BOUNDARY VALUE PROBLEMS ON A NETWORK AND ITS DUAL

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Abstract. In previous work, it was found that the *Neumann-to-Dirichlet* map, $\mathbf{H}$, and the *Dirichlet-to-Neumann* map, $\mathbf{\Lambda}$, on a network may be expressed in terms of each other. Here, we use the concept of γ-harmonic conjugates on connected circular planar networks to relate Dirichlet boundary data on a network to Neumann boundary data on the dual and vice versa. With these relations, we find ways to express $\mathbf{H}$ and $\mathbf{\Lambda}$ on the network and its dual in terms of each other. In addition, we briefly explore the possibility of a natural correspondence between two mixed problems, one on a network and the other on its dual.

Contents

1. Basic Definitions and Notation

We start with the usual definitions and notation for a circular planar network, as stated in [1]. In the definitions below, "graph" is used to refer to an undirected graph.

Definition 1.1. A *graph with boundary* is a triple $G = (V, \partial V, E)$, where V is the set of vertices and E is the set of edges in a finite graph. ∂V is a nonempty subset of V, whose members are referred to as the *boundary vertices*. All other vertices are referred to as *interior vertices*.

Date: December 1, 2008.

Definition 1.2. A *circular planar graph* is a graph with boundary that may be embedded in a disc D in the plane so that the boundary vertices lie on the circle which bounds D and the rest of the vertices lie in the interior of D.

Definition 1.3. A *circular planar network* is a pair $\Gamma = (G, \gamma)$, where $G = (V, \partial V, E)$ is a circular planar graph and $\gamma : E \to \mathbb{R}^+$ is a function on the edges of the graph, referred to as the *conductivity*.

Now that we have defined the type of networks we are considering, we may consider vertex functions on the networks, which represent potentials at each of these vertices. As such, given a network $\Gamma = (G, \gamma)$ with associated graph $G = (V, \partial V, E)$, we consider vertex functions of the form $u : V \to \mathbb{R}$. Given such a potential function on the vertices, the current out of each vertex can be calculated with the expression below, derived from Ohm's Law:

$$\mathcal{I}(p) = \sum_{q \in \mathcal{N}(p)} \gamma(p,q)\big(u(p) - u(q)\big)$$

Above, $\mathcal{N}$(p) denotes the set of vertices in G that are adjacent to p, that are connected to p by an edge. We can define a linear operation K on the potential function that will give us the resulting current out of every vertex.

Definition 1.4. The *Kirchhoff matrix* is the linear operator that takes a potential function on the graph to a vertex function c, which describes the current out of each vertex with u as the potential function. In other words, $c = Ku$. The entries of the Kirchhoff matrix are:

$$K_{ij} = \begin{cases} \sum_{k \neq i} \gamma(i,k) & \text{if } i = j \\ -\gamma(i,j) & \text{if } i \neq j \end{cases}$$

In the above, if no edge (i, j) exists, then $\gamma(i, j) = 0$.

In the coming discussions, we also consider K partitioned in the following manner, where the vertices of the graph are indexed with the boundary vertices preceding the interior vertices:

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

We may now define a special type of potential function, one for which Kirchhoff's current law is satisfied for all interior nodes.

Definition 1.5. A vertex function u is γ-harmonic, if the following holds:

$$\mathcal{I}(p) = \sum_{q \in N(p)} \gamma(p,q)\big(u(p) - u(q)\big) = 0 \qquad \forall p \in V - \partial V$$

or equivalently

$$Ku = \begin{bmatrix} \psi \\ 0 \end{bmatrix}$$

where ψ denotes the current out of the boundary vertices.

It is shown in [1] that for a given set of potential values on the boundary vertices, there exists a unique γ-harmonic potential function on the graph, which results in a unique ψ. They also develop the map that takes these boundary potentials to their resulting boundary currents.

Definition 1.6. The *Dirichlet-to-Neumann* map, or *response matrix*, is the unique linear operator that sends specified boundary potentials to the resulting boundary currents. It can be easily expressed in terms of the submatrices of K, under the usual partition:

$$\mathbf{\Lambda} = A - BC^{-1}B^T$$

For a more complete explanation of and elaboration on the above concepts, see [1].

2. Preliminaries on γ-harmonic Conjugates

The idea of γ-harmonic conjugates to a γ-harmonic potential function on a network was introduced by Karen Perry in 2003 in [2] and further investigated by Owen Biesel and Amanda Rohde in 2005 in [3]. In order to discuss this concept further, we must first define the dual graph for connected circular planar networks.

