ON THE DISCRETIZED CAUCHY PROBLEM

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1. Getting acquainted

In the continuous case, Cauchy problems are often defined in the following way:

Problem 1. *Consider a κ-th ordinary differential equation in u, of the form*

$$\frac{d^\kappa u}{dx^\kappa} = F\left(x, u, \frac{du}{dx}, \ldots, \frac{d^{\kappa-1}u}{dx^{\kappa-1}}\right),$$

where F is some polynomial in κ variables. Given some compact interval $I \subset \mathbb{R}$, find a solution u on I of the given differential equation, satisfying also the initial conditions

$$\begin{aligned} u(x_0) &= \alpha_0 \\ \left.\frac{du}{dx}\right|_{x=x_0} &= \alpha_1 \\ &\vdots \\ \left.\frac{d^{\kappa-1}u}{dx^{\kappa-1}}\right|_{x=x_0} &= \alpha_{\kappa-1}, \end{aligned}$$

for some $x_0 \in I$.

This Cauchy problem can be generalized in many ways in the continuous case: for instance, let u satisfy a partial differential equation in n variables, work on subset $I \subset \mathbb{R}^n$, and require $n\kappa$ conditions to be satisfied on some subset of ∂I. In this paper, I will explore a special case of the generalized Cauchy problem on finite discrete networks.

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Definition 1. *Let $\Gamma = (G, \gamma)$ be a finite network, where G represents a graph and γ a conductivity function defined on the edges of G. Let $G = (V, E)$ be a finite graph, where the set of vertices V is partitioned into three sets of boundary nodes P, S, Q, and $\operatorname{int} V$, such that $P \cup S \cup Q \cup \operatorname{int} V = V$.*

Definition 2. *A function $u : V \to \mathbb{R}$ is said to be γ-harmonic on Γ if, for every $j \in \operatorname{int} V$, the equation*

$$\sum_{i \sim j} \gamma_{i,j} \left(u_i - u_j\right) = 0$$

holds.

Now that we have defined our network Γ and γ-harmonicity, we can state the Cauchy problem.

Problem 2. *Fix a network Γ. Under what conditions on G is it true that, for every conductivity function γ, for every function ϕ, defined on $P \cup S$, and for every function ψ, defined on P, there exists a unique function u, γ-harmonic on Γ, such that $u|_{P \cup S} = \phi$ and*

$$\sum_{i \sim j} \gamma_{i,j} \left(u_i - u_j\right) = \psi,$$

for all $j \in P$?

The uniqueness requirement may seem restrictive. There are two arguments for this restriction. First, consider the continuous statement of the Cauchy problem. In general, when solving physically-relevant systems of differential equationSSolutions of non-pathological problems satisfy some local uniqueness condition. Second, there is a strong physical motivation for this problem. Say that we wish to know the voltages on the surface of a patient's heart, based on measurements of voltages and currents on the patient's chest. Well, if we knew that the voltages on the heart were given either by Φ_A or Φ_B, but that no further information could be gotten, we're out of luck! In order for our answer to be meaningful, we must be able to uniquely determine the voltages on the heart.

Another bit of notation: let $\phi_1 = \phi|_P$ and $\phi_2 = \phi|_S$.

2. Onward and upward

First, a quick definition.

Definition 3. *Define the current at j, denoted as I_j, as*

$$\sum_{i \sim j} \gamma_{i,j} \left(u_i - u_j\right).$$

Note that we are using a sign convention opposite to [1]. Next, two formal definitions.

Definition 4. *Define the linear map $K : \mathbb{R}^{|V|} \to \mathbb{R}^{|V|}$ to be the function taking $u|_V$ to $I|_V$. Define the linear map $\Lambda : \mathbb{R}^{|\partial V|} \to \mathbb{R}^{|\partial V|}$ to be the function taking $u|_{\partial V}$ to $I|_{\partial V}$.*

It is clear that

$$K = \begin{pmatrix} -\sum_{j\neq 1}\gamma_{1,j} & \gamma_{PS} & \gamma_{PQ} & \cdots & \gamma_{1,|V|} \\ \gamma_{2,1} & -\sum_{j\neq 2}\gamma_{2,j} & \gamma_{SQ} & \cdots & \gamma_{2,|V|} \\ \gamma_{3,1} & \gamma_{3,2} & -\sum_{j\neq 3}\gamma_{3,j} & \cdots & \gamma_{3,|V|} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{|V|,1} & \gamma_{|V|,2} & \gamma_{|V|,3} & \cdots & -\sum_{j\neq |V|}\gamma_{|V|,j} \end{pmatrix}.$$

