HOW EXTRA CONNECTIONS RUIN THE CUT POINT LEMMA

ZACK GEBALLE

ABSTRACT. This paper explains the challenges in trying to generalize the Cut Point Lemma that I find to be most interesting. In particular, I tried to see how connections that are not counted in m can change the quantity $m-n+r$. There are lots of examples to help future students see what doesn't work and hopefully to see what might still work, plus a few pieces of advice about how to count and draw. See the papers by Rachel[4] and Ming[3] for other examples concerning the same kinds of problems.

CONTENTS

1. INTRODUCTION

Curtis and Morrow's Cut Point Lemma gives an invariant for circular planar critical graphs. No one has been able to come up with a good definition for "critical" in the non circular planar case, so it is hard to predict what a generalized Cut Point Lemma might look like. Nevertheless, it seems that there could be lots to say about large classes of circularly embedded graphs. For example, Rachel, Ming and I considered graphs whose medial graphs have no region bounding lenses, graphs with no interior nodes, and graphs with fixed z-sequences. Unfortunately, we found no interesting invariants involving the numbers m (the maximum connection for a fixed cut XY), r (the number of re-entrant geodesics on the arc XY) and n (the number of black intervals wholly contained in XY where boundary nodes are place in the black intervals). We considered other numbers to add or subtract such as the number of geodesics that cross themselves, the number of non-region-bounding lenses (see Ming's paper for a detailed classification of lenses), and the number of ways a maximum connection can be obtained. There are many possibilities for the additional numbers that might be involved in a cut point lemma for a large class of higher genus circularly embedded graphs. A good way to determine which numbers

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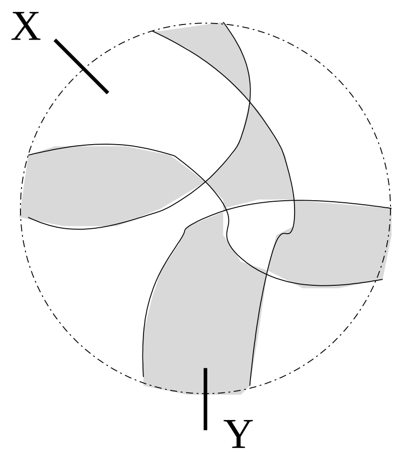


FIGURE 1. $n_{XY} - r_{XY} = 1 - 0 = 3 - 2 = n_{YX} - r_{YX}$

are important is to start with graphs for which $m - n + r = 0$ and add or delete something that changes the quantity $m - n + r$. In most of the following examples, I chose to add edges that give an extra connection.

2. A note about how to count more efficiently

Lemma 2.1. *Suppose X, Y is a cut on a medial graph of a circularly embedded graph of any genus. Let n_{XY} and n_{YX} be the number of black intervals on each side. Let r_{XY} and r_{YX} be the number of re-entrant geodesics. Then $n_{XY} - r_{XY} = n_{YX} - r_{YX}$.*

See Figure 1 for an example. Note that m is independent of side, so as an immediate corollary, $m - n + r$ is too, so there is no need to count both sides. .

Proof. Let c be the number of geodesics that have one endpoint in XY and one in YX. Let k be the number of cuts in black intervals (so k is $0, 1$, or 2). Note that neither c nor k is associated to a single side.