Definition 2.1. A *face* of a connected circular planar graph G is a connected set of points in the set $D - \mathcal{G}$, where D is the unit disc, and $\mathcal{G}$ is the set of points that is the image of G under the embedding associated with it. If the closure of the face includes points in ∂D, then the face is referred to as an *exterior face*. Other faces are referred to as *interior faces*.

Definition 2.2. Given a connected circular planar graph G, its *dual graph*, denoted G^*, is constructed by associating a vertex with each face of G. For each interior face, we place a vertex within the face, and this vertex will be an interior vertex of G^*. For each exterior face, we place a vertex on ∂D between the two boundary vertices of G in the closure of the face, and this vertex will be a boundary vertex of G^*. If any two faces of G share an edge in their closures, then for each shared edge there is an edge in the dual between the two dual vertices associated with the two faces. An example is shown below.

The definition of a dual graph above is not the same as the typical dual defined in graph theory texts, but is closely related and is well-defined in the case of connected circular planar graphs. As you can note from the example, for each edge in G, there is a dual edge that crosses it. Given this definition, we may now define the dual network.

Definition 2.3. Given a network $\Gamma = (G, \gamma)$, we have a dual network $\Gamma^* = (G^*, \frac{1}{\gamma})$, where the conductivity of a dual edge is simply the inverse of the conductivity of its associated edge in the original graph.

Now, in order to define a γ-harmonic conjugate, we need to have a convention which associates a direction on an edge e in Γ to a certain perpendicular direction on its corresponding edge e' in Γ^*. As our convention, we will use the right-hand rule. First, take one's right hand standing upright on the pinky edge, align it with e with the fingers pointing in the direction under consideration. Now, sweep inward with the hand until it aligns with a direction on e'. This direction on e' is the perpendicular direction to the original direction on e. A figure below illustrates an example of two perpendicular directions:

Note in the above that the correspondence is not a duality. The perpendicular direction to $p'q'$ is not pq, but $qp = -pq$.

Definition 2.4. Given a γ-harmonic potential function u defined on Γ, a γ-harmonic conjugate is a vertex potential function v defined on Γ^* such that the following is satisfied:

$$\text{(DCR)} \qquad \gamma(e = pq)\big(u(q) - u(p)\big) = v(q') - v(p') \qquad \forall e \in E$$

where p' and q' are oriented as in the figure above, according to the right-hand convention.

The above equation is known as the discrete γ-Cauchy-Riemann equation. Note that u is the $-\gamma^*$-harmonic conjugate of v where $\gamma^* = \frac{1}{\gamma}$. We can now show a few simple facts.

Lemma 2.5. *Given a γ-harmonic potential function u on a network $\Gamma = (G, \gamma)$, a γ-harmonic conjugate v will be γ^*-harmonic on Γ^*, the dual network, where $\gamma^* = \frac{1}{\gamma}$.*

Proof. Consider any interior vertex p' of Γ^*. This vertex will be associated with a face F of Γ. The figure above illustrates the situation. As can be seen, each edge with p' as an endpoint has an associated edge in the boundary of F. Thus, we see that:

$$\begin{aligned}
\mathcal{I}(p') &= \sum_{q' \in \mathcal{N}(p')} \gamma^*(p', q')\big(v(p') - v(q')\big) \\
&= \sum_{pq \in \partial F} u(q) - u(p) \\
&= 0
\end{aligned}$$

where the orthogonal edge of pq is $p'q'$. The second equality follows, as u and v satisfy (DCR), and the last equality follows, as we are summing the differences in u around a closed loop and u is well-defined. $\square$

Lemma 2.6. *Given a γ-harmonic potential function u on a network $\Gamma = (G, \gamma)$, consider a potential function v on Γ^*, the dual network, that is a γ-harmonic conjugate to u. Then the following will hold.*

$$\sum_{p'q' \in \partial F'} v(q') - v(p') = 0$$

where F' is any interior face of Γ^*.

Proof. Consider the diagram above, where we are summing the differences in v in the clockwise direction around F'. We've let p denote the vertex in Γ that is associated with F'. Each edge in $\partial F'$ has an associated edge in Γ with p as one of its endpoints. The clockwise directions that we are summing in are the orthogonal directions to those heading into p. As v is a γ-harmonic conjugate to u, (DCR) is satisfied for each edge in the sum, so we have that:

$$\begin{aligned}\sum_{p'q' \in \partial F'} v(q') - v(p') &= \sum_{q \in \mathcal{N}(p)} \gamma(p,q)\big(u(p) - u(q)\big) \\ &= \mathcal{I}(p) \\ &= 0\end{aligned}$$

with the last equality following because u is γ-harmonic and p is an interior vertex. Note that summing in the counterclockwise direction will simply give us $-\mathcal{I}(p)$, which will again be zero for an interior vertex p. $\square$

The two lemmas above essentially show that Kirchhoff's voltage law on a network implies Kirchoff's current law on the dual network and vice versa. With the above lemma in hand, we can see that given a γ-harmonic potential u, we can construct a well-defined γ-harmonic conjugate v simply by specifying the value of v at one vertex in G^*.