For our purposes, there are two useful partitions of K. Firstly, we can partition K as

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where the first row corresponds to ∂V and the second to $\operatorname{int} V$; the columns correspond in the same fashion to this partition of V. As proven in [1], under this partitioning of K, Λ is given by $K/C = A - BC^{-1}B^T$. Secondly, we can partition K as

$$\begin{pmatrix} A_{PP} & A_{PS} & A_{PQ} & B_P \\ {A_{PS}}^T & A_{SS} & A_{SQ} & B_S \\ {A_{PQ}}^T & {A_{SQ}}^T & A_{QQ} & B_Q \\ {B_P}^T & {B_S}^T & {B_Q}^T & C \end{pmatrix},$$

where the first row corresponds to P, the second to S, the third to Q, and the fourth row to $\operatorname{int} V$; the columns correspond in the same fashion to this partition of V. The first partitioning is useful, in that the response matrix Λ can then be written in a simple, intuitive form (take the Kirchhoff matrix and take the Schur complement with respect to the submatrix corresponding to $\operatorname{int} V$-$\operatorname{int} V$ connections), whereas the second partitioning is useful, in that it is explicit in the partitioning of the boundary nodes.

An analogous partitioning of Λ is very useful. From now on, I will write Λ as

$$\begin{pmatrix} \Lambda_{PP} & \Lambda_{PS} & \Lambda_{PQ} \\ {\Lambda_{PS}}^T & \Lambda_{SS} & \Lambda_{SQ} \\ {\Lambda_{PQ}}^T & {\Lambda_{SQ}}^T & \Lambda_{QQ} \end{pmatrix}.$$

By the definition of the response matrix, this partitioning allows us to write

$$\begin{pmatrix} \Lambda_{PP} & \Lambda_{PS} & \Lambda_{PQ} \\ {\Lambda_{PS}}^T & \Lambda_{SS} & \Lambda_{SQ} \\ {\Lambda_{PQ}}^T & {\Lambda_{SQ}}^T & \Lambda_{QQ} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ x \end{pmatrix} = \begin{pmatrix} \psi \\ y \\ z \end{pmatrix}.$$

Here ϕ_1, ϕ_2, and ψ are known, whereas x, y, and z — the voltage on Q, the current on S, and the current on Q, respectively — are not.

By the first line of this matrix equation, we have $\Lambda_{PP}\phi_1 + \Lambda_{PS}\phi_2 + \Lambda_{PQ}x = \psi$. Since we require x to be determined uniquely, we can now state a useful theorem.

Theorem 1. *The voltage vector x can be determined uniquely if Λ_{PQ} is square and invertible.*

This is a sufficient algebraic condition for a unique solution of the Cauchy problem to exist. However, with the help of the following theorem, proved in [1], some geometric intuition can be gleaned from this algebraic condition.

Theorem 2. *Take U and V to be disjoint subsets of V. If the submatrix $\Lambda(U;V)$ of the response matrix Λ is square and invertible, then there exists a connection between U and V. If the graph we are working on is circular planar, then this condition is both necessary and sufficient.*

If it assumed that $\det \Lambda_{PQ} \neq 0$ is also a necessary condition for the unique solubility of the Cauchy problem, then a necessary condition for the existence of a unique solution of the Cauchy problem is that $|P| = |Q|$, and that there exists a connection between P and Q. However, it is difficult to formulate a necessary and sufficient geometric condition.

Remark. Denote the set of neighboring nodes of a vertex $v \in V$ by $\mathcal{N}(v)$. Operating on physical intuition, derived from the problem of determining the voltages on the surface of the heart, it seems reasonable to assume that $P \cap \mathcal{N}(Q) = \mathcal{N}(P) \cap Q = \emptyset$. Specifically, it follows that A_{PQ} is a zero matrix. Since $\Lambda = A - BC^{-1}B^T$, $\Lambda_{PQ} = A_{PQ} - B_P C {B_Q}^T$; in this case, it follows that $\Lambda_{PQ} = -B_P C {B_Q}^T$. We can now observe another nice aspect of the solvability of the Cauchy problem: if it assumed that P and Q do not border on one another, then the solvability of the Cauchy problem is determined solely by edges connecting nodes in P to nodes in $\operatorname{int} V$, edges connecting nodes in Q to nodes in $\operatorname{int} V$, and edges connecting nodes in $\operatorname{int} V$ to nodes in $\operatorname{int} V$.