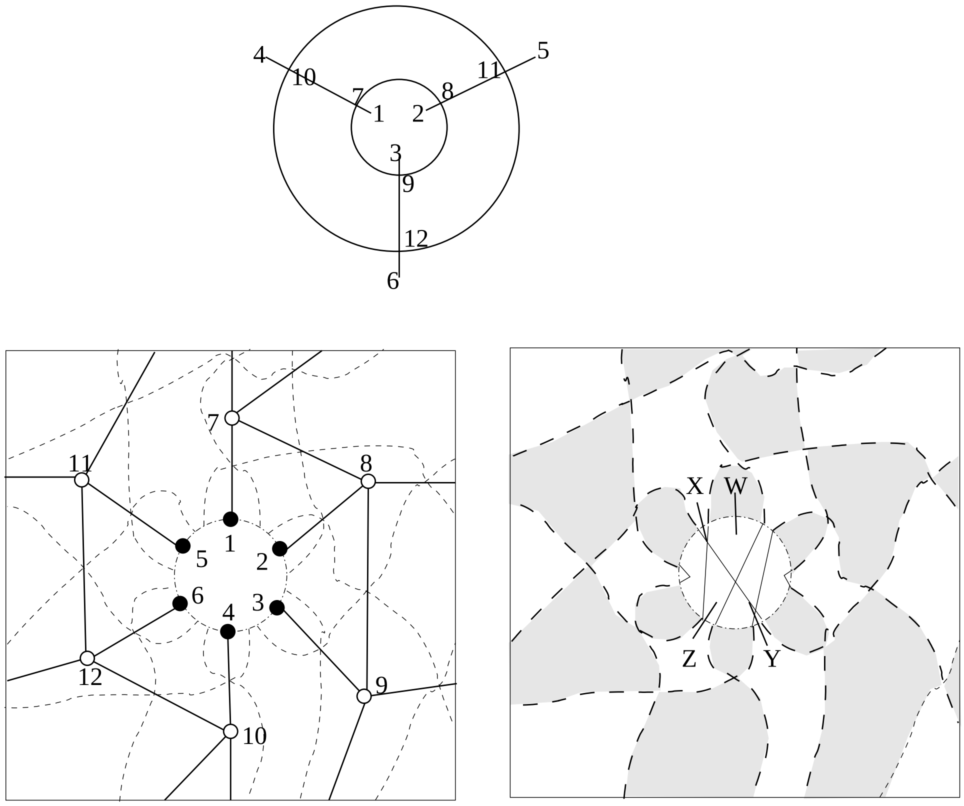
The only identity necessary for this proof is to equate two ways of counting the number of times a geodesic hits an interval. One way is $2n_{XY} + k$ which follows from definitions. The other is $c + 2r_{XY}$ because intersections between geodesics come from exactly two sources: re-entrant geodesics which hit one interval twice, and non-re-entrant geodesics which hit each interval once. Therefore,

$$2n_{XY} + k = c + 2r_{XY}$$
$$2(n_{XY} - r_{XY}) = c - k$$
$$\text{Likewise, } 2(n_{YX} - r_{YX}) = c - k$$
$$\text{Therefore, } n_{XY} - r_{XY} = n_{YX} - r_{YX}$$

□

3. The Cut Point Lemma fails for many graphs embedded on toruses

I will give organized examples of graphs later, but to see that the Cut Point Lemma fails for some important graphs, see Figures 3 and 2.



Graph with dotted medial graph

Shaded medial graph, with z−sequence represented by solid lines. X,Y,W, and Z mark cuts.

FIGURE 2. Two circles Three rays embedded on a torus. Cuts such as X,Y and Z,W give $m - n + r \neq 0$, but even they don't give the same number. For the smaller side of ZW, $m = 2$, $n = 2$ and $r = 2$. For either side of XY, $m = n = 3$ and $r = 1$

The proof of the Cut Point Lemma fails in exactly one place. We have no higher genus equivalent of Curtis and Morrow's Lemma 8.6 that for a set of geodesics all of which intersect some other geodesic, but which make no lenses, there are three boundary triangles.[2] The proof of the Cut Point Lemma reduces every critical graph to a simple case for which $m = n$ and $r = 0$. The two reduction steps are eliminating geodesics start and end in adjacent spots, and uncrossing boundary triangles. It is important that there are three triangles at all times because up to two could surround the intervals which are chosen to be cut. (Uncrossing a boundary triangle across a cut could change r without changing m or n, so it is not allowed as a reduction step). This is the only place the proof fails to generalize, so it is tempting to seek an analog of Lemma 8.6. See Figure 4 parts A and B for some examples of what could happen at the boundary circle.

