COMPLEXERS FROM STARS

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Abstract. A complexer is a plexer on a graph with an admittance function γ such that the positive real-valued entries in the response correspond exactly to the members of Π^K, the known set of the plexer's partition. A star network whose admittances conform to certain conditions becomes a complexer via a $\bigstar - K$ transformation. This paper defines those conditions, thereby providing a way to cook up a complexer on a star by choosing an appropriate admittance function.

Contents

1. Simple Parallel Edges and their Admittances

In this paper I use $i = \sqrt{-1}$. All indices therefore appear as j, k, l.

Admittance is the periodic-voltage analogue of conductance. Admittances of circuit elements are represented on the complex plane as follows, with Y representing admittance and ω representing the angular frequency of the voltage across the circuit element:

For a resistor with resistance $R \in \mathbb{R}^+$, $Y = \frac{1}{R}$. It is independent of ω.
For a capacitor with capacitance $C \in \mathbb{R}^+$, $Y = i\omega C$
For an inductor with inductance $L \in \mathbb{R}^+$, $Y = \frac{1}{i\omega L} = \frac{-i}{\omega L}$.

I will consider networks where the potentials are periodic functions of one ω.

When circuit elements are combined in parallel or in series along an edge, that edge has a well-defined equivalent admittance (denoted Y_{eq}) according to two rules of addition:

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- For m admittances $Y_1, Y_2, \cdots, Y_m$ in parallel, $Y_{eq} = Y_1 + Y_2 + \cdots + Y_m$.
- For n admittances $Y_1, Y_2, \cdots, Y_n$ in series, $Y_{eq} = (Y_1^{-1} + Y_2^{-1} + \cdots + Y_n^{-1})^{-1}$.

Definition 1.1. An *R-edge* is an edge consisting only of resistors in series; an *L-edge* is an edge consisting only of inductors in series, and a *C-edge* is an edge consisting only of capacitors in series. Likewise, an *RC-edge* consists of resistors and capacitors in series; an *RL-edge* consists of resistors and inductors in series; an *LC-edge* consists of inductors and capacitors in series; and an *RLC-edge* consists of all three types of elements in series. Each edge has a well-defined Y_{eq} found by adding the elements in series.

Definition 1.2. A *simple parallel edge* is a parallel connection consisting of at least one of the following: an R-edge, an RC-edge, an RL-edge, an RLC-edge. It may also consist of C-edges, L-edges and LC-edges, though that is not required. This paper considers networks made up of simple parallel edges. Such networks have a unique Dirichlet solution [1].

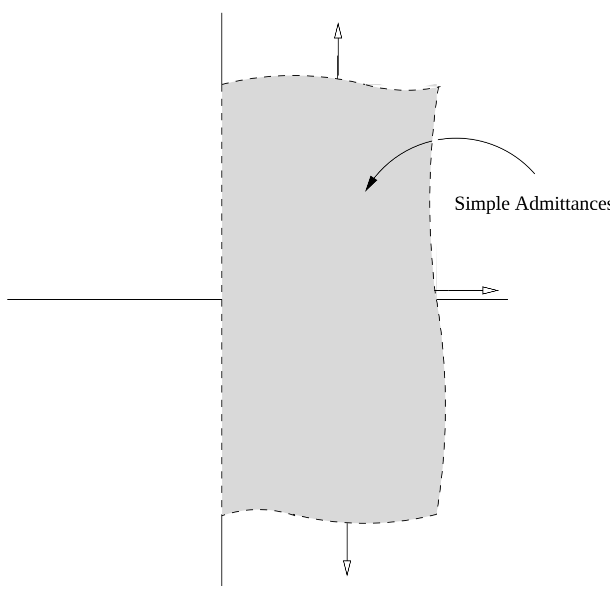

FIGURE 1. The set of all possible simple admittances for fixed ω in the complex plane. Simple admittances always lie in the right half-plane.

Definition 1.3. A *simple admittance* is the associated Y_{eq} of a simple parallel edge.