From now on, assume that Λ_{PQ} is square and invertible. Assuming this, the first line of the matrix equation gives

$$x = {\Lambda_{PQ}}^{-1}\left(\psi - \Lambda_{PP}\phi_1 - \Lambda_{PS}\phi_2\right).$$

Substituting this expression into the second and third lines of the matrix equation, it follows that

$$\begin{aligned} y &= {\Lambda_{PS}}^T\phi_1 + \Lambda_{SS}\phi_2 + \Lambda_{SQ}{\Lambda_{PQ}}^{-1}\left(\psi - \Lambda_{PP}\phi_1 - \Lambda_{PS}\phi_2\right) \\ z &= {\Lambda_{PQ}}^T\phi_1 + {\Lambda_{SQ}}^T\phi_2 + \Lambda_{QQ}{\Lambda_{PQ}}^{-1}\left(\psi - \Lambda_{PP}\phi_1 - \Lambda_{PS}\phi_2\right). \end{aligned}$$

These equations can be elegantly rewritten, using Schur complements:

$$\begin{aligned} y &= \Lambda_{SQ}{\Lambda_{PQ}}^{-1}\psi + \left({\Lambda_{PS}}^T - \Lambda_{SQ}{\Lambda_{PQ}}^{-1}\Lambda_{PP}\right)\phi_1 + \left(\Lambda_{SS} - \Lambda_{SQ}{\Lambda_{PQ}}^T\Lambda_{PS}\right)\phi_2 \\ &= \Lambda_{SQ}{\Lambda_{PQ}}^{-1}\psi + \left(\Lambda\left(P\cup S; P\cup Q\right)/\Lambda_{PQ}\right)\phi_1 + \left(\Lambda\left(P\cup Q; S\cup Q\right)/\Lambda_{PQ}\right)\phi_2 \\ z &= \Lambda_{QQ}{\Lambda_{PQ}}^{-1}\psi + \left({\Lambda_{PQ}}^T - \Lambda_{QQ}{\Lambda_{PQ}}^{-1}\Lambda_{PP}\right)\phi_1 + \left({\Lambda_{SQ}}^T - \Lambda_{QQ}{\Lambda_{PQ}}^T\Lambda_{PS}\right)\phi_2 \\ &= \Lambda_{QQ}{\Lambda_{PQ}}^{-1}\psi + \left(\Lambda\left(P\cup Q; P\cup Q\right)/\Lambda_{PQ}\right)\phi_1 + \left(\Lambda\left(P\cup Q; S\cup Q\right)/\Lambda_{PQ}\right)\phi_2. \end{aligned}$$

These equations can be cleaned up considerably. If we write our equations for the unknown quantities in terms of known quantities, we have

$$M\begin{pmatrix}\phi_1\\ \phi_2\\ \psi\end{pmatrix} = \begin{pmatrix}x\\ y\\ z\end{pmatrix},$$

where

$$M = \begin{pmatrix} {\Lambda_{PQ}}^{-1} & -{\Lambda_{PQ}}^{-1}\Lambda_{PP} & -{\Lambda_{PQ}}^{-1}\Lambda_{PS} \\ \Lambda_{SQ}{\Lambda_{PQ}}^{-1} & \Lambda\left(P\cup S; P\cup Q\right)/\Lambda_{PQ} & \Lambda\left(P\cup S; S\cup Q\right)/\Lambda_{PQ} \\ \Lambda_{QQ}{\Lambda_{PQ}}^{-1} & \Lambda\left(P\cup Q; P\cup Q\right)/\Lambda_{PQ} & \Lambda\left(P\cup Q; S\cup Q\right)/\Lambda_{PQ} \end{pmatrix}.$$

This equation can be rewritten as

$$\begin{pmatrix} {\Lambda_{PQ}}^{-1} & -{\Lambda_{PQ}}^{-1}\Lambda\left(P; P\cup S\right) \\ \Lambda\left(S\cup Q; Q\right){\Lambda_{PQ}}^{-1} & \Lambda/\Lambda_{PQ} \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \psi\end{pmatrix} = \begin{pmatrix}x\\ y\\ z\end{pmatrix}.$$

I will refer to this matrix M as the Cauchy map.