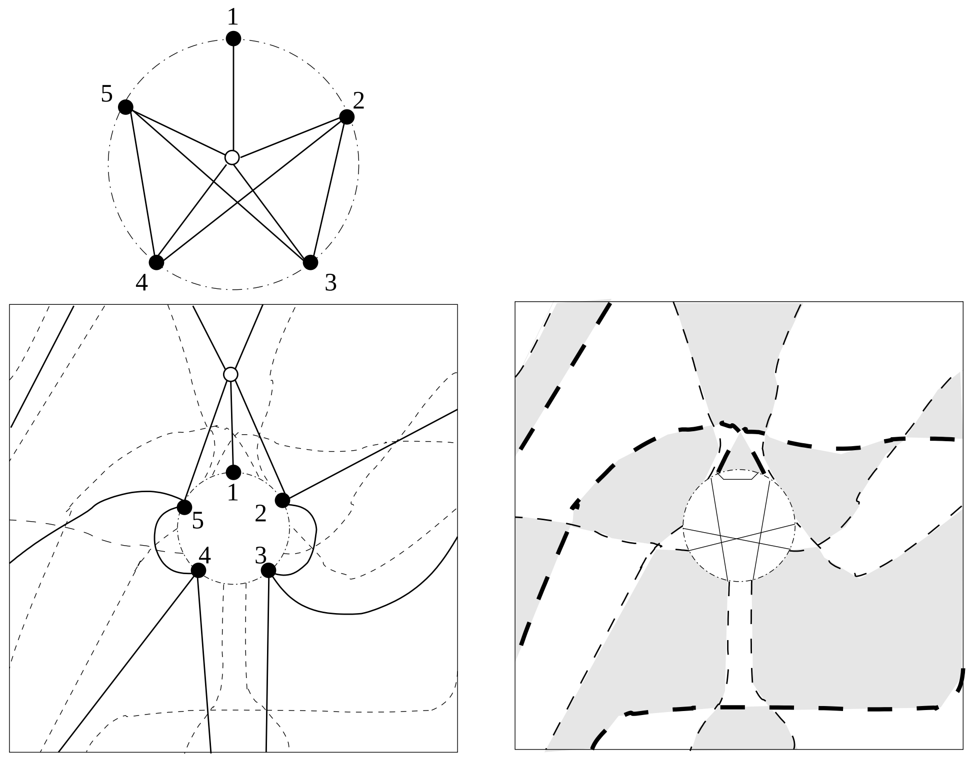


FIGURE 3. A graph with all possible set-connections (see Definition 4.1) is embedded on a torus. The z-sequence is not 1234512345, so following the logic of Remark 4.5, there are cuts for which the smaller side has a re-entrant geodesic, and yet $m = n$. Therefore, $m - n + r = 0 + 1 = 1$ for certain cuts. Also, this graph is recoverable, so it would be best to find a cut point lemma that allows for crazy geodesics like the bold one.

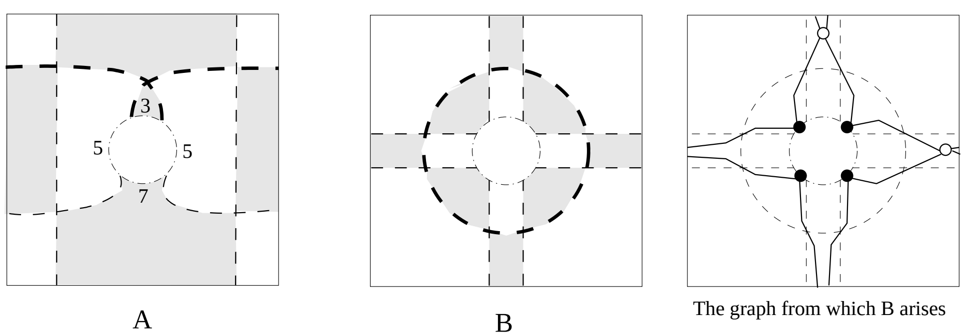


FIGURE 4. Two medial graphs (see [5] for how to draw legitimate medial graphs without first drawing a graph.) In "A", the boundary polygons have 3,5,7,and 5 sides. In "B" there are eight boundary squares

Idea for Future Research 3.1. Perhaps we shouldn't allow the kinds of lenses that the bold geodesics cause in Figure 4 part A and B. Maybe it is best to consider medial graphs with a geodesics surrounding the boundary circle separately from the others. Hopefully any geodesics that are not allowed correspond to non-recoverable graphs, but that is a more distant problem. See Ming's paper [3] for more on lenses.