A simple admittance γ as a function of ω is of the form:

$$\begin{aligned}\gamma(\omega) &= f + \omega g + \frac{-h}{\omega} + \left[\frac{1}{p_1(\omega-\xi_{p_1})} + \cdots + \frac{1}{p_d(\omega-\xi_{p_d})}\right] \\ &+ \left[\frac{1}{q_1(\omega-\beta_{q_1})} + \cdots + \frac{1}{q_m(\omega-\beta_{q_m})}\right] + \left[\frac{\omega r_1}{(\omega-\delta_{r_1})(\omega-\psi_{r_1})} + \cdots + \frac{\omega r_n}{(\omega-\delta_{r_n})(\omega-\psi_{r_n})}\right] \\ &+ \left[\frac{\omega s_1}{(\omega-\mu_{s_1})(\omega-\tau_{s_1})} + \cdots + \frac{\omega s_t}{(\omega-\mu_{s_t})(\omega-\tau_{s_t})}\right]\end{aligned}$$

where

- $f \in \mathbb{R}^+$
- $g, h \in i\mathbb{R}$
- $p_j = iL$ and
  $\xi_{p_j} = \frac{iR}{L}$ for each RL-edge from 1 to d
- $q_j = \frac{1}{iC}$ and
  $\beta_{q_j} = \frac{1}{iRC}$ for each RC-edge from 1 to m
- $r_j = C$ and
  δ_{r_j}, ψ_{r_j} are the roots of the quadratic equation $iLC\omega^2 - i$ for each LC-edge from 1 to n
- $s_j = C$ and
  μ_{s_j}, τ_{s_j} are the roots of the quadratic equation $iLC\omega^2 + CR\omega - i$ for each RLC-edge from 1 to t

Note that in the remainder of this paper I will refer to simple admittances as admittances of the form $\alpha_0 + \alpha_1 i$, where $\alpha_0 \in \mathbb{R}^+$ and $\alpha_1 \in \mathbb{R}$ as in Figure 1.

Remark 1.4. A sum of simple admittances is also a simple admittance; the set is closed under addition.

2. Nifty Results from Complex Algebra

It is useful to identify complex numbers with vectors in $\mathbb{R}^2$. The symbol $\sim$ represents identification.

Lemma 2.1 (The Parallel Lemma)**.** *For two complex numbers*

$$\begin{aligned} x = x_0 + ix_1 &\sim \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \\ y = y_0 + iy_1 &\sim \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}\end{aligned}$$

xy is real if and only if $x \parallel \bar{y}$ in $\mathbb{R}^2$.

Proof. Take $\bar{y} \sim \begin{bmatrix} y_0 \\ -y_1 \end{bmatrix}$. Define a new vector $\bar{y}'$ orthogonal to the vector $\bar{y}$ by multiplying the complex number $\bar{y}$ by i so that $\bar{y}' \sim \begin{bmatrix} y_1 \\ y_0 \end{bmatrix}$.

The product xy is real if and only if $\Im(xy) = x_0y_1 + x_1y_0 = 0$. But it is also the case that $x \cdot \bar{y}' = x_0y_1 + x_1y_0$. So xy is real if and only if the vector x is orthogonal to $\bar{y}'$. Since $\bar{y}'$ is also orthogonal to $\bar{y}$, it follows that xy is real if and only if $x \parallel \bar{y}$ in $\mathbb{R}^2$. □

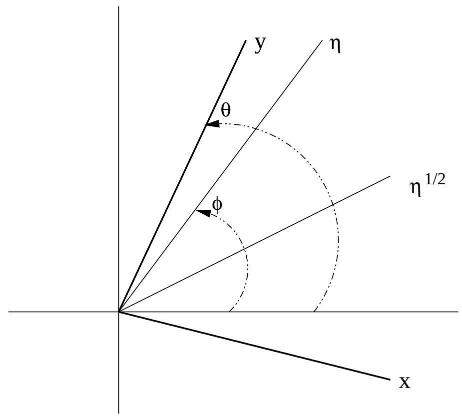

FIGURE 2. The Conjugate Lemma: $x \parallel \eta\bar{y}$ if and only if x is parallel to y conjugated over the line proportional to $\sqrt{\eta}$.

From here on, the notation $x \parallel y$ means that $\arg(x) = \arg(y)$.