Theorem 2.7. *Given a γ-harmonic potential function u on a connected circular planar network $\Gamma = (G, \gamma)$ and a value of a vertex function v at some vertex p' of G^*, there is a well-defined value for v at all other vertices of G^* such that v is a γ-harmonic conjugate of u.*

Proof. We show this result by construction of v. With (DCR) and u given to us, we have information on the difference of v along each edge, so to find the value of v at any vertex a, we simply need to trace a path through G^* to a from p'. We can see that the resulting value $v(a)$ will be independent of the path taken, because we can take two paths and sum the differences up one path and back down another. The values arising from edges in common will cancel, and the loops in the dual graph are composed of summing around the faces, which by the lemma just shown, will also have a zero contribution to the sum. Thus, the sum will be zero and taking the different paths will lead to the same value for $v(a)$. An illustrative diagram is shown below.

□

As we may specify the value of v at p' to be anything and still obtain a well-defined γ-harmonic conjugate, the correspondence differs by a constant.

3. Preliminaries on $\mathbf{H}$

The derivation of $\mathbf{H}$, the *Neumann-to-Dirichlet* map, and investigation of some of its properties was carried out by Nate Bottman and James McNutt in 2007 [4]. It is a linear transformation which acts on given boundary currents to give the corresponding boundary potentials that will result in the given boundary currents, with the extra stipulation that the boundary potentials sum to zero.

Here, we simply cite their results in relating $\mathbf{\Lambda}$ to $\mathbf{H}$ on the graph, because we use it to come up with some interesting ways to write the maps on the graph in terms of their counterparts on the dual. These results are:

$$\mathbf{H} = (\mathbf{\Lambda}^2 + E)^{-1}\mathbf{\Lambda} \tag{1}$$

and

$$\mathbf{\Lambda} = (\mathbf{H}^2 + E)^{-1}\mathbf{H} \tag{2}$$

where E denotes the matrix that contains 1 in every entry. For the derivation of the above results, see [4].

4. Boundary Data on Γ and Γ^*

Now, we use the concept of γ-harmonic conjugates to relate boundary data on a network, Γ, to boundary data on Γ^*, the dual network. The basic idea is that boundary data on a network, whether it be potentials or currents, will specify a γ-harmonic potential on the network that is at least unique up to a constant. This potential on the network will have a γ-harmonic conjugate which is unique up to a constant, and will have particular boundary data on the boundary of Γ^*. The correspondence will also work similarly in going from Γ^* to Γ, as we are dealing with a duality here. Important details are ironed out in the section below, but the goal of the section can be neatly summarized as completion of the following commutative diagram.

$$
\begin{tikzcd}
\phi \arrow[r, bend left, "\mathbf{\Lambda}"] \arrow[d, bend left, "?"] & \psi \arrow[l, bend left, "\mathbf{H}"] \arrow[d, bend left, "?"] \\
\psi^* \arrow[u, bend left, "?"] \arrow[r, bend left, "\mathbf{H}^*"] & \phi^* \arrow[l, bend left, "\mathbf{\Lambda}^*"] \arrow[u, bend left, "?"]
\end{tikzcd}
$$

Above, ϕ and ψ denote boundary potentials and boundary currents on Γ, while ϕ^* and ψ^* denote the corresponding quantities on Γ^*. In order for the diagram to truly commute, we would need to restrict ϕ and ϕ^* to refer to boundary potentials for which the potentials sum to zero. The work done on **H** shows why this must be done. Consider taking a set of boundary potentials whose potentials do not sum to zero and allowing Λ to act on it to obtain the corresponding set of boundary currents. If we were then to allow **H** to act on this set of boundary currents, we would not obtain our original set of boundary potentials, but a set that would differ by a constant. We will elaborate on this necessary restriction on ϕ and ϕ^* later.

Before we begin, let us define a standard indexing of the boundary vertices for both Γ and Γ^*. We start numbering the boundary vertices of Γ by choosing an arbitrary boundary vertex to be vertex 1. Then, we begin traversing the boundary in a clockwise direction and label the boundary vertices of Γ in ascending order. Now, to number each of the boundary vertices of Γ^*, we simply traverse counterclockwise from each vertex until we reach a boundary vertex of Γ, and give the boundary vertex of Γ^* the same label. A diagram is shown below for clarification.