3. Properties of the Cauchy map

The Cauchy map derived here is strikingly similar to the analogous map described in [2] (the problem discussed in [2] is another statement of the Cauchy problem, except that exactly one of voltage and current is known at each boundary node). As in [2], we can investigate the injectivity of M by considering $\det M$.

$$\begin{aligned}
\det M &= \frac{\det\left(M/{\Lambda_{PQ}}^{-1}\right)}{\det \Lambda_{PQ}} \\
&= \frac{\det\left(\Lambda/\Lambda_{PQ} - \left(\Lambda\left(S\cup Q;Q\right){\Lambda_{PQ}}^{-1}\right)\Lambda_{PQ}\left(-{\Lambda_{PQ}}^{-1}\Lambda\left(P;P\cup S\right)\right)\right)}{\det \Lambda_{PQ}} \\
&= \frac{\det\left(\Lambda/\Lambda_{PQ} + \Lambda\left(S\cup Q;Q\right){\Lambda_{PQ}}^{-1}\Lambda\left(P;P\cup S\right)\right)}{\det \Lambda_{PQ}} \\
&= \frac{\det\begin{pmatrix} {\Lambda_{PS}}^T & \Lambda_{SS} \\ {\Lambda_{PQ}}^T & {\Lambda_{SQ}}^T \end{pmatrix}}{\det \Lambda_{PQ}} = \frac{\det\begin{pmatrix} \Lambda_{PS} & \Lambda_{PQ} \\ \Lambda_{SS} & \Lambda_{SQ} \end{pmatrix}}{\det \Lambda_{PQ}}.
\end{aligned}$$

That is, M is injective if and only if the sub-block $\Lambda\left(P\cup S;S\cup Q\right)$ is invertible.

4. The dual problem

Let me now propose a new Cauchy problem, which is slightly different than the Cauchy problem we have been working with so far.

Problem 3. *Fix a network Γ. Under what conditions on G is it true that, for every conductivity function γ, for every function x, defined on Q, for every function y, defined on S, and for every function z, defined on some Q, there exists a unique function u, γ-harmonic on Γ, such that $u|_Q = x$, $I|_S = y$, and $I|_Q = z$?*

This is a sort of dual problem of the Cauchy problem we have been working with: instead of seeking a map from (ϕ_1, ϕ_2, ψ) to (x, y, z), we are now seeking a map from (x, y, z) to (ϕ_1, ϕ_2, ψ). Let's try to apply our arguments for the Cauchy problem to the dual problem (don't get impatient; exciting things are in store!).

Recall our matrix equation:

$$\begin{pmatrix} \Lambda_{PP} & \Lambda_{PS} & \Lambda_{PQ} \\ {\Lambda_{PS}}^T & \Lambda_{SS} & \Lambda_{SQ} \\ {\Lambda_{PQ}}^T & {\Lambda_{SQ}}^T & \Lambda_{Q.Q} \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ x \end{pmatrix} = \begin{pmatrix} \psi \\ y \\ z \end{pmatrix}.$$

Here x, y, and z are known, whereas ϕ_1, ϕ_2, and ψ are not.

By the first and second lines of this matrix equation, we have

$$\begin{aligned} {\Lambda_{PS}}^T\phi_1 + \Lambda_{SS}\phi_2 + \Lambda_{SQ}x &= y \\ {\Lambda_{PQ}}^T\phi_1 + {\Lambda_{SQ}}^T\phi_2 + \Lambda_{Q.Q}x &= z \end{aligned}.$$

This can be rewritten as a matrix equation:

$$\begin{pmatrix} \Lambda_{PS} & \Lambda_{PQ} \\ \Lambda_{SS} & \Lambda_{SQ} \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} y - \Lambda_{SQ}x \\ z - \Lambda_{QQ}x \end{pmatrix}.$$

Since we require x to be determined uniquely, we can now state a useful theorem.

Theorem 3. *The voltage vector $\phi = (\phi_1, \phi_2)^T$ can be determined uniquely if and only if $\Lambda\,(P \cup S; S \cup Q)$ is square and invertible.*

Computing

$$L : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \phi_1 \\ \phi_2 \\ \psi \end{pmatrix},$$

which takes the known data to the solution of the dual problem, is not really necessary: provided that M^{-1} is defined, $L = M^{-1}$. That is, $\det L = 1/\det M$, so when M is invertible, L is invertible.

References

[1] Edward B. Curtis and James A. Morrow. *Inverse Problems for Electrical Networks*. World Scientific, 2000.
[2] Jeremiah Jones and Jaime Ramos. Mixed problems. 2007.