Lemma 2.2 (The Conjugate Lemma)**.** *For complex numbers x, y, η, $x \parallel \eta\bar{y}$ if and only if x is parallel to y conjugated over the line proportional to $\sqrt{\eta}$.*

Proof. Take y and η as in Figure 2, where

$$y = y_0 e^{i\theta}, y_0 \in \mathbb{R}$$
$$\eta = \eta_0 e^{i\phi}, \eta_0 \in \mathbb{R}$$

Note that $\sqrt{\eta} \parallel e^{\frac{1}{2}\phi}$. Now take $x \parallel \eta\bar{y}$. This is true exactly when $x \parallel e^{i(\phi-\theta)}$, so $\arg(x) - \arg(\sqrt{\eta}) = \frac{1}{2}\phi - \theta$. Also, $\arg(y) - \arg(\sqrt{\eta}) = \theta - \frac{1}{2}\phi$. So x is parallel to the conjugate of y across the line proportional to $\sqrt{\eta}$ if and only if $x \parallel \eta\bar{y}$. □

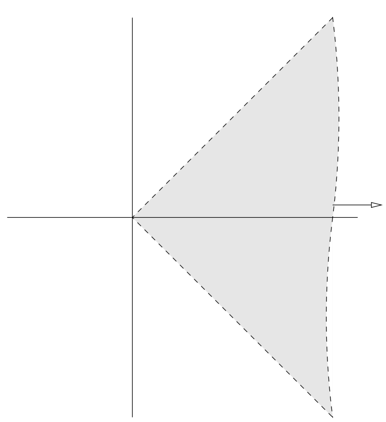

FIGURE 3. When η is a simple admittance, the open shaded region in the complex plane represents all possible values for $\sqrt{\eta}$.

3. Complex-Valued $\bigstar - K$ Transformations

Definition 3.1. An *n-star* , denoted $\bigstar_n$, is a connected graph with exactly one interior node, n boundary nodes, and no boundary-to-boundary connections.

In this section I will consider networks consisting of a $\bigstar_n$ with an associated simple admittance function γ on its edges. For such a network, the $\bigstar - K$ transformation defines the complete graph that corresponds to the response matrix Λ according to the formula in [2]:

$$\lambda_{jk} = \frac{\gamma_j \gamma_k}{\sigma} \tag{1}$$

where $\sigma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$.

4. Complexers

Definition 4.1. A *plexer* is an ordered pair $P = (G, \Pi)$ where G is a graph with boundary and Π is a nontrivial partition of the set of all distinct unordered pairs of boundary vertices. The partition $\Pi = (\Pi^U, \Pi^K)$ defines two sets: the unknown set (consisting of unknown pairs) and the known set (consisting of known pairs). P has the following properties:

1. For a valid response matrix on G, given only the values of the entries corresponding to the known pairs, it is not possible to determine the values of any entry corresponding to an unknown pair.
2. For a valid response matrix on G, given only the values of the entries corresponding to known pairs and one unknown pair, we can recover the entire response matrix.

A catalogue of plexers appears in [3].

Definition 4.2. A *complexer* is an ordered triple $C = (G, \Pi, \gamma)$ where (G, Π) is a plexer and γ is a simple admittance function on G's edges. Additionally, the known set Π^K in P must correspond exactly to the positive real entries in the response matrix Λ for G; equivalently, the unknown set Π^U must correspond exactly to the union of the nonreal and negative real entries in Λ.

A k-complexer is a complexer such that $|\Pi^U| = k$.

The following definitions are adapted from [3]:

Definition 4.3. A $\Pi_{m \oplus (n-m)}$ *complexer* is a complexer such that the nonreal entries, or Π^U, in Λ correspond exactly to a K_m and a K_{n-m}, disjoint, on the boundary nodes.

Definition 4.4. A $\Pi_{m,n-m}$ *complexer, where* $m = 1$ *or* 2*, is a complexer such that the real entries* Π_K *in* Λ *correspond exactly to a* K_n *and a* K_{n-m}*, disjoint, on the boundary nodes.*

Remark 4.5. Strictly speaking, $(\bigstar_4, \Pi_{2,2}, \gamma)$, where γ a simple admittance function, is not a complexer because it is not the case that, given one unknown entry in Λ, one may recover all the unknown entries in Λ. This triple still has interesting algebraic properties that I discuss in Section 6.

Definition 4.6. A $\Pi_{l\oplus(m-l),n-m}$ *complexer* is a complexer such that the real entries Π^K in Λ correspond exactly to a K_{n-m} and a $K_{l,m-l}$, disjoint, on the boundary nodes.