4. Connections

Definition 4.1. There is an *ordered connection* between two ordered sets of n boundary nodes if the i^{th} elements of each set are connected by a path for all $1 \leq i \leq n$ such that all paths are disjoint.
There is a *set connection* between two sets of n boundary nodes if there is an ordered connection between some ordering of them.

Remark 4.2. The traditional way to denote an ordered connection between two sets of boundary vertices is to write one set and then the other so that the nodes that can be connected are in the same place in their respective lists. For example $(1, 3; 5, 6)$ means that there are disjoint paths from 1 to 5 and 3 to 6. I find it easier to denote the same thing (15)(36), because then it is easy when two connections are equivalent. (15)(36) is obviously the same as (63)(51) in my notation, whereas I find it hard to see that $(6, 1; 3, 5)$ is the same as $(1, 3; 5, 6)$.

In my opinion, the biggest hurdle in generalizing the Cut Point Lemma is that there is no easy way to account for non-circular connections. The number m is merely the maximal connection between XY and YX. In particular, the original Cut Point Lemma does not depend on the number of maximal connections or any lesser connection. For this reason $m - n + r = 0$ is something of a miracle to me. On the other hand, the Cut Point Lemma holds only for critical graphs − those for which removal of any edge breaks a connection − so maybe it is not such a miracle. Perhaps the biggest impediment to generalizing the Cut Point Lemma is that we have no good analog of critical for higher genus surfaces. Unfortunately, past students have had a very hard time deciding how to define "critical" for non-circular planar graphs.

Anyway, I would be surprised if the number of ways a maximal set-connection can be connected (say (1,4)(2,5)(3,6) and (1,5)(2,6)(3,4) were two ways to connect the set {1,2,3}in XY with the set {4,5,6} in YX) was not a factor in a generalized cut point lemma. My best evidence is that adding connections that change only the number of ways to connect maximal sets of nodes but not connectedness of sets of nodes often changes the value of $m - n + r$. See figures 5 and 6 for examples.

Idea for Future Research 4.3. It might be fruitful to try to figure out how geodesics change when extra combinations of nodes are connected in already connected sets. I would start with graphs on which all sets are connected and add connections by going around handles of a torus. In addition to Figures 5 and 6, there is an easy example in Remark 4.5.

4.1. A note about maximum anti-circular connections on genus n Riemann surfaces.

Definition 4.4. An anti-circular n-connection is one that (by suitable labeling of boundary nodes) can be denoted $(1, n + 1)(2, n + 2)...(n, 2n)$ where $1, 2, ..., 2n$ are in circular order with any number of other nodes mixed in.