4.1. **Boundary Potentials to Boundary Currents on the Dual.** Let's begin by working out the $\phi \to \psi^*$ map. Consider a set of three boundary vertices, as shown below.

Suppose that we know ϕ_i and ϕ_{i+1}, the potentials at the vertices in Γ, and consider ψ_i^*, the current out of vertex i in Γ^*. We have the following:

$$\begin{aligned}\psi_i^* &= \sum_{q' \in \mathcal{N}(i)} \gamma^*(i,q')\big(v(i) - v(q')\big)\\ &= \sum_{pq \in \partial F} u(q) - u(p)\\ &= u(i) - u(i+1)\\ &= \phi_i - \phi_{i+1}\end{aligned}$$

where q' denotes vertices in Γ^* that are adjacent to i and F denotes the exterior face in Γ associated with i. The second equality follows since u and v satisfy (DCR) and we've used the right-hand convention for the directionality of the differences. The second to last equality follows by the fact that the series telescopes, leaving only the first and last terms.

With the above example, we can see now that the $\phi \to \psi^*$ map is simply equivalent to multiplication by a difference matrix, which we'll call D.

$$D\phi = \psi^* \text{ where } D = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -1 \\ -1 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{3}$$

The $\phi^* \to \psi$ map may be worked out analogously, and is left as an exercise to the reader. The result is shown below.

$$D^T \phi^* = \psi$$

Before we continue completing our commutative diagram, let us note that the arguments used above to equate a difference in boundary potentials on a network to a boundary current on the dual network may be easily generalized. Suppose that we know the value of the potential at two points on the boundary of a network. The situation is illustrated below.

An arbitrary path is chosen between vertices a and b above. Each edge in the path has a dual edge associated with it. If we sum the potential differences in the direction shown, we can see that this is equivalent to the current out of the region in the dual graph, as (DCR) is satisfied. The sum of potential differences telescopes, leaving it equal to $\phi_a - \phi_b$ and as for the current out of the region bounded by the path, the only possible current sources are the boundary vertices of the dual between a and b. Thus, we have the following theorem.

Theorem 4.1. *Given the value of the potential at two boundary vertices, a and b, in a network Γ, the following equation holds:*

$$\phi_a - \phi_b = \sum_{b' \in \mathcal{B}} \psi'_b$$

where $\mathcal{B}$ denotes boundary vertices of Γ^* that we encounter as we traverse from a to b in a clockwise direction along the boundary.

There is also a twin theorem when we want to relate the difference of two boundary potentials on Γ^* to the sum of the boundary currents for vertices in Γ. Its proof is analogous and is left as an exercise for the reader.

Theorem 4.2. *Given the value of the potential at two boundary vertices, a' and b', in a dual network Γ^*, the following equation holds:*

$$\phi'_b - \phi'_a = \sum_{b \in \mathcal{B}} \psi_b$$

where $\mathcal{B}$ denotes boundary vertices of Γ that we encounter as we traverse from a' to b' in a clockwise direction along the boundary.

Also, we present two short lemmas on the nullspace of D.

Lemma 4.3. *The nullspace of D is the space of constant vectors.*

Proof. If we have a constant vector x, a simple calculation shows that $Dx = 0$ as $(Dx)_i$ is equal to the difference of two entries of x. For the other direction, suppose $Dx = 0$. Then the ith row of D states that $x_i = x_{i+1}$ (except for the nth row, which states that $x_n = x_1$). Thus, we have that $x_1 = x_2 = \cdots = x_n$, and we have our result. ☐

The twin lemma is proven analogously, and is left as an exercise.

Lemma 4.4. *The nullspace of D^T is the space of constant vectors.*

4.2. **Boundary Currents to Boundary Potentials on the Dual.** Let us consider the $\psi^* \to \phi$ map first, for which we utilize (3). As noted, D has a nonempty nullspace, so we cannot invert it, as we might like to do. This reflects the physical fact that for any two sets of boundary potentials on a network Γ which differ by a constant, their corresponding boundary currents on Γ^* are the same.

So, we must place an extra requirement on the set of boundary potentials that we map to. As $\mathbf{H}$ maps to boundary potentials whose sum is zero, we impose the same extra condition here. This leads us to the following system of equations, where ϵ is the column vector of all 1s.

$$\begin{bmatrix} D \\ \epsilon^T \end{bmatrix} \phi = \begin{bmatrix} \psi^* \\ 0 \end{bmatrix}$$

Premultiplying both sides by $\begin{bmatrix} D^T & \epsilon \end{bmatrix}$ we get:

$$\begin{aligned} \begin{bmatrix} D^T & \epsilon \end{bmatrix} \begin{bmatrix} D \\ \epsilon^T \end{bmatrix} \phi &= \begin{bmatrix} D^T & \epsilon \end{bmatrix} \begin{bmatrix} \psi^* \\ 0 \end{bmatrix} \\ (D^T D + E)\phi &= D^T \psi^* \end{aligned}$$

where $E = \epsilon\epsilon^T$ is the matrix where every entry is 1. To proceed, we need the following lemma.