5. Cooking up Complexers on Stars

Some stars become complexers via a $\bigstar - K$ transformation. All and only these stars satisfy certain conditions on their admittances, so it is possible to cook up complexers by choosing the appropriate admittance function on a star and then transforming it into a K.

Remark 5.1. First, an explanation of my notation. In the figures that follow, the labels a_j, b_j and c_j that appear next to each node represent the values of the admittances on the corresponding edges of the original star. (Figure 4)

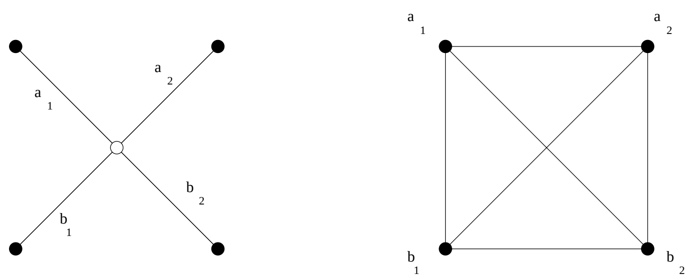

Figure 4. The graph on the left is the original $\bigstar_4$ with a simple admittance function on its edges; it transforms to the K_4 on the right. The labels a_1, a_2, b_1, b_2 on the K are the admittances on the corresponding edges of the original star.

It is important to remember that every complexer started as a $\bigstar_n$ and underwent a $\bigstar - K$ transformation.

In this section, take a $\bigstar_n$ with associated simple admittances γ_j on its edges. Recall that $\sigma = \gamma_1 + \cdots + \gamma_n$ is in the right half-plane.

Theorem 5.2. *When $n \geq 4$, a $\bigstar_n$ will transform to a $\Pi_{m\oplus n-m}$ complexer (Figure 5) if and only if it satisfies the following conditions:*

(1) $a_1 \parallel a_2 \parallel \cdots \parallel a_m$
(2) $b_1 \parallel b_2 \parallel \cdots \parallel b_{n-m}$
(3) $a_1 \parallel \sigma \bar{b}_1$
(4) $b_1 \nparallel a_1$

Proof. For all the edges between the a_j and b_k to be real-valued, it must be the case that for all j from 1 to m and for all k from 1 to $n-m$,

$$\begin{aligned} & \frac{a_j b_k}{\sigma} \in \mathbb{R} \\ \Leftrightarrow \quad & a_j \parallel \overline{\left(\frac{b_k}{\sigma}\right)} \\ \Leftrightarrow \quad & a_j \parallel \sigma \bar{b}_k \end{aligned}$$

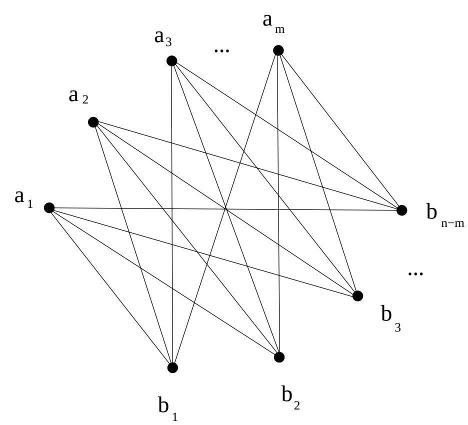

FIGURE 5. This is Π^K of a $\Pi_{m\oplus(n-m)}$ complexer. It is the complement of Π^U. All and only these edges in the complete graph on these n vertices correspond to response entries in $\mathbb{R}^+$.

In other words, Conditions 1, 2 and 3 taken together. This guarantees that every edge in Π^K is real-valued. Now we must guarantee that no edge in Π^U is real. That is to say, for all j and k,

$$\begin{aligned} &\frac{b_j b_k}{\sigma} \notin \mathbb{R} \\ \Leftrightarrow\quad & b_j \nparallel \overline{\left(\frac{b_k}{\sigma}\right)} \\ \Leftrightarrow\quad & b_j \nparallel a_1, \text{ because } a_1 \parallel \overline{\left(\frac{b_k}{\sigma}\right)} \end{aligned}$$

This is Condition 4. It guarantees that no edge in Π^U is real. By definition this is a $\Pi_{m\oplus(n-m)}$ complexer. □

Corollary 5.3. *When $n \geq 4$, a $\bigstar_n$ that transforms to a $\Pi_{m\oplus(n-m)}$ complexer has no real-valued admittances.*