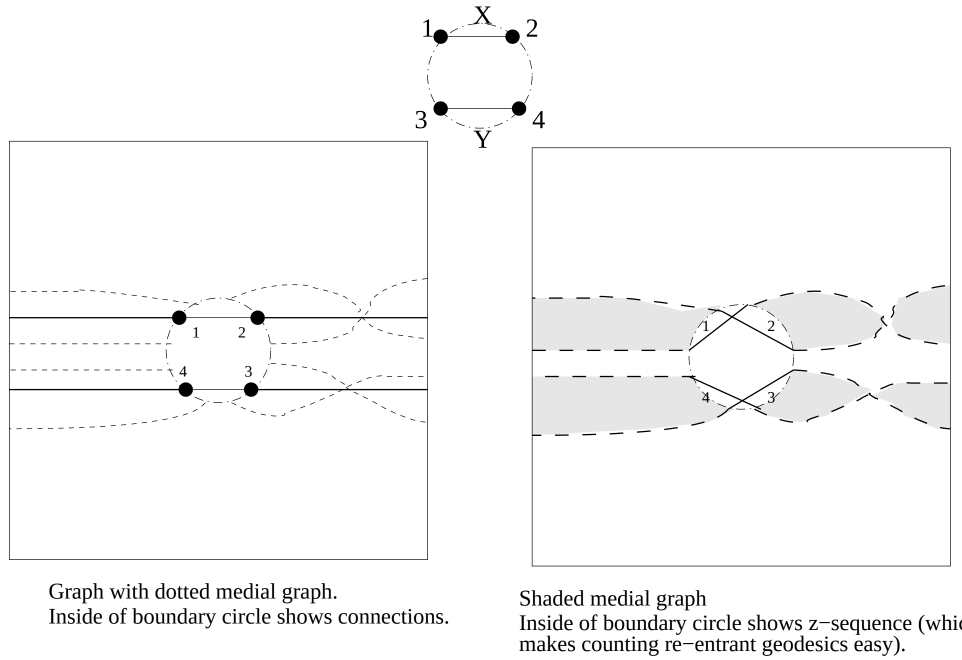
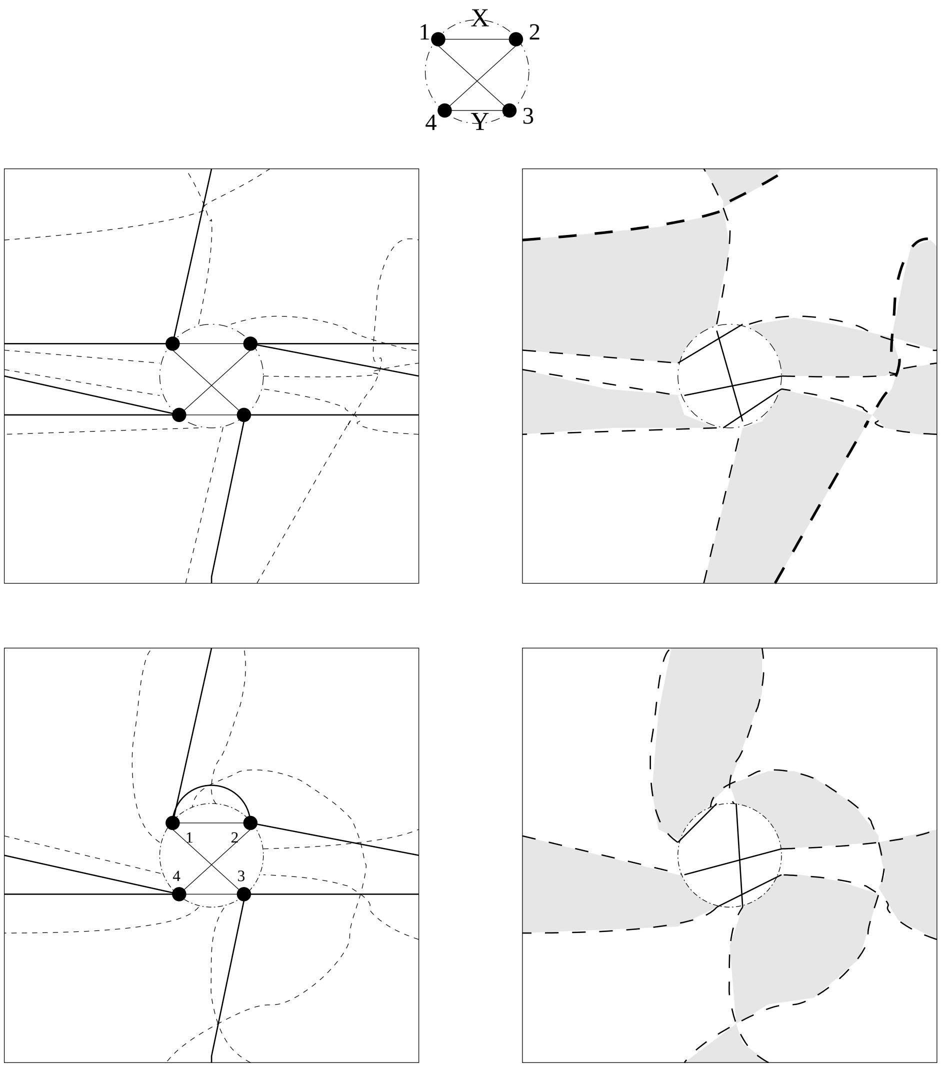


FIGURE 5. An embedding of the top graph on a torus. (12)(34) is the only 2-connection from XY to YX. It might not be a coincidence that $m - n + r = 2 - 2 + 0 = 0$