Lemma 4.5. *$D^T D + E$ is invertible.*

Proof. Note that the right nullspace of $L = \begin{bmatrix} D^T & \epsilon \end{bmatrix}^T$ is empty, because the kernel of D consists of the space of constant vectors, and ϵ^T forces the sum of the entries to be zero, leaving us with just the zero vector in the right nullspace. Now, consider the kernel of $L^T L$ and suppose we have a vector x such that $L^T L x = 0$. Then either $Lx = 0$, which implies that $x = 0$, or Lx is in the right nullspace of L^T. In the second case, by the Fredholm alternative, this means that $Lx \in im(L)^\perp$, which in turn implies that $x = 0$. Thus, $L^T L = D^T D + E$ is invertible. $\square$

So, we get our desired map, which we denote with an S.

$$S\psi^* = \phi \text{ where } S = (D^T D + E)^{-1} D^T$$

An analogous set of steps allows us to get the $\psi \to \phi^*$ map, which we'll call Q for the moment, and it takes the form below.

$$Q\psi = \phi^* \text{ where } Q = (DD^T + E)^{-1} D$$

In developing this equation, there is no need for another lemma on the invertibility of $DD^T + E$, because it is a quick exercise to show that:

$$DD^T = D^T D = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \ddots & \ddots & 0 \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ 0 & \ddots & \ddots & -1 & 2 & -1 \\ -1 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

Using this fact, we may show that $D(D^T D + E)^{-1} = (D^T D + E)^{-1} D$.

$$\begin{aligned} D(D^T D + E)^{-1} &= (D^T D + E)^{-1} D \\ \Leftrightarrow (D^T D + E) D &= D(D^T D + E) \\ \Leftrightarrow (D^T D) D &= (DD^T) D \end{aligned}$$

With this, we can better characterize Q and show that $Q = S^T$.

$$\begin{aligned} S^T &= \left((D^T D + E)^{-1} D^T\right)^T \\ &= D(D^T D + E)^{-1} \\ &= (D^T D + E)^{-1} D \\ &= (DD^T + E)^{-1} D = Q \end{aligned}$$

Thus, we have completed our commutative diagram.

$$\begin{tikzcd}
\phi \arrow[r, bend left, "\mathbf{\Lambda}"] \arrow[d, bend left, "D"] & \psi \arrow[l, bend left, "\mathbf{H}"] \arrow[d, bend left, "S^T"] \\
\psi^* \arrow[u, bend left, "S"] \arrow[r, bend left, "\mathbf{H}^*"] & \phi^* \arrow[l, bend left, "\mathbf{\Lambda}^*"] \arrow[u, bend left, "D^T"]
\end{tikzcd}$$

The extra conditions that we added in the derivation of S and S^T, that the boundary potentials ϕ and ϕ^* that we map to from ψ^* and ψ must have entries that sum to zero, also force us to restrict the kinds of boundary potentials that ϕ and ϕ^* refer to. As in the case with $\mathbf{\Lambda}$ and $\mathbf{H}$, if we take a set of boundary potentials whose entries do not sum to zero and act D, then S (or D^T, then S^T) upon it, we will get a different set of potentials that differ by a constant.

This is reflected in the fact that $DS = SD = \mathbf{\Lambda H} = \mathbf{H\Lambda} = I - \frac{1}{n}E$.

$$\begin{aligned}
DS = D(D^TD+E)^{-1}D^T = (D^TD+E)^{-1}DD^T &= I - \frac{1}{n}E \\
\Leftrightarrow DD^T &= (D^TD+E)(I-\frac{1}{n}E) \\
&= D^TD + E - \frac{1}{n}E^2 \\
&= D^TD + E - \frac{1}{n}nE \\
&= D^TD
\end{aligned}$$

$$\begin{aligned}
SD = (D^TD+E)^{-1}D^TD &= I - \frac{1}{n}E \\
\Leftrightarrow D^TD &= (D^TD+E)(I-\frac{1}{n}E) \\
&= D^TD
\end{aligned}$$

The results for $\mathbf{H\Lambda}$ and $\mathbf{\Lambda H}$ are obtained with analogous relations proved in [4], and are left as an exercise to the reader. Note also that $D^TS^T = (SD)^T = I - \frac{1}{n}E$ and $S^TD^T = (DS)^T = I - \frac{1}{n}E$. This matrix, $I - \frac{1}{n}E$, is just the projection matrix onto the space of vectors orthogonal to ϵ, which explains the behavior observed when we act on a vector whose entries do not sum to zero.

