

Consider the G(3,2) Network with boundary nodes numbered as in Figure 10. There are 15 edges and thus 15 conductivities to recover. There are also 15 entries in the response matrix. We have 13 sign conditions and 1 relation and with that information should be able to uniquely determine 1 entry.

Remark 3.1 There are 3 entries in the response which are not involved in the relation. They are $\lambda_{1,4}$, $\lambda_{2,5}$, and $\lambda_{3,6}$. By the notation in equation 14 they are denoted c, h, and s. Our method is to pick 11 entries in the relation, satisfaying any necessary determinantal inequatlities as we go. Then, we will satisfy the relation with our 12^{th} pick. Finally, we will use c, h, and s to fix any left over sign conditions.

The following is the order in which we can pick entries in the response matrix.

1. pick $e < 0$

2. pick $p < 0$
3. pick $d < 0$
4. pick $u < 0$
5. pick $r < 0$
6. pick $v < 0$
7. pick $t < 0$
8. pick $q < 0$
9. pick $f < 0$
   and $f < \frac{pr}{v}$ (satisfies sign condition 13)
10. pick $b < 0$
    and $b < \frac{eq}{u}$ (satisfies sign condition 12)
    and $b < \frac{-dfu-dpq+eft}{pt}$ (will help to satisfy sign conditions 3 and 5)
    and $b \neq \frac{re}{v}$ (will help satisfy the relation)
11. pick $a < 0$

    If $(ur - qv) = 0$ (or) $(ur - qv)$ has a different sign than $(re - vb)$ (or) both $(ur - qv)$ and $(re - vb)$ both have positive sign and $d(ur - qv) < t(re - vb)$, then pick $a < \frac{d(-fdu-bpt+tef+dpq}{t(re-vb)(1-\frac{d(ur-qv)}{t(re-vb)}}$.

    If $(ur - qv)$ and $(re - vb)$ both have negative sign (including the case where $\frac{d(ur-qv)}{t(re-vb)} = 1$) then pick $a > \frac{d(-fdu-bpt+tef+dpq}{t(re-vb)(1-\frac{d(ur-qv)}{t(re-vb)}}$.
12. determine $g = \frac{-aqv-fdu-bpt+tef+ura+dpq}{re-vb}$ (satisfies both the relation and sign condition 11)

    We know that $(re - vb)$ will not equal zero because of the way in which we picked b.

    We also know that $(-aqv - fdu - bpt + tef + ura + dpq)$ will always be negative because of our choices of both a and b.
13. pick $h < 0$
    and $h > \frac{-bpt-dfu+dpq+eft}{eq-bu}$ (satisfies sign conditions 3 and 5)

    We know that $(eq - bu)$ will always be positive by sign condition 12.

    We also know that $(-bpt - dfu + dpq + eft)$ will always be negative because of the way in which we picked b.

14. pick $c < 0$
    and $c < \frac{dg}{h}$ (satisfies sign condition 8)
    and $c < \frac{bpt+egr-bgv-eft}{rp-fv}$ (satisfies sign condition 2)
    and $c < \frac{aru+dpq-aqv-dfu}{rp-fv}$ (satisfies sign condition 7)

    We know that $(rp - fv)$ is negative by sign condition 13.

15. pick $s < 0$
    and $s < \frac{pr}{h}$ (satisfies sign condition 9)
    and $s < \frac{eq}{c}$ (satisfies sign condition 10)
    and $s < \frac{gbv+pqd-aqv-pbt}{dg-at}$ (satisfies sign condition 7)
    and $s < \frac{dfu+erg-aru-eft}{dg-at}$ (satisfies sign condition 6)
    and $s < \frac{dgs+ehq-dpq-egr}{ch-dg}$ (satisfies sign condition 1)

    We know that $(dg - at)$ is neqative by sign condition 11.

    We also know that $(ch - dg)$ is positive by sign condition 4.

Remark 3.2 The above is not a parameterization for all G(3,2) response matrices.

The determinant corresponding to $(re-vb)$ can in fact equal zero, but we didn't allow this case which enabled us to determing g. But, this is not a problem. We can account for this case by picking a different entry to uniquely determine. When determining a different entry in the relation we will end up with a different determinant in the denominator that can also be positive, negative, or equal zero. This parameterization will allow for the case that $(re - vb) = 0$. We will then do as we did before and not allow this new determinant to equal zero. Thus, if we parameterize the matrix 12 times, each time determining a different entry in the relation, we will be restricting different determinants with each parameterization. Every possibility among these restricted determinants will be accounted for.

We also have one problem that does not have a clear solution. When picking a we didn't account for the case when $(ur - qv)$ and $(re - vb)$ both have positive sign and $d(ur - qv) > t(re - vb)$. At first it appeared that this case could never occur, but it was verified numerically using MATLAB that although it's very rare this situation will arise. If we had account for the above case we would be forced to pick $a >$ something positive and $a < 0$, which is an obvious contradiction. There are a couple possible explanations for this problem.

1. Our method for picking entries in the response matrix was incorrect and there is a better ordering.

2. The conjecture made in [3] was incorrect. There might be more sign conditions involved the the characterization for the G(3,2) response matrix.

4 Further Research

1. The most obvious question is how to give a full parameterization for the G(3,2) network. It appeared that the method used above was the best way to pick entries, but there may be something better. But, before looking for different parameterizations it would be much more useful to either prove the characterization conjecture in [3], or find more conditions for a characterization of the G(3,2) response matrix.

2. It would also be interesting to look at the G(n,n/2) networks, and provide parameterizations for their response matrix. But, before tackling any parameterization it is important to find a prove their characterization.

3. There are other networks with lots of symmetry that would probably produce nice parameterizations. This notion of picking parameters could be applied to both 3-d lattice networks and hexagonal networks.

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