The Simplest Mixed Problem

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Abstract

This paper will introduce the concept of a "mixed problem." Knowledge of Curtis and Morrow's inverse problem will be assumed throughout the paper. Only the simplest case of a mixed problem will be considered and it will be shown that the solution to this problem exists and is unique. We will also introduce a new map called the "mixed map" and investigate some of its more interesting properties. Finally, it will be shown that the inverse problem can be solved by using the mixed map as the input data.

1 Solution from the Kirchhoff Matrix

Let $\Gamma = (G, \gamma)$ be an electrical network where G is a connected graph with boundary. Suppose that G has $m + n \geq 1$ boundary vertices and $d \geq 1$ interior vertices for a total of $m + n + d$ vertices. Consider the case where the voltages of the first $m \geq 1$ vertices are known and the currents at the other n vertices are known. As always, no information about the interior is known. We now propose the following problem: Given the specified data, is it possible to determine the currents at vertices $\{p_1, p_2 ... p_m\}$ and the voltage at vertices $\{p_{m+1}, p_{m+2} ... p_{m+n}\}$? In the classical inverse problem, the *Kirchhoff matrix* is partitioned into four submatrices so that

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}.$$

The motivation for partitioning K in this manner is to separate the data relating to the boundary from the data related to the interior. However, the mixed problem contains two different types of boundary nodes, the "voltage" nodes and the "current" nodes. In order to approach the mix problem, we must also partition A and B so that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

Note that the submatrix A_{11} comes from the entries in K corresponding to the m vertices where the voltages are known and A_{22} comes from the entries in K corresponding to the n vertices where the currents are known. With this notation, we can now express the linear system that maps voltages to currents via the Kirchhoff matrix:

$$\begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{12}^T & A_{22} & B_2 \\ B_1^T & B_2^T & C \end{bmatrix} \begin{bmatrix} v \\ x \\ y \end{bmatrix} = \begin{bmatrix} \phi \\ \psi \\ 0 \end{bmatrix}, \tag{1}$$

where v is the vector containing the m known voltages, x is the vector containing the n unknown voltages, y is the vector containing the d interior voltages, ϕ contains the m unknown currents and ψ contains the n known currents. The zero in the third cell of the right hand side of the equation is a d-vector with all entries equal to zero. This is a result of *Kirchhoff's Law* which says that the sum of all currents into any interior node is equal to zero. From (1), we get the two equations

$$A_{12}^T v + A_{22} x + B_2 y = \psi$$

$$B_1^T v + B_2^T x + C y = 0$$

with two unknowns, x and y. Rearranging this equation results in

$$\begin{bmatrix} A_{22} & B_2 \\ B_2^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \psi - A_{12}^T v \\ -B_1^T v \end{bmatrix}. \tag{2}$$

Theorem 1. *The solution to equation (2) exists and is unique.*

Proof. Notice that the matrix in equation (2) is a principle proper submatrix of K and is therefore invertible. Thus, the solution of the system is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A_{22} & B_2 \\ B_2^T & C \end{bmatrix}^{-1} \begin{bmatrix} \psi - A_{12}^T v \\ -B_1^T v \end{bmatrix}.$$

$\square$

Likewise, it is possible to solve for the unknown currents, ϕ. From the first equation of (1), we get

$$\begin{aligned} \phi &= A_{11}v + A_{12}x + B_1 y \\ &= A_{11}v + \begin{bmatrix} A_{12} & B_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= A_{11}v + \begin{bmatrix} A_{12} & B_1 \end{bmatrix} \begin{bmatrix} A_{22} & B_2 \\ B_2^T & C \end{bmatrix}^{-1} \begin{bmatrix} \psi - A_{12}^T v \\ -B_1^T v \end{bmatrix}. \end{aligned}$$

We can express all three of the unknown quantities explicitly but the notation gets rather messy. To simplify the notation, we introduce the following submatrices of K.

$$K_1 = \begin{bmatrix} A_{11} & B_1 \\ B_1^T & C \end{bmatrix}, K_2 = \begin{bmatrix} A_{12} & B_1 \\ B_2^T & C \end{bmatrix}, K_3 = \begin{bmatrix} A_{12}^T & B_2 \\ B_1^T & C \end{bmatrix}, K_4 = \begin{bmatrix} A_{22} & B_2 \\ B_2^T & C \end{bmatrix}$$

The following equations now express x, y and ϕ in terms of *Schur complements* of the submatrices listed above.