Given this diagram, and the results from [4], we see that we now have a way to obtain any of our maps $\mathbf{\Lambda}$, $\mathbf{H}$, $\mathbf{\Lambda}^*$, or $\mathbf{H}^*$ in terms of each other. As an example, let's find $\mathbf{\Lambda}$ in terms of $\mathbf{H}^*$. Instead of going from ϕ to ψ directly, we first go from ϕ to ψ^*, then through $\mathbf{H}^*$, and then up to ψ from ϕ^* to obtain $\mathbf{\Lambda} = D^T\mathbf{H}^*D$. We can now use (1) and substitute to get $\mathbf{\Lambda}$ in terms of $\mathbf{\Lambda}^*$. Similar manipulations will give us any such equality we desire.

4.3. **Characterizing S.** You may have noted that we've found S in terms of D, but that the expression is rather messy, and we have not found a nice form for it. As an example, let's consider S for a network with 3 boundary vertices.

$$S = \frac{1}{3}\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

As can be seen, the first row is a decreasing sequence of three numbers, all spaced apart by a difference of 1, centered around 0, and then scaled by a factor of $\frac{1}{n}$. The other rows are just shifted versions of the first row, so that S is a circulant matrix, like D. A theorem below states this for n boundary vertices, and the proof follows.

Theorem 4.6. *For a network with n boundary vertices, the first row of S, the $\psi^* \to \phi$ map, takes the form shown below.*

$$S_{1i} = \frac{1}{n}\begin{bmatrix} \frac{n-1}{2} & \frac{n-3}{2} & \dots & -\frac{n-3}{2} & -\frac{n-1}{2} \end{bmatrix}$$

The rest of the rows are just cyclic permutations, where the jth row is the 1st row with the permutation $(1\ 2\ \dots\ n)$ applied $j-1$ times.

Proof. We can see that this is the right form for S by considering its action on the standard basis and utilizing our knowledge of the relationship between boundary data on the network and its dual. Acting on the ith element of the standard basis would give us the ith column of S. However, note that elements of the standard basis are not valid boundary currents, as the entries sum to 1, not 0, so we cannot use the relationship between boundary data on the network and its dual. We save ourselves by noting that we can still get the ith column of S by acting on the ith column of $I - \frac{1}{n}E$, as the space of constant vectors is in the nullspace of $S = (D^TD + E)^{-1}D^T$. The columns of $I - \frac{1}{n}E$ are valid boundary currents, as they are just the projections of the standard basis onto the space of vectors orthogonal to ϵ, the space of valid boundary currents. Noting the form of the ith column of S, we get the following:

$$S\begin{bmatrix} -1/n \\ -1/n \\ \vdots \\ -1/n \\ (n-1)/n \\ -1/n \\ -1/n \\ \vdots \\ -1/n \end{bmatrix} = \begin{bmatrix} S_{1i} \\ S_{1(i-1)} \\ \vdots \\ S_{12} \\ S_{11} \\ S_{1n} \\ S_{1(n-1)} \\ \vdots \\ S_{1(i+1)} \end{bmatrix}$$

Here, we recall the formula $\phi_i - \phi_{i+1} = \psi_i^*$ and we see that this relation holds for the two vectors above, where the ψ^* is the ith column of $I - \frac{1}{n}E$ and ϕ is the ith column of S, of course. Furthermore, the ith column of S is exactly the set of boundary potentials that satisfies this formula and has entries that sum to zero. So, we have that the matrix equation above must hold, and we get our desired form for S. $\square$

We can also show another interesting property of S.

Theorem 4.7. *The equation $D = (S^TS + E)^{-1}S^T$ holds.*

Proof.

$$\begin{aligned}
D &= (S^TS + E)^{-1}S^T \\
\Leftrightarrow D &= \left((DD^T + E)^{-2}DD^T + E\right)^{-1}(DD^T + E)^{-1}D \\
\Leftrightarrow D &= \left((DD^T + E)^{-1}DD^T + E^2\right)^{-1}D \\
\Leftrightarrow (DD^T + E)^{-1}DD^TD &= D \\
\Leftrightarrow DD^TD &= (DD^T + E)D = DD^TD
\end{aligned}$$

□

At this point, the reader who has also perused [4] will note that the relationship between D and S closely mirrors the relationship between $\mathbf{\Lambda}$ and $\mathbf{H}$. All of this follows from three shared characteristics. The first is that both D and $\mathbf{\Lambda}$ share the same nullspace, the space of constant vectors. The second is that both D and $\mathbf{\Lambda}$ commute with their transposes. The last that both S and $\mathbf{H}$ map to sets of boundary potentials whose entries sum to zero.

