Connections, Determinants, and the Differential of the Response Matrix

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July 2006

Abstract

This paper explores the relationship between connections and determinants by looking at the derivatives of response matrices with respect to the conductances of their corresponding graphs. We begin with the simple 3-star and develop a method for construction of the differential that works for all n-stars. This method will serve as a basis for understanding the relationship between connections and determinants in such 2 to 1 networks as the n-gon in n-gon. We demonstrate that if a functional relationship exists between entries in the response matrix of a graph, then the determinant of the differential is equal to zero.

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1 Introduction

This paper is motivated by a result presented in Jenny and Jerry's 2004 paper $2^n to 1$ *Graphs*, which states that n-gon in n-gon networks have a unique solution if and only if certain discriminantal conditions hold. For the triangle-in-triangle network shown in Figure 1, this condition for recoverability is that $\gamma_{6,8}\gamma_{3,8}\gamma_{2,7}\gamma_{5,7}\gamma_{1,9}\gamma_{4,9} - \gamma_{3,9}\gamma_{6,9}\gamma_{1,7}\gamma_{4,7}\gamma_{5,8}\gamma_{2,8} = 0$. In 2005, Ernie demonstrated that when this condition holds, there is a drop in the rank of the differential of the map from the 12 conductances in the graph to the response matrix. In order to understand why this happens and what it means about the recoverability of a graph, we must take a closer look at the differential of various networks and determine how its structure relates to the structure of the graph.

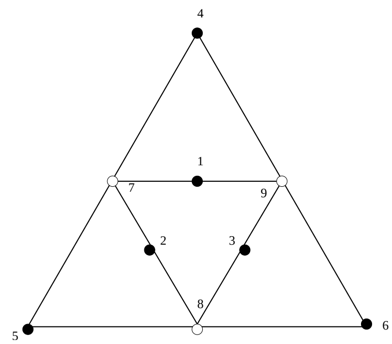

Figure 1: Triange-in-Triangle Network

2 The Differential

We begin by with a network $\Gamma = (G, \gamma)$ where G is a graph and γ is a positive conductivity function defined on all cables in G and consider the map F from the conductances γ to the entries in the response matrix Λ. F maps $(\mathbb{R}^+)^m$ to $\mathbb{R}^n$, where m is the number of entries abpve the diagonal in the response matrix and n is the number of edges in G.

Definition 2.1 The *differential* dF is the derivative of each entry in the response matrix with respect to the conductances. $dF$is an nxm matrix.

We compute the the determinant of the differential to determine whether or not F has full rank.

3 N-stars

We initially look at stars because their symmetry makes computation of the differential fairly simple. Furthermore, their differential properties should generalize to be helpful in understanding star-K transformed networks. In the following examples we take the differential of each map from conductances to response matrices. In the case of the 3-star network, the differential is a square matrix, because the number of entries above the diagonal of the response matrix is equal to the number of ednges in the graph. This makes the computation of the determinant straightforward. As we add boundary nodes to the star networks, however, the number of entries in Λ increases more quickly than the number of edges in G. The differential of the 4-star, for example, is a 4x6 matrix. In these cases we make use of a result given in [**?**] that the determinant of a matrix multiplied by its transpose is equal to the sum of the squares of the determinants of its square submatrices. This means that information can be cleaned about a retangular differential matrix by examining the determinants of its square submatrices.

3.1 The 3-star

In the case of the 3-star, with the nodes and edges labelled as in Figure 2 and σ equal to the a+b+c, the differential is a 3x3 matrix:

$$dF = \frac{1}{\sigma^2} \begin{bmatrix} b(\sigma - a) & c(\sigma - a) & -bc \\ a(\sigma - b) & -ac & c(\sigma - b) \\ -ab & a(\sigma - c) & b(\sigma - c) \end{bmatrix}$$

The determinant of the differential turns out to be $abc(a + b + c)^3$.

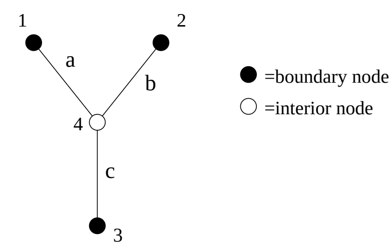

Figure 2: The 3-star network

3.2 The 4-star

We extend the results of the previous example to the slightly larger 4-star network and notice that the resulting differential is a 4x6 matrix:

$$dF = \frac{1}{\sigma^2} \begin{bmatrix} b(\sigma - a) & c(\sigma - a) & d(\sigma - a) & -bc & -bd & -cd \\ a(\sigma - b) & -ac & -ad & c(\sigma - b) & d(\sigma - b) & -cd \\ -ab & a(\sigma - c) & -ad & b(\sigma - c) & -bd & d(\sigma - c) \\ -ab & -ac & a(\sigma - d) & -bc & b(\sigma - d) & c(\sigma - d) \end{bmatrix}$$

Again, σ is equal to the sum of the conductances on the graph. Since dF is a rectangular matrix we compute the determinant of dF multiplied by its transpose to get $(a^4b^2c^2 + a^4b^2d^2 + a^4c^2d^2 + b^4a^2c^2 + b^4a^2d^2 + b^4c^2d^2 + c^4a^2b^2 + c^4a^2d^2 + c^4b^2d^2 + d^4a^2b^2 + d^4a^2c^2 + d^4b^2c^2)(a+b+c+d)^8$. Although this appears to be of a similar form to the determinant of the 3-star differential, it is too large to tell us very much about the graph, so we look at the determinants of the 15 4x4 submatrices of dF. Interestingly, three of the determinants turn out to be zero. The sets of columns that give zero determinants are

1. $\lambda_{1,2}\lambda_{1,3}\lambda_{2,4}\lambda_{3,4}$
2. $\lambda_{1,2}\lambda_{1,4}\lambda_{2,3}\lambda_{3,4}$
3. $\lambda_{1,3}\lambda_{1,4}\lambda_{2,3}\lambda_{2,4}$

It is clear that these combinations of elements correspond to the different combinations of the circular pairs $(1,2)$,$(1,3)$, and $(1,3)$. Of course, in the 3-star network, there are no existing 2-connections between any of these pairs and we know the determinant of these entries in the response matrix are zero.