$$\begin{aligned} x &= (K_4/C)^{-1}[\psi - (K_3/C)v] \\ y &= -C^{-1}[B_1^T v + B_2^T (K_4/C)^{-1}(\psi - (K_3/C)v)] \\ \phi &= [(K_1/C) - (K_2/C)(K_4/C)^{-1}(K_3/C)]v + (K_2/C)(K_4/C)^{-1}\psi \end{aligned}$$

2 Solution from the Response Matrix

In the previous section, it was shown that the voltages and currents at all vertices of G can be recovered from the *Kirchhoff Matrix*. However, the *Kirchhoff Matrix* is not necessary to find just the unknown quantities on the boundary. All information regarding the boundary is contained in the response matrix, $\Lambda = \Lambda_\gamma$, and therefore we should be able to find the unknown vectors pertaining to the boundary, x and ϕ, strictly from Λ. Following the same method as in the previous section, we partition Λ so that

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix}.$$

Then the following matrix equation results:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} = \begin{bmatrix} \phi \\ \psi \end{bmatrix}. \tag{3}$$

Note that Λ_{11} is a square matrix of order m and Λ_{22} is a square matrix of order n.

Lemma 1. *The matrices Λ_{11} and Λ_{22} are both invertible.*

Proof. Recall that G is a connected graph. Let K' be a principle proper submatrix of K. Since G is connected, K' is invertible and moreover, any *Schur complement* of K' in C is invertible. By definition of Λ, we have

$$\Lambda_{11} = A_{11} - B_1 C^{-1} B_1^T$$

and

$$\Lambda_{22} = A_{22} - B_2 C^{-1} B_2^T$$

which are both *Schur complements* of principle proper submatrices of K in C. Therefore, Λ_{11} and Λ_{22} are both invertible. □

By verifying that Λ_{22}^{-1} exists, we can now solve for the unknown quantities, x and ϕ. After performing some matrix algebra on equation (3), we end up with the following solution:

$$\begin{aligned} \phi &= (\Lambda/\Lambda_{22})v + \Lambda_{12}\Lambda_{22}^{-1}\psi \\ x &= -\Lambda_{22}^{-1}\Lambda_{12}^T v + \Lambda_{22}^{-1}\psi. \end{aligned}$$

Or, in matrix form, we have

$$\begin{bmatrix} \Lambda/\Lambda_{22} & \Lambda_{12}\Lambda_{22}^{-1} \\ -\Lambda_{22}^{-1}\Lambda_{12}^T & \Lambda_{22}^{-1} \end{bmatrix} \begin{bmatrix} v \\ \psi \end{bmatrix} = \begin{bmatrix} \phi \\ x \end{bmatrix}. \tag{4}$$

3 The Mixed Map

We now define the matrix from equation (4) as a linear operator $M : \mathbf{R}^{m+n} \mapsto \mathbf{R}^{m+n}$ that maps the space of the known data into the space of the unknown data. We will refer to M as the "mixed map." Note that M is a square matrix of order $m+n$ and is block skew-symmetric.

Theorem 2. *M is an injective linear map.*

Proof. We will show that M is injective (or one-to-one) by showing that $\det M \neq 0$. Using the *Schur complement* determinant identity we have,

$$\begin{aligned} \det(M) &= \det(M/\Lambda_{22}^{-1})\det(\Lambda_{22}^{-1}) \\ &= \det(\Lambda/\Lambda_{22} + \Lambda_{12}\Lambda_{22}^{-1}\Lambda_{22}\Lambda_{22}^{-1}\Lambda_{12}^T)\det(\Lambda_{22}^{-1}) \\ &= \det(\Lambda/\Lambda_{22} + \Lambda_{12}\Lambda_{22}^{-1}\Lambda_{12}^T)\det(\Lambda_{22}^{-1}) \\ &= \det(\Lambda/\Lambda_{22} + (\Lambda_{11} - \Lambda/\Lambda_{22}))\det(\Lambda_{22}^{-1}) \\ &= \frac{\det(\Lambda_{11})}{\det(\Lambda_{22})}. \end{aligned} \tag{5}$$

It was previously shown in Lemma 1 that Λ_{11} and Λ_{22} are both invertible, i.e. their determinants are non-zero. Therefore, $\det M$ exists and is not zero, which implies that M is injective. $\square$

Actually, M^{-1} can be calculated quite easily and is given by

$$M^{-1} = \left[\begin{array}{cc} \Lambda_{11}^{-1} & -\Lambda_{11}^{-1}\Lambda_{12} \\ \Lambda_{12}^{T}\Lambda_{11}^{-1} & \Lambda/\Lambda_{11} \end{array} \right].$$