Proof. First assume that one of the a_j is real. Then all the other a_j are real too. By the Conjugate Lemma, all the b_k are parallel to σ. Then σ and all the b_k must be real; if they were not, then σ could not be the sum of the admittances. But then $a_1 \parallel b_1$, which contradicts Condition 4. So none of the a_j is real. Similarly, none of the b_k is real. □

Theorem 5.4. *When $n - m \geq 3$ and $l, m - l \geq 1$, a $\bigstar_n$ will transform to a $\Pi_{l\oplus(m-l),n-m}$ complexer (Figure 6) if and only if it satisfies the following conditions:*

(1) $a_1 \parallel a_2 \parallel \cdots \parallel a_l$
(2) $b_1 \parallel b_2 \parallel \cdots \parallel b_{m-l}$
(3) $a_1 \parallel \sigma\bar{b}_1$
(4) $c_1 \parallel c_2 \parallel \cdots \parallel c_{n-m} \parallel \sqrt{\sigma}$
(5) $a_1 \nparallel b_1$

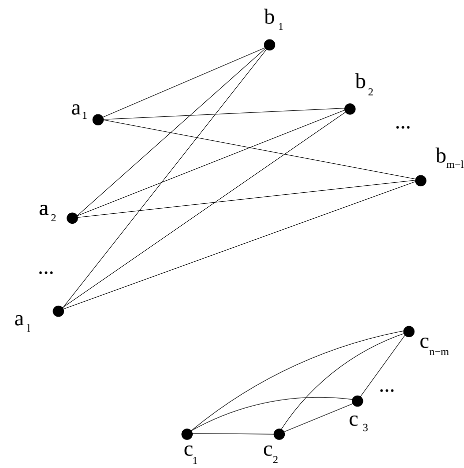

FIGURE 6. This is Π^K of a $\Pi_{l\oplus(m-l),n-m}$ complexer. It is the complement of Π^U. All and only these edges in the complete graph on these n vertices correspond to response entries in $\mathbb{R}^+$.

Proof. Conditions 1, 2 and 3 follow from the proof of Theorem 5.2. In addition, it must be the case that for all j, k from 1 to r,

$$\begin{aligned} & \frac{c_j c_k}{\sigma} \in \mathbb{R} \\ \Leftrightarrow \quad & c_j \parallel \sigma \bar{c_k} \\ \Leftrightarrow \quad & c_j \parallel \sigma \bar{c_j} \\ \Leftrightarrow \quad & c_j^2 \parallel \sigma \\ \Leftrightarrow \quad & c_j \parallel \sqrt{\sigma} \end{aligned}$$

This is Condition 4. Additionally, it must be the case that no edge from an a_j to a c_l or from a b_k to a c_l is real:

$$\begin{aligned} & \frac{a_j c_l}{\sigma} \notin \mathbb{R} \text{ and } \frac{b_k c_l}{\sigma} \notin \mathbb{R} \\ \Leftrightarrow \quad & a_j \nparallel \sigma \bar{c_l} \text{ and } b_k \nparallel \sigma \bar{c_l} \\ \Leftrightarrow \quad & a_j \nparallel \sqrt{\sigma} \text{ and } b_k \nparallel \sqrt{\sigma} \\ \Leftrightarrow \quad & a_j \nparallel b_k, \text{ because } a_j \parallel b_k \Leftrightarrow b_k \parallel \sqrt{\sigma} \quad \square \end{aligned}$$

Corollary 5.5. *When $n - m \geq 3$ and $l, m - l \geq 1$, a $\bigstar_n$ that transforms to a $\Pi_{l\oplus(m-l),n-m}$ complexer has no more than $n - m$ real-valued admittances.*

Proof. If the star has more than $n - m$ real-valued admittances, then at least one a_j or b_k is real-valued. Without loss of generality, assume a_1 is real. Then all the a_j are real. By the Conjugate Lemma, all the b_k are parallel to σ. Then, because all the c_l are parallel to $\sqrt{\sigma}$ and σ is the sum of the admittances, $\sqrt{\sigma}$ and σ must also be real. But then b_1 is real, and $a_1 \parallel b_1$, which contradicts Condition 5. So none of the a_j is real. Similarly, none of the b_k is real.