3.3 The 5-star

The differential of the 5-star network is a 5x10 matrix and the determinant is too large to include in this paper. It does however appear to follow the pattern of the 3- and 4-star determinants. Fortunately, the determinants of the square submatrices prove to be more interesting and englightening. There are 252 square submatrices, whose determinants can be categorized as follows:

1. Choose five columns $\lambda_{i,j}$, where $i, j \leq 5$, so that each index appears twice and get $abcde(a+b+c+d+e)^5$
2. Choose five columns $\lambda_{i,j}$ such that one index appears four times, two others appear twice, and the last one only once and get something of the form $a^3bc(a+b+c+d+e)^5$
3. Choose five columns $\lambda_{i,j}$ such that two indices appear three times, one appears twice, and the last two only once and get something of the form $a^2b^2c(a+b+c+d+e)^5$
4. Choose five columns $\lambda_{i,j}$ such that one index appears three times, three appear twice, and the last only once and get something of the form $a^2bcd(a+b+c+d+e)^5$
5. Choose four columns $\lambda_{i,j}$ such their indices correspond to a circular pair on the graph. The fifth column can be any of the remaining columns. This combination will result in a zero determinant.

Notice that the indices one through five correspond to boundary nodes in the graph and that in the first four cases, the number of times an index is chosen is reflected in the determinant. If indices one and two are chosen three times, then the conductances a and b appear in the determinant in powers of two. As an example, the combination of columns $\lambda_{1,2}\lambda_{1,3}\lambda_{1,4}\lambda_{2,4}\lambda_{2,5}$ gives the determinant $a^2b^2d(a+b+c+d+e)^5$. In the last case, where the determinant is zero, the indices chosen must indicate a circular pair. Due to the fact that there are 15 circular pairs, and that each set of set of four columns can be grouped with one of six remaining columns, there are 90 instances where the determinant of a 5x5 submatrix is zero.

3.4 A Method for Computing the Determinants of N-star Differentials

The preceding examples show that the zero determinants arise when the λs chosen are part of a circular pair. That is, when the determinant of the corresponding entries in the response matrix is zero. This leads to the conjecture that

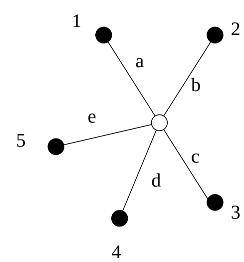

Figure 3: The 5-star network

there is a relationship between the connections on a graph and the determinant of the differential.

Theorem 3.1 *If a 2-connection between circular pairs does not exist, then the determinant of the differential of those circular pairs will be equal to zero.*

Proof: This has yet to be proved. We suspect that this conclusion is a result of the Rank Theorem [**?**], which states that if there exists a functional relationship between elements of a map, the the map will not have full rank. The columns of the Jacobian will thus be dependent, and the determinant will be zero. □

We have developed a useful tool for computing the determinant of the differential when there is no functional relationship between the elements of the maps. Consider the 5-star. In order to construct the determinant of the submatrix involving columns $\lambda_{1,2}\lambda_{1,3}\lambda_{1,4}\lambda_{1,5}\lambda_{2,3}$, for example, simply look at the graph of the 5-star and draw in connections between the boundary nodes indicated by column indices. Write the conductance of the adjacent edge next to each boundary node and then look at how many connections that node is involved in. Figure 4 shows that boundary node one is involved in four connections. Corresponding conductance a will appear one less time in the determinant. Boundary node two is involved in two connections, so it will appear once in the determinant. This tool allows us to correctly construct the determinant, which is equal to $a^3bc(a+b+c+d+e)^5$.

4 Conclusions

Thus far, we have developed a method for constructing the determinant of the differential of N-star networks. It is clear that the determinants follow a pattern and that they provide some information about connections in the network. Specifically, if a 2-connection between circular pairs does not exist, then the determinant of the differential of those circular pairs is be equal to zero. We also know if there is a functional relationship between the entries of a response matrix, then the rank of the differential will not be full.

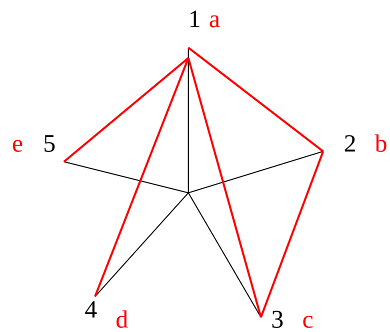

Figure 4: Picture Tool For Computing Determinants

5 Future Work

1. Prove that if a 2-connection between circular pairs does not exist, then the determinant of the differential of those circular pairs will be equal to zero.

2. Look at networks with other functional relationships (besides the determinantal relationship investigated here) and determine whether or not such relationships are reflected in the differential.

3. Look at n-gon in n-gon networks in terms of the rank of the differential; see if it increases our understanding of their 2-to-1ness.

References

[1] F.R. Gantmacher *The Theory of Matrices.* Chelsea Publishing, ©1977.

[2] Walter Rudin *Principles of Mathematical Analysis.* McGraw-Hill, ©1964.