All of the results in this section can be easily generalized to include graphs with multiple connected components, each one of which is circular planar, but I've chosen not to include these details, as I feel that they would simply obscure the ideas presented.

5. Future Research

In developing the above work, several other ideas surfaced that were either left unpursued, or no significant progress was made. Two of these ideas are presented below.

5.1. **Mixed Boundary Data.** Mixed boundary data for a network is data that is composed of both boundary currents and boundary potentials. Jamie Ramos and Jeremiah Jones investigated mixed boundary data problems where each boundary vertex has a value associated with it, either its potential or current in [5]. For problems of this type, they found an injective linear map M which acts on a vector of the given boundary values to give the opposite boundary values for each boundary vertex.

$$M \begin{bmatrix} \phi_k \\ \psi_k \end{bmatrix} = \begin{bmatrix} \psi_u \\ \phi_u \end{bmatrix} \text{ where } M = \begin{bmatrix} \mathbf{\Lambda}/\mathbf{\Lambda}_{22} & \mathbf{\Lambda}_{12}\mathbf{\Lambda}_{22}^{-1} \\ -\mathbf{\Lambda}_{22}^{-1}\mathbf{\Lambda}12^T & \mathbf{\Lambda}_{22}^{-1} \end{bmatrix}$$

In the expression above, ϕ_k and ψ_k denote the known boundary potentials and currents, respectively, while ϕ_u and ψ_u denote the unknown boundary potentials and currents, respectively. The subscripts on the $\mathbf{\Lambda}$ terms denote the submatrices that result when we order the indices so that vertices with known boundary potential come before those with known boundary current and then partition based on these differences. $\mathbf{\Lambda}/\mathbf{\Lambda}_{22} = \mathbf{\Lambda}_{11} - \mathbf{\Lambda}_{12}\mathbf{\Lambda}_{22}^{-1}\mathbf{\Lambda}_{12}^T$ just denotes the Schur complement.

The above map can be viewed as analogous to either $\mathbf{\Lambda}$ or $\mathbf{H}$ in that it replaces known boundary potentials at vertices with their boundary currents and vice versa. This suggests that we search for a reverse map, and M is in fact injective, so the expression for M^{-1} is also rather easily found.

$$M^{-1}\begin{bmatrix}\phi_u\\ \psi_u\end{bmatrix}=\begin{bmatrix}\psi_k\\ \phi_k\end{bmatrix}\text{ where }M^{-1}=\begin{bmatrix}\mathbf{\Lambda}_{11}^{-1} & -\mathbf{\Lambda}_{11}^{-1}\mathbf{\Lambda}_{12}\\ \mathbf{\Lambda}12^{T}\mathbf{\Lambda}_{11}^{-1} & \mathbf{\Lambda}/\mathbf{\Lambda}_{11}\end{bmatrix}$$

This time ϕ_u and ψ_u denote the known values, while ϕ_k and ψ_k denote the unknown quantities. After the investigations presented on a correspondence between boundary data on the network and its dual, a natural question to ask is whether or not a similar sort of correspondence exists for mixed boundary data.

One can easily note that given a set of mixed boundary values like those noted above, one could use the mixed map to obtain the missing boundary potentials and the missing boundary currents. With these values, one could then use the maps found in section 4 to switch over to the dual and consider any mixed problem on the dual. The linear map that gives ψ, the vector of boundary currents, is shown below.

$$\Xi\begin{bmatrix}\phi_k\\ \psi_k\end{bmatrix}=\begin{bmatrix}\psi_u\\ \psi_k\end{bmatrix}=\psi\text{ where }\Xi=\begin{bmatrix}\mathbf{\Lambda}/\mathbf{\Lambda}_{22} & \mathbf{\Lambda}_{12}\mathbf{\Lambda}_{22}^{-1}\\ 0 & 1\end{bmatrix}$$

When we try and obtain the boundary potentials, denoted by ϕ, we should also ensure that the sum of the boundary potentials is 0. To do this, we apply the projection operator $I-\frac{1}{n}E$ as well.

$$\Theta\begin{bmatrix}\phi_k\\ \psi_k\end{bmatrix}=\begin{bmatrix}\phi_u\\ \phi_k\end{bmatrix}=\phi\text{ where }\Theta=(I-\frac{1}{n}E)\begin{bmatrix}1 & 0\\ -\mathbf{\Lambda}_{22}^{-1}\mathbf{\Lambda}_{12}^{T} & \mathbf{\Lambda}_{22}^{-1}\end{bmatrix}$$

