Circular Duals of Circularly Embedded Graphs

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Abstract

This paper discusses construction of a dual graph for circularly embedded graphs, providing a definition of a circular dual. We will give necessary and sufficient conditions for determining whether a graph is the dual of its dual.

Contents

1 Introduction **3**

2 Applications of the Euler-Poincare Characteristic Formula **5**

3 Definition and Significance of Supercellular **6**

1 Introduction

Before beginning a discussion of circular dual graphs, it is important to give a few definitions concerning our embeddings of graphs and the construction of their duals.

Definition 1.1 For our purposes the graphs will only be embedded on orientable Reimann surfaces of genus g, where edges in the graph meet only at vertices to which they are incident.

Definition 1.2 If we remove the vertices and edges of the graph from the surface and the remainder is a disjoint union of topological discs, we call this a *cellular embedding.* The discs left behind after removal of vertices and edges will be referred to as *faces.*

Definition 1.3 When embedding a graph with boundary, if all the boundary vertices lie on the arcs of an empty disc, we will call this a *circular embedding.* The empty disc will be referred to as the *boundary disc,* and its boundary is called the *boundary circle.*

Remark 1.1 In the case of a circular cellular embedding, the boundary disc will not be counted as a face in the graph. Also, arcs of the boundary circle will not be counted as edges in the graph, but they will be used as psuedo-edges to form faces of the graph which border on the boundary circle.

Definition 1.4 In a circular cellular embedding if a face borders on the boundary circle, we will call it a *boundary face,* denoted ∂F, and a face not bording the boundary circle is an *interior face,* denoted V^o.

Instead of adopting the topologist's definition of a dual graph, we will define a *circular dual* in a way that is analogous to Curtis and Morrow's construction of dual graphs in the circular planar case. [1]

Definition 1.5 Given a circular cellular embedding of a graph with boundary, construct the *circular dual* as follows:

- Let $V^\perp$ be the set of vertices in the circular dual graph, with each vertex corresponding to a face in the primal graph. Also, let the circular dual boundary vertices (denoted $\partial(V^\perp)$) correspond to primal boundary faces, and the circular dual interior vertices (denoted $(V^\perp)^o$) correspond to primal interior faces. Place each element of $(V^\perp)^o$ in an interior face in the primal graph. As for the elements of $\partial(V^\perp)$, place each vertex for the circular dual graph on the arc of the boundary circle bording a primal boundary face. In the event that a face in the primal graph borders on the boundary circle in more than one arc, there is a choice of arcs on which to place the boundary vertex for the dual graph.

- Let $E^{\perp}$ be the set of edges in the dual graph, where each edge is constructed between dual vertices which correspond to faces in the primal graph which border on a common edge. Thus, in drawing an edge in the dual graph you are effectivley crossing each edge in the primal graph exactly once.

  This is a construction of a *circular dual* with boundary, which is embedded on the same surface as the primal graph.

Lemma 1.1 Let $(F^{\perp})^o$ be the set of interior faces in the dual graph. By our construction, each interior face in the dual graph corresponds to an interior vertex in the primal graph.

Proof: When constructing the edges of the dual graph, we are effectively cutting the faces of the primal graph into wedges. Each of these wedges has exactly one node from both the primal and dual graph as part of its boundary.
The first line of Figure 1, shows that concatenating two wedges with identified

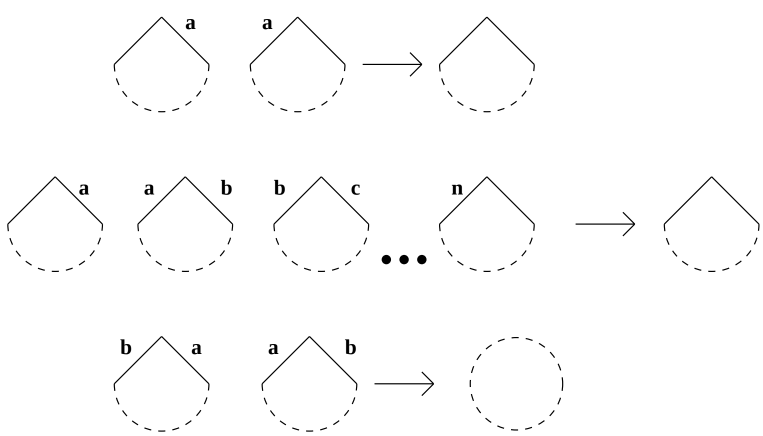

Figure 1: Concatenating Two Wedges with Identified Edges

edge a, will give back a wedge. Then, by induction it is easy to see that concatenating a series of wedges, will also result in a wedge. Thus, if we combine two wedges with two identified edges, the result will be a topological disc.
The same arguement can be applied to the wedges formed by the the construction of the dual graph. The result is a topological disc, or a face in the dual graph (see figure 2). □

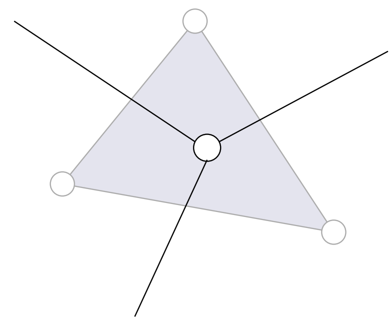

Figure 2: An Interior Face in the Dual Graph Corresponding to an Interior Vertex in the Primal Graph

2 Applications of the Euler-Poincare Characteristic Formula

Theorem 2.1 *(Poincare Formula)*

$$V - E + F = 2 - 2g$$

where g is the genus of the polyhedron.