An upper bound on the genus of the simplest graphs that have anti-circular connections is the minimal genus surface on which a complete graphs can be circularly embedded. K-1 through K-7 can be circularly embedded on a torus. Therefore, anti-circular 1, 2 and 3-connections are possible on a torus. See Figure 7 for the toroidal graph with an anti-circular 3-connection. It can be created by deleting one boundary node and lots of edges of a toroidal embedding of K-7, but I do not recommend doing that. Since it is the simplest one possible, I call it "Antipodal 3"

If you can convince yourself that Figure 7 shows the only way to have an anti-circular 3-connection, then I can convince you that there is no anti-circular 4-connection. Color in the regions of the graph (not the medial graph) of Antipodal 3. Since no antipodal intervals of the boundary circle border the same region, there is no way to add a pair of antipodal boundary nodes and connect them without crossing another edge. (If you cross an edge , you must add an interior node that the two edges share, so the 2-connections are not disjoint, so they don't count as part of a larger connection).

Remark 4.5. Antipodal graphs provide nice examples of how adding edges can change $m - n + r$. Starting with antipodal graphs, m , n, and r are easy to count. Any cut separates the boundary nodes into sets for which the maximum possible connection exists (i.e. $m = n$ on the smaller side). In this way, for the cuts that divide the boundary circle into two connected components (the only cuts I have considered), antipodal graphs have the same values of m and n as complete graphs. Additionally, they always have the z-sequence 123..n123...n . All together,



Graphs with dotted medial graphs.

Shaded medial graphs
Insides of boundary circles show z−sequence (whic
makes counting re−entrant geodesics easy).

Figure 6. Two (of several) embeddings of the top graph on a torus. The graph is the one in Figure 5 with two additional edges that create an extra 2-connection. (12)(34) and (13)(24) are both 2-connections from XY to YX. Hence, m = 2, as in Figure 5, but the extra 2 connection across the cut seems to make the cut point lemma fail. $m - n + r = 1$ for both graphs

$m - n + r = (m - n) + r = 0 + 0 = 0$ on the smaller side. (There is no need to count the larger side by Lemma 2.1). Next, it is easy to change r without changing m or n. Simply add an edge between two boundary vertices. The z-sequence must

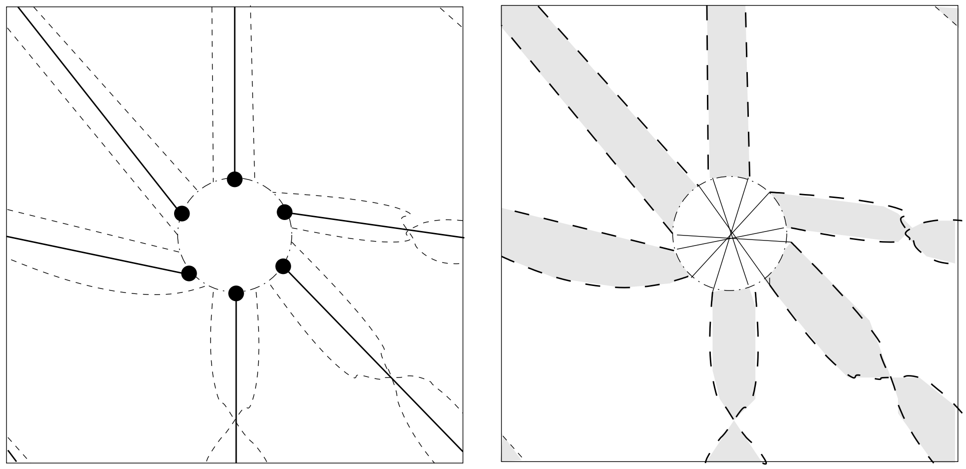


FIGURE 7. Antipodal 3

change, but it is still true that $m = n$ on the smaller side of the cut. Therefore, for some cuts, $m - n + r = 1$.