So a star that transforms to a $\Pi_{l\oplus(m-l),n-m}$ complexer has no more than $n - m$ real-valued admittances. $\square$

Remark 5.6. A star with $n - m$ real-valued admittances can transform to a $\Pi_{l\oplus(m-l),n-m}$ complexer. Assume one of the star's admittances is real-valued and, without loss of generality, it is called c_1. Then all the c_k are real and $\sqrt{\sigma}$ is real. So σ is real. As long as $\sum_{j=1}^{l} a_j$ is conjugate to $\sum_{k=1}^{m-l} b_k$ over the real axis, all the conditions are satisfied.

6. Unusual Small Partitions

Further discussion of the $\Pi_{1,n-1}$ complexer and the $\Pi_{2,2}$ partition is useful, because they are special cases of the larger theorems.

The $\Pi_{1,n-1}$ complexer is a special case of the $\Pi_{l\oplus(m-l),n-m}$ complexer, where $m = 1$.

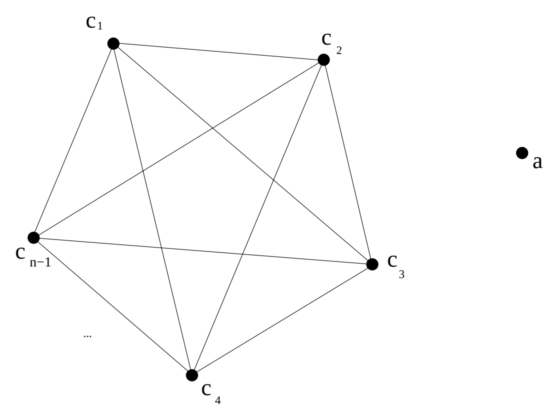

Figure 7. This is Π^K of a $\Pi_{1,n-1}$ complexer. It is the complement of Π^U. All and only these edges in the complete graph on these n vertices correspond to response entries in $\mathbb{R}^+$.

Lemma 6.1. *A $\bigstar_n$ transforms to a $\Pi_{1,n-1}$ complexer (Figure 7) if and only if it conforms to the following conditions:*

(1) $c_1 \parallel c_2 \parallel \cdots \parallel c_{n-2} \parallel \sqrt{\sigma}$
(2) $c_1 \nparallel \sigma\bar{a}$

Proof. Condition 1 follows from the proof of Theorem 5.4; it guarantees that every edge in the K_{n-1} is real-valued. Additionally, it must be the case that no edge between a and any c_j is real-valued, which is Condition 2. □

Corollary 6.2. *A $\bigstar_n$ that transforms to a $\Pi_{1,n-1}$ complexer has no real-valued admittances.*

Proof. First assume one of the c_j is real. Then all the c_j are real. By the Conjugate Lemma, $\sqrt{\sigma}$ and σ are real. Then a is real, because σ is the sum of the admittances. But then no entry in the response is nonreal-valued, and the star cannot transform to a complexer. So none of the c_j is real-valued. Similarly, a is not real. □

This result is significant because a star that transforms to a $\Pi_{l\oplus(m-l),n-m}$ with $l, m - l \geq 1$ may have some real-valued admittances, as shown in Corollary 5.5.

$(\bigstar_4, \Pi_{2,2})$ is not a plexer [3], so $(\bigstar_4, \Pi_{2,2}, \gamma)$ is not a complexer. But it is still possible to discuss the conditions a $\bigstar_4$ must satisfy if there is to be a $\Pi_{2,2}$ partition on its response entries. These conditions are interesting because they are less stringent than the conditions a $\bigstar_n, n \geq 5$, must satisfy if it is to transform to a $\Pi_{2,n-2}$ complexer.

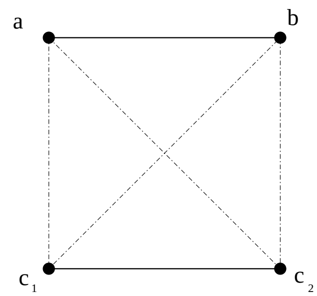

FIGURE 8. This is a $\bigstar_4$ with a $\Pi_{2,2}$ partition on its response entries. The solid lines denote the entries in $\mathbb{R}^+$ in Λ; the dashed lines represent the entries not in $\mathbb{R}^+$.