By definition this formula requires the faces to cover the entire Riemann surface, thus counting the inside of our boundary circle as a face and the intervals between boundary vertices as edges. The V, E, and F used in Theorem 2.1 are counting the total number of vertices, edges, and faces embedded on the surface of genus g. So, we can modify the Euler-Poincare formula in the following way:

We can write the total number of vertices V in terms of boundary nodes denoted ∂V and interior nodes denoted V^o. We can also write the total number of edges E as the number of edges in our graph denoted E plus the number of ∂V. Finally, we can write the total number of faces F in terms of the number of faces in our graph denoted F plus 1 for the face contributed by the boundary circle. Thus the formula can be written as:

$$(\partial V + V^o) - (E + \partial V) + (F + 1) = 2 - 2g$$

Then simplify:

$$V^o - E + F = 1 - 2g \tag{1}$$

From this equation we can derive an equation for circular dual graphs. From Definition 1.5 of circular dual graphs the number of interior faces in the circular dual equals the number of primal interior vertices. Also, the total number of vertices in the dual equals the number of primal faces.

$$(F^{\perp})^o - E^{\perp} + V^{\perp} = V^o - E + F = 1 - 2g$$

So, for any circular dual graph

$$F^o - E + V = 1 - 2g \tag{2}$$

3 Definition and Significance of Supercellular

Definition 3.1 We will call a circular cellular embedded graph *supercellular* provided that no face borders the boundary circle in more than one arc.

Remark 3.1 A circular cellularly embedded graph having no face bordering on the boundary circle in more than one arc implies that the number of $\partial V = \partial F$. Thus, $\partial V = \partial F \Leftrightarrow$ *supercellular.*

Lemma 3.1 Given a supercellular graph G, primal boundary vertices correspond to dual boundary faces.

Proof: Primal boundary faces border the boundary circle on exactly one arc, so there is a well defined placement of each boundary vertex.

By an induction arguement similar to the one in proof of Lemma 1.1, if there

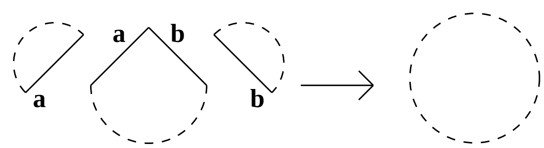

Figure 3: Concatenating A Wedge and Two Half-Discs with Identified Edges

were more than one wedge they could easily be contracted into one. The wedges are constructed as in the interior case. As for the half-discs, their construction comes from part of the boundary circle, and edges being placed in the dual graph. The combining of wedges and half-circles with identified edges will yield a topological disc, which is a face bordering on the boundary circle. □

In constructing the edges of the dual graph, we effectively cut the entire surface into a combination of wedges and half circles, which we proved can be contracted into a topological disc. So, if we remove the vertices and edges our surface is covered in a disjoint union of topological dics. So, we now can say that the circular dual of a supercellular embedded graph also has a cellular embedding.

Lemma 3.2 *If G is a circular dual graph, then G is supercellular.*

Proof: G must satisfy both equations 1 and 2. So we can write:

$$\begin{aligned}
&\Rightarrow \quad V^o - E + F = 1 - 2g = F^o - E + V \\
&\Rightarrow \quad V - V^o = F - F^o \\
&\Rightarrow \quad \partial V = \partial F \\
&\Leftrightarrow \quad G \text{ is Supercellular.}
\end{aligned}$$

□

Definition 3.2 A graph G is called *reflexive* if it is the dual of its dual.

Theorem 3.1 *A circular cellular embedded graph is reflexive* $\Leftrightarrow$ *it is supercellular.*

Proof:
$\Rightarrow$ If reflexive then supercellular:
If G is reflexive, then it is both a primal graph and a circular dual. By Lemma 3.2, it follows that G is supercellular.

$\Leftarrow$ If supercellular then reflexive:
By Definition 1.5:
$F \implies V^{\perp}$
By Definition 1.5 and Lemma 1.1:
$F^o \implies (V^{\perp})^o$ and $V^o \implies (F^{\perp})^o$
By Definition 3.1 and Lemma 3.1:
$\partial F \implies \partial(V^{\perp})$ and $\partial V \implies \partial(F^{\perp})$
Thus, if we embed our supercellular graph G and its dual on the same surface, it is clear that G satisfies the conditions to be the dual of the dual graph. □

Example 3.1 *Below, see a supercellular graph and its dual embedded on a torus.*

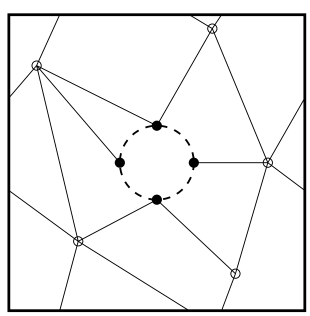

Figure 4: A Supercellular Graph G

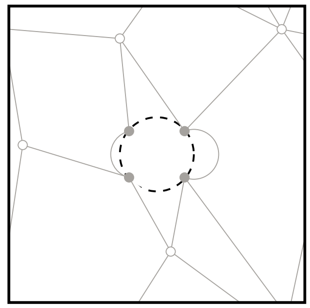

Figure 5: The Circular Dual of G

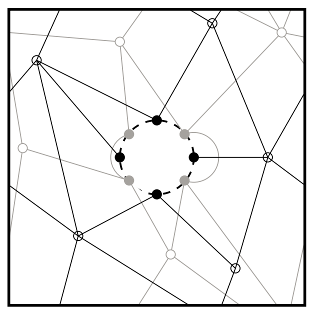

Figure 6: Reflexive Graph

References

[1] Edward B. Curtis and James A. Morrow *Inverse Problems for Electrical Networks.* Series on applied mathematics - Vol. 13. World Scientific, ©2000.