4.2. **The Sad Truth.** Examples cited above show that adding connections can easily change the quantity $m - n + r$. It would be nice to show that among graphs with a fixed number of boundary nodes, those that have the same large ordered connections tend to have the same value for $m - n + r$ for each cut. (The quantity $m - n + r$ may differ from cut to cut, but the point is that hopefully the essense of the Cut Point Lemma - that there is a relationship between n, r and the connections - still exists for non circular planar graphs). Sadly, there is no easy pattern.

Example 4.6. Graphs with the same ordered connections can have different z-sequences. See figures 8, 9 and 10.

5. OTHER IDEAS FOR FUTURE RESEARCH

1. Look at different embeddings of a single graph. (see Figure 6 for an example)
   (a) Follow Rachel's line of research by including "winding number" in z-sequences to differentiate between embeddings.
   (b) Consider Ming's idea of requiring graphs to be embedded cellularly on the highest genus surface possible. No lemma with this requirement could reduce to the original Cut Point Lemma, but at least the requirement might associate a single z-sequence to each graph.
2. Allow cuts that break the boundary circle into more than two connected components. In some sense, this would be the most honest approach because theordering of nodes on the boundary circle is less important in the non circular planar case.
3. As an alternative to embedded on higher genus Riemann surfaces, decompose every graph into circular planar layers where each layer includes all the nodes, but only edges that are not contained in any other layer. See

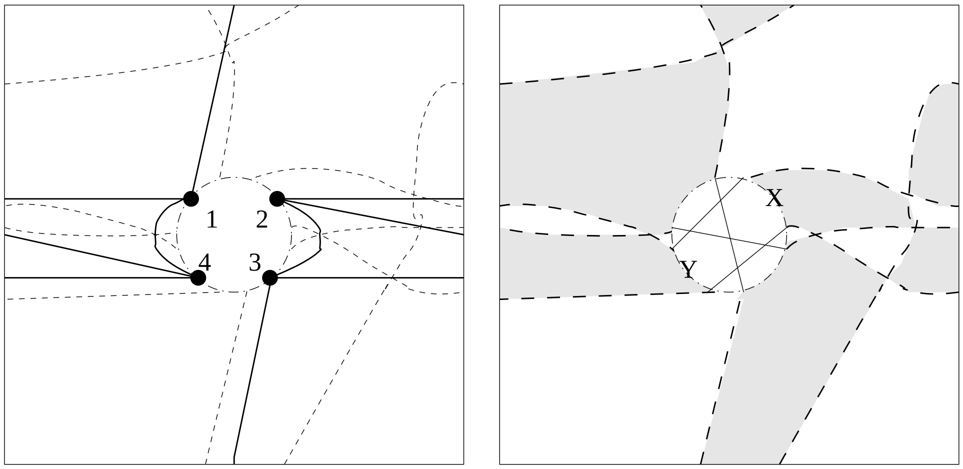


FIGURE 8. K-4 has every possible ordered connection, so $m = n$ on the smaller side of any cut. Therefore, $m - n + r = 0$ for every cut except X,Y, for which $m = n = 1$ and $r = 1$