So we have the following partially complete commutative diagram, where we let ζ_{mn} denote mixed boundary data with m boundary potentials and n boundary currents:

$$\begin{tikzcd}
\phi \arrow[r, bend left, "?"] \arrow[d, bend left, "D"] & \zeta_{mn} \arrow[l, bend left, "\Theta"] \arrow[r, bend left, "\Xi"] \arrow[d, dotted, <->] & \psi \arrow[l, bend left, "?"] \arrow[d, bend left, "S^{T}"] \\
\psi^{*} \arrow[u, bend left, "S"] \arrow[r, bend left, "?"] & ? \arrow[l, bend left, "\Xi^{*}"] \arrow[r, bend left, "\Theta^{*}"] & \phi^{*} \arrow[l, bend left, "?"] \arrow[u, bend left, "D^{T}"]
\end{tikzcd}$$

As you may have noted, things are again complicated by the issue that the same set of boundary potentials which differ by a constant give the same boundary currents. As such, when me map mixed boundary problems to the space of boundary currents, we see that the map is not injective. If we were to differ the boundary potential values by a constant, both problems result in the same boundary current, will map to the same set of boundary currents. However, we can use our work with $\mathbf{H}$ to create a map that takes boundary currents to a mixed boundary problem whose boundary potentials (at all boundary vertices, not just those at which it's specified) will sum to zero.

$$\Pi \begin{bmatrix} \psi_u \\ \psi_k \end{bmatrix} = \begin{bmatrix} \phi_k \\ \psi_k \end{bmatrix} = \zeta_{mn} \text{ where } \Pi = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ 0 & 1 \end{bmatrix}$$

Likewise, when we map mixed boundary problems to the space of boundary potentials, we mapped it to the space of boundary potentials which sum to zero. As such, this map is not injective either. So, for a return map, we can't have an inverse but we may use $\mathbf{\Lambda}$ to create a map that will take a set of boundary potentials to a mixed boundary problem where those boundary potentials that remain will not change.

$$\Omega \begin{bmatrix} \phi_k \\ \phi_u \end{bmatrix} = \begin{bmatrix} \phi_k \\ \psi_k \end{bmatrix} = \zeta_{mn} \text{ where } \Omega = \begin{bmatrix} 1 & 0 \\ \mathbf{\Lambda}_{12}^T & \mathbf{\Lambda}_{22} \end{bmatrix}$$

The bigger question is whether or not the space in the diagram can be filled, or in other words, is there a natural corresponding mixed problem on the dual for each mixed problem on the network? Further work could investigate this problem, and explore in greater detail the maps mentioned above. Lastly, it would be interesting to see how M and M^{-1} compare to $\mathbf{\Lambda}$ and $\mathbf{H}$ as the mixed problem gets successively closer to either a Neumann or Dirichlet problem (as we have many boundary currents and few boundary potentials, or vice versa).

5.2. **Discrete Analytic Functions.** The above work was completed in the last two weeks of the program, as I spent most of my time pursuing another idea, which unfortunately did not result in any significant progress. My initial interest was in continuing the work done by Karen Perry, Owen Biesel, and Amanda Rohde in defining a notion of a discrete analytic function and to find analogues of concepts and theorems from complex analysis.

In the process of attempting to do so, I scanned the mathematical literature for other concepts of discrete analytic functions. I stumbled upon a book, *Introduction to Circle Packing: The Theory of Discrete Analytic Functions* by Kenneth Stephenson (ISBN: 0521823560). In it, he defines a discrete analytic function as a map between two circle packings. A circle packing is informally a set of circles that are tangent to each other with the restriction that its associated graph has each face as a triangle. This associated graph is formed by representing each circle with a vertex and forming an edge for each tangency relation. An example of a circle packing is shown below.

Much of my time was spent in an attempt to unite our concept of discrete analytic function with the definition presented in this book. A particularly useful section is Chapter 18, Random Walks on Circle Packings.

Even if attempts to unify the two notions are unsuccessful, it is likely that there will be some insight gained into the stengths and weaknesses of each approach, as one notion seems to have been constructed by building on a discrete Cauchy-Riemann equation, while the other has built on discrete analogies of the conformal mapping properties of analytic functions.

6. Acknowledgements

I would like to extend thanks to Jim Morrow, Owen Biesel, and Ernie Esser for discussing this material with me throughout the program. Their ideas contributed to key steps in the development of the work and their support made this work possible. In addition, the authors of the papers that I've cited deserve my gratitude, as their research set the foundation for mine.

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