THE NEUMANN-TO-DIRICHLET MAP DETERMINES CONNECTION STRUCTURE FOR CIRCULAR PLANAR NETWORKS

NATE BOTTMAN

Fix a circular planar network $\Gamma = (G, \gamma)$, where the underlying graph G is an ordered pair (V, E), where V is disjointly partitioned as $\partial V \cup \text{int } V$. Let $\mathbf{H}$ denote the Neumann-to-Dirichlet map, which maps a boundary voltage to the unique normalized boundary current. Consider the two following theorems.

Theorem 1. *Let $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_k)$ be two sequences of boundary nodes. Define*

$$\mathbf{L} = \begin{pmatrix} (\eta_{p_1q_1} - \eta_{p_1q_k}) - (\eta_{p_kq_1} - \eta_{p_kq_k}) & \cdots & (\eta_{p_1q_{k-1}} - \eta_{p_1q_k}) - (\eta_{p_kq_{k-1}} - \eta_{p_kq_k}) \\ \vdots & \ddots & \vdots \\ (\eta_{p_{k-1}q_1} - \eta_{p_{k-1}q_k}) - (\eta_{p_kq_1} - \eta_{p_kq_k}) & \cdots & (\eta_{p_{k-1}q_{k-1}} - \eta_{p_{k-1}q_k}) - (\eta_{p_kq_{k-1}} - \eta_{p_kq_k}) \end{pmatrix}.$$

If $\mathbf{L}$ is singular, there does not exist a connection between P and Q.

Proof. To start off, some notation: let $S = \partial V \setminus (P \cup Q)$. Assume that $\mathbf{L}$ is singular. Then there exists $\mathbf{x} = \begin{pmatrix} x_1 & \ldots & x_{k-1} \end{pmatrix}^T \neq 0$ such that $\mathbf{L}\mathbf{x} = \mathbf{0}$. It follows that

$$\begin{pmatrix} (\eta_{p_1q_1} - \eta_{p_1q_k}) - (\eta_{p_kq_1} - \eta_{p_kq_k}) & \cdots & (\eta_{p_1q_{k-1}} - \eta_{p_1q_k}) - (\eta_{p_kq_{k-1}} - \eta_{p_kq_k}) \\ \vdots & \ddots & \vdots \\ (\eta_{p_{k-1}q_1} - \eta_{p_{k-1}q_k}) - (\eta_{p_kq_1} - \eta_{p_kq_k}) & \cdots & (\eta_{p_{k-1}q_{k-1}} - \eta_{p_{k-1}q_k}) - (\eta_{p_kq_{k-1}} - \eta_{p_kq_k}) \\ 0 & \cdots & 0 \end{pmatrix} \mathbf{x} = \mathbf{0}.$$

Next, it follows that

$$\begin{pmatrix} \eta_{p_1q_1} - \eta_{p_1q_k} & \cdots & \eta_{p_1q_{k-1}} - \eta_{p_1q_k} \\ \ldots & \ddots & \ldots \\ \eta_{p_kq_1} - \eta_{p_kq_k} & \cdots & \eta_{p_kq_{k-1}} - \eta_{p_kq_k} \end{pmatrix} \mathbf{x} = \alpha\mathbf{e},$$

where $\alpha = \begin{pmatrix} \eta_{p_kq_1} - \eta_{p_kq_k} & \ldots & \eta_{p_kq_{k-1}} - \eta_{p_kq_k} \end{pmatrix} \mathbf{x}$ and $\mathbf{e}$ is the ones vector of appropriate length. Finally, it follows that

$$\begin{pmatrix} \eta_{p_1q_1} & \cdots & \eta_{p_1q_k} \\ \vdots & \ddots & \vdots \\ \eta_{p_kq_1} & \cdots & \eta_{p_kq_k} \end{pmatrix} \mathbf{y} = \alpha\mathbf{e},$$

where

$$\mathbf{y} = \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ -\sum_{j=1}^{k-1} x_j \end{pmatrix}.$$

It follows that there exists $\mathbf{y} \neq 0$ such that $\mathbf{H}(P; Q)\,\mathbf{y} = \alpha\mathbf{e}$, with the property that the element sum over $\mathbf{y}$ is zero.

This work implies that

$$\begin{pmatrix} \mathbf{H}(P; P) & \mathbf{H}(P; S) & \mathbf{H}(P; Q) \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{y} \end{pmatrix} = \alpha\mathbf{e}.$$

This equation in turn implies that

$$\begin{pmatrix} \mathbf{H}(P;P) & \mathbf{H}(P;S) & \mathbf{H}(P;Q) \\ \mathbf{H}(P;S)^T & \mathbf{H}(S;S) & \mathbf{H}(S;Q) \\ \mathbf{H}(P;Q)^T & \mathbf{H}(S;Q)^T & \mathbf{H}(Q;Q) \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} + \alpha \mathbf{e},$$

where $\mathbf{z}_1 = \mathbf{H}(S;Q)\,\mathbf{x} - \alpha\mathbf{e}$ and $\mathbf{z}_2 = \mathbf{H}(Q;Q) - \alpha\mathbf{e}$. That is,

$$\mathbf{H} \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} + \alpha \mathbf{e}.$$

Note that since the element sum over $\mathbf{y}$ is zero, we have shown that the potential $\begin{pmatrix} \mathbf{0} & \mathbf{z}_1 & \mathbf{z}_2 \end{pmatrix}^T + \alpha\mathbf{e}$ solves the Neumann problem for the input current $\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{y} \end{pmatrix}^T$. Since no normalization is required for the Dirichlet problem, it follows that the potential $\begin{pmatrix} \mathbf{0} & \mathbf{z}_1 & \mathbf{z}_2 \end{pmatrix}^T$ induces the current $\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{y} \end{pmatrix}^T$. It follows that the vector $\begin{pmatrix} \mathbf{z}_1 & \mathbf{z}_2 \end{pmatrix}^T$ is nonzero: if it was the zero vector, the input potential $\begin{pmatrix} \mathbf{0} & \mathbf{z}_1 & \mathbf{z}_2 \end{pmatrix}^T$ would be constant, which would imply that the output current $\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{y} \end{pmatrix}^T$ is constant, which would imply that $\mathbf{y}$ is the zero vector.

This work shows that the equation

$$\mathbf{\Lambda} \begin{pmatrix} \mathbf{0} \\ \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{y} \end{pmatrix}$$

holds, which implies that

$$\mathbf{\Lambda}(P,S;S,Q) \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} = \mathbf{0}.$$

Since $\begin{pmatrix} \mathbf{z}_1 & \mathbf{z}_2 \end{pmatrix}^T$ is nonzero, it follows that $\mathbf{\Lambda}(P,S;S,Q)$ is singular. The connection-determinant formula implies that there does not exist a connection between P and Q. □

Theorem 2. *Let $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_k)$ be two sequences of boundary nodes. Define $\mathbf{L}$ in the previous fashion. If $\mathbf{L}$ is nonsingular, there exists a connection between P and Q.*

Proof. Let's prove the contrapositive of the claim. Assume that there does not exist a connection between P and Q. Note that the proof of the previous claim is reversible; that is, under the assumption that there does not exists a connection between P and Q, the previous argument implies that $\mathbf{L}$ is singular. The contrapositive follows. □

The title of this paper is now justified.