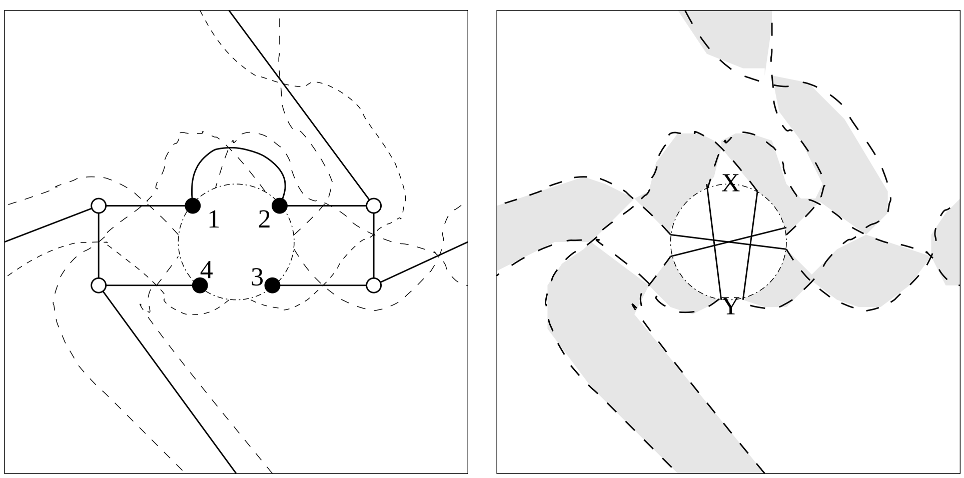


FIGURE 9. This graph is like K-4 in that there is a way to make every possible ordered 2-connection:(12)(34), (13)(24) and (14)(23). To be thorough, they differ in that here there is more than one way to make the 2-connection (12)(34) and 1-connections (12), (13) and (24). Above, all ordered connections are unique. Again, the z-sequence is not 12341234, so for the cut XY, $(m-n)+r = 0+1$

Figure 11. It is possible to draw medial graphs on each layer, but it is unclear how different decompositions of a single graph are related.

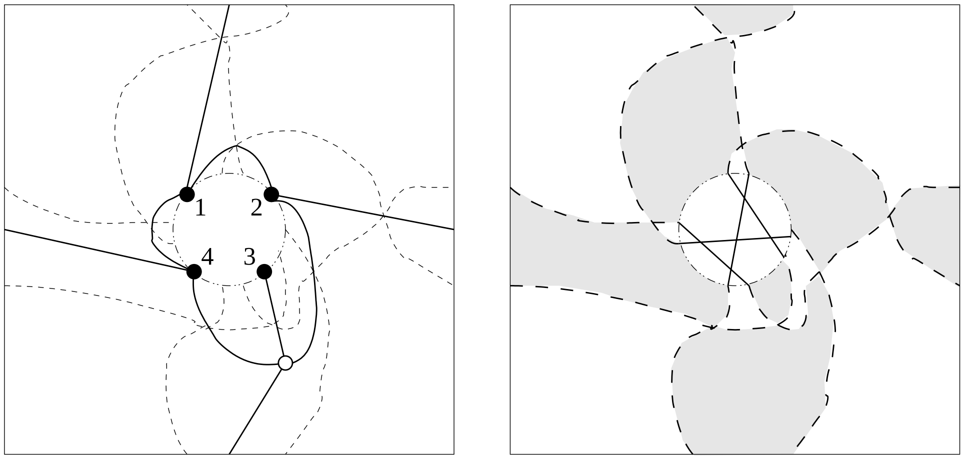


FIGURE 10. This graph is very much like K-4. (It is really an embedding of K-4 with a boundary spike added). There is a unique way to make every possible ordered 2-connection:(12)(34), (13)(24) and (14)(23). A difference is that a few 1-connections are not unique. In particular, there are two ways to connect each of (12), (14) and (24). Unfortunately, the z-sequence is different from both Figures 8 and 9. Here, as above, one cut yields $m - n + r = 1$.

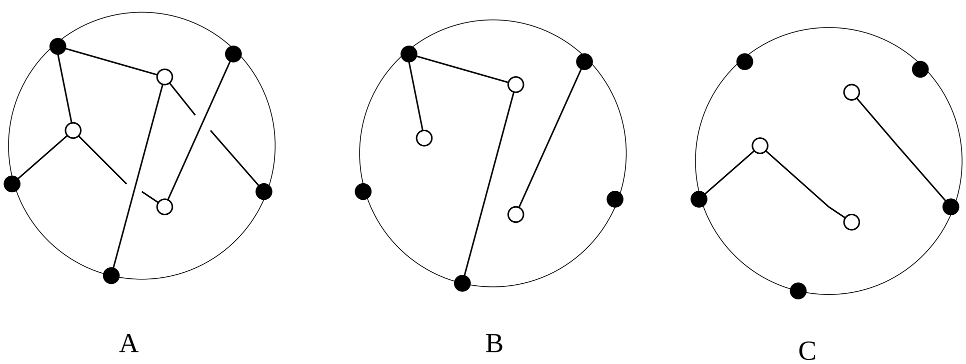


FIGURE 11. A is decomposed into B and C

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UNIVERSITY OF MICHIGAN
E-mail address: zgeballe@umich.edu