Lemma 6.3. *A $\bigstar_4$ transforms to a $\Pi_{2,2}$ complexer (Figure 8) if and only if it conforms to the following conditions:*

(1) $a \parallel \sigma\bar{b}$
(2) $c_1 \parallel \sigma\bar{c_2}$
(3) $a \nparallel \sigma\bar{c_1}, a \nparallel \sigma\bar{c_2}, b \nparallel \sigma\bar{c_1}, b \nparallel \sigma\bar{c_2}$

Proof. It must be the case that

$$\frac{ab}{\sigma}, \frac{c_1c_2}{\sigma} \in \mathbb{R}$$
$$\Leftrightarrow \quad a \parallel \sigma\bar{b} \text{ and } c_1 \parallel \sigma\bar{c_2}$$

These are Conditions 1 and 2. Now to ensure that the other four response entries are not real-valued:

$$\frac{ac_1}{\sigma} \notin \mathbb{R}, \ \frac{ac_2}{\sigma} \notin \mathbb{R}, \ \frac{bc_1}{\sigma} \notin \mathbb{R}, \ \frac{bc_2}{\sigma} \notin \mathbb{R}$$
$$\Leftrightarrow \quad a \nparallel \sigma\bar{c_1}, a \nparallel \sigma\bar{c_2}, b \nparallel \sigma\bar{c_1}, b \nparallel \sigma\bar{c_2} \quad \square$$

Note that it is not required that $a_1 \parallel a_2 \parallel \sqrt{\sigma}$ on a $\bigstar_4$. For such a partition to result on a $\bigstar_n, n \geq 5$, it is required that all the c_j are parallel to $\sqrt{\sigma}$, as shown in Theorem 5.4.

7. DEMONSTRATIONS

Example 7.1. *A $\Pi_{1\oplus 4}$ complexer on a $\bigstar_5$. (Figure 9)*

To cook up this complexer, first choose a ray from the origin in the right half-plane that will be proportional to σ. This will define the ray proportional to $\sqrt{\sigma}$, because $\arg(\sqrt{\sigma}) = \frac{1}{2}\arg(\sigma)$.

It follows from the conditions in Theorem 5.2 that a never lies within the region

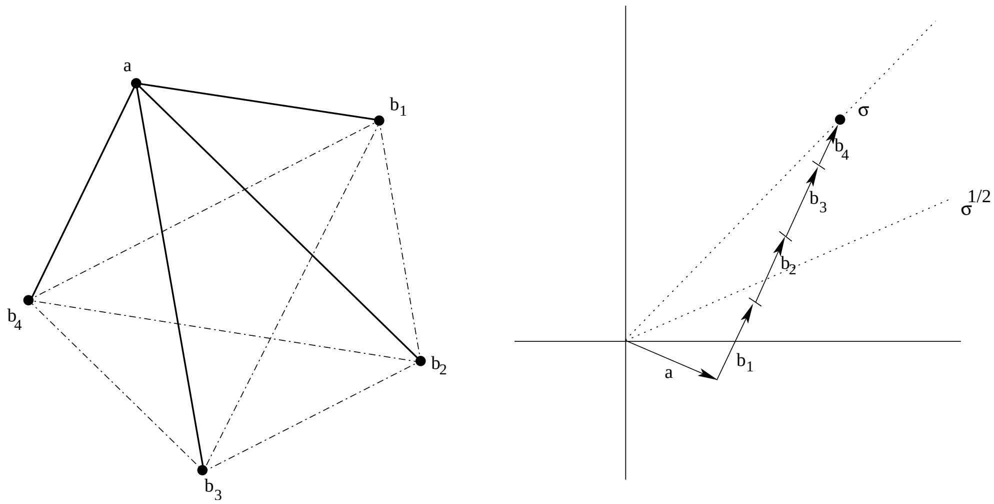

FIGURE 9. The solid edges in the complete graph are real-valued and correspond to Π^K; the dashed edges are not real-valued and correspond to Π^U.

between the real axis and the line proportional to σ. Also, the angle between a and the line proportional to $\sqrt{\sigma}$ must be smaller than the angle between the line proportional to $\sqrt{\sigma}$ and the imaginary axis. This follows from the stipulation that all the admittances are in the right half-plane. Choose any simple admittance a in accordance with these directions, as in Figure 9.

This move determines everything else. The b_k are all conjugate to a over the line proportional to $\sqrt{\sigma}$, so the argument of every b_k is determined. So the line proportional to the b_k is determined. σ is also determined; it is the intersection of the line proportional to σ and the line proportional to the b_k. Additionally, $a + \sum_{k=1}^{4} b_k = \sigma$, so the modulus of $(b_1 + b_2 + b_3 + b_4)$ is determined. All that remains, then, is to divide this the value of this modulus among the four b_k.

Example 7.2. *A $\Pi_{2\oplus 3,3}$ complexer on a $\bigstar_7$. (Figure 10)*
To cook up this complexer, first choose a ray from the origin in the right half-plane that will be proportional to σ. This will define the ray proportional to $\sqrt{\sigma}$, because $\arg(\sqrt{\sigma}) = \frac{1}{2}\arg(\sigma)$.

Next, choose a simple admittance a_1 in accordance with the conditions in Example 1. $a_1 \parallel a_2$, so the argument of a_2 is determined. Choose a modulus for a_2. The argument of the b_k is determined, because a_1 is conjugate to the b_k over the line proportional to $\sqrt{\sigma}$. The modulus of $(b_1 + b_2 + b_3)$ must be great enough to allow a simple admittance from $(a_1 + a_2 + b_1 + b_2 + b_3)$ and parallel to $\sqrt{\sigma}$ to intersect the line proportional to σ, as in Figure 10. Choose the modulus of each b_k in this way. Then σ is determined; it is the intersection of the simple admittance from $(a_1 + a_2 + b_1 + b_2 + b_3)$ parallel to $\sqrt{\sigma}$ and the line proportional to σ. Additionally, $a_1 + a_2 + \sum_{k=1}^{3} b_k + \sum_{k=1}^{3} c_l = \sigma$, so the modulus of $(c_1 + c_2 + c_3)$ is determined.

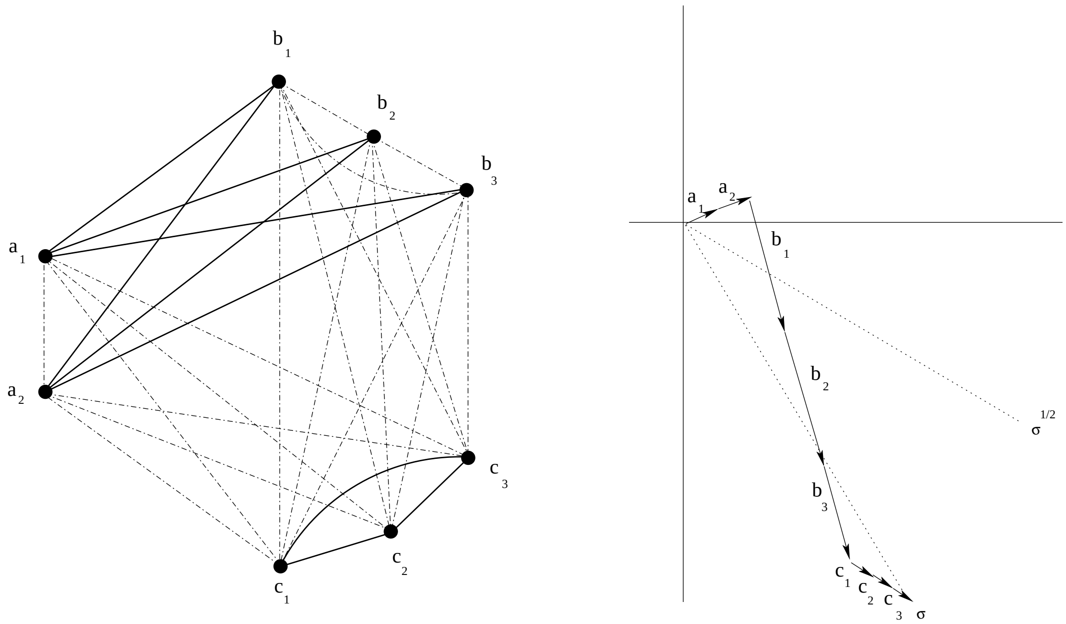

FIGURE 10. The solid edges in the complete graph are real-valued and correspond to Π^K; the dashed edges are not real-valued and correspond to Π^U.

All that remains, then, is to divide this the value of this modulus among the three c_l.

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