Discrete Sobolev spaces

- Discuss analogies to Sobolev spaces
- Show where they can be useful
- Main example: Semidiscrete approximation to semilinear heat equation in 1D. $u_t = \Delta u + |u|^{\gamma-1} u$, $\gamma > 1$

(Refs: Ball, Tao)

By working with discrete Sobolev spaces, can show existence on $[0,T)$ for T indep. of h. (very analogous to continuous argument)

(Ref: Tomanoic)

- Also note estimates for convergence rates of finite difference schemes (in terms of discrete Sobolev norms)

Review Sobolev spaces: Basically a generalization of C^s.
want to measure derivs in L^p norms.

On $\Omega \subset \mathbb{R}^n$ define $\|u\|_{W^{s,p}} = \left(\sum_{|\alpha| \leq s} \int |D^\alpha u|^p dx \right)^{1/p}$

Want all weak derivs up to order s in L^p

However C^s not complete in $\|\cdot\|_{W^{s,p}}$, so include L^2 limits of sequences Cauchy in $W^{s,p}$ norm.

Result is $W^{s,p}$, a Banach space. Also completion of C^s.
(ie: smooth fns dense in Sobolev space)
(However, classical differentiability not required.)

$W^{s,2} = H^s$, a Hilbert space

Completeness essential for existence arguments:
ie: fixed point contraction mapping approach

$\|Ax - Ay\| \leq c\|x-y\|$, $c < 1$, $A^n x_0$ Cauchy sequence, need completeness to say converge to x

Get fixed point by continuity: $x_n \to x \Rightarrow A x_n \to Ax = x$

Sobolev spaces especially useful for studying weak formulations of PDEs.

Ex: $\nabla\cdot(\sigma\nabla u) = f$

Integrate by parts $\Rightarrow$ $\int_\Omega \nabla\cdot(\sigma\nabla u)v = \int_\Omega fv$ $\qquad u, v \in H_0^1(\Omega)$

$$\Rightarrow \int_\Omega (\nabla\sigma\cdot\nabla u + \sigma\Delta u)v = \int_\Omega fv$$

$$\int_\Omega \nabla\sigma\cdot\nabla u\, v - \nabla(\sigma v)\cdot\nabla u + \int_{\partial\Omega} \cancelto{0}{\sigma v \frac{\partial u}{\partial n}} = \int_\Omega fv$$

$$\int_\Omega \sigma\nabla v\cdot\nabla u = -\int_\Omega fv \qquad \forall\, v\in H_0^1(\Omega)$$

symmetric bilinear form on <u>Hilbert space</u>
satisfying elliptic estimate

Lax Milgram $\Rightarrow$ $\exists!$ solution

Fractional derivatives: if on $\mathbb{R}^n$ or periodic domain,
can define Sobolev norms with Fourier Transform $\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx$

$$\|f\|_{H^s} = \|(1+|\xi|^2)^{s/2}\hat{f}(\xi)\|_{L^2}$$

s can be a fraction

Sobolev embedding:

$$H^s \subset C^k \quad \text{if} \quad s > k + \frac{n}{2}$$

dimension

Ex: in 1D, H^1 functions are continuous

Discrete Sobolev Spaces: $\Omega \subset \mathbb{R}^d$

First finite differences: $D_i^{\pm} u = \dfrac{u(x \pm h e_i) - u(x)}{h}$ difference in ith direction

Define nodes on uniform mesh Ω_h with boundary $\partial\Omega_h$

$$\|u\|_{L_h^2}^2 = h^d \sum_{x \in \Omega_h} u(x)^2$$

Could have used D^+ ...

$$\|u\|_{H_h^1} = \|u\|_{L_h^2} + \sum_{i=1}^{d} \|D_i^- u\|_{L_h^2}$$

$$\|u\|_{H_h^2} = \|u\|_{H_h^1} + \sum_{i,j=1}^{d} \|D_i^- D_j^- u\|_{L_h^2}$$ and so on.

Don't include differences that overflow $\bar{\Omega}_h$.

(Not an issue on periodic domain)

Some analogies: Poincare inequality

$$\|u\|_{L^2}^2 \le C_\Omega \int_\Omega |\nabla u|^2 \, dx \quad , u \in H_0^1(\Omega)$$

Discrete: $\|u\|_{L_h^2} \le 2L \|D^- u\|_{L_h^2}$ where $\Omega = [0, L]$

pf: $\langle u, u \rangle = \langle u^2, 1 \rangle = \langle u^2, D_j^+ j \rangle = -\langle D^- u^2, j \rangle = -\sum_{j=0}^{N-1} (u_j^2 - u_{j-1}^2) j h = -\sum_j (u_j - u_{j-1})(u_j + u_{j-1}) j h$

$= \sum_{j=0}^{N-1} \frac{(u_j - u_{j-1})}{h} u_j \, j h^2 + \sum_{j=0}^{N-1} \frac{(u_{j-1} - u_j)}{h} u_{j-1} \, j h^2$. Cauchy Schwarz + $j \le N-1$

$\Rightarrow \|u\|_{L_h^2}^2 \le h(N-1)\left(\|D^- u\|_{L_h^2} \|u\|_{L_h^2} + \|D^+ u\|_{L_h^2} \|u\|_{L_h^2}\right)$ $\quad h(N-1) = L$

$\Rightarrow \boxed{\|u\|_{L_h} \le 2L \|D^- u\|_{L_h^2}}$

Discrete version of Sobolev embedding theorem:

special case: $\|u\|_\infty^2 \le \left(1+\frac{1}{L}\right)\|u\|_{H_h^1}^2$ $\qquad \Omega=[0,L]$

Pf: $\langle u,v\rangle_r^s = h\sum_{j=r}^{s} u_j v_j$

$\langle u, D^+ v\rangle_r^s = -\langle D^- u, v\rangle_{r+1}^{s+1} + \left[u_{s+1}v_{s+1} - u_r v_r\right]$

Let M be index where max u attained. $u_M = \|u\|_\infty$ $\quad u_m$ min

$u_M^2 - u_m^2 = \langle u, D^+ u\rangle_m^{M-1} + \langle u, D^- u\rangle_{m+1}^{M}$

$\le 2\|u\|_{L_h^2}\|D^+ u\|_{L_h^2} \le \|u\|_{L_h^2}^2 + \|D^+ u\|_{L_h^2}^2$

Now $u_m^2 \le \frac{1}{hN}\sum_{j=0}^{N-1} u_j^2 h$ and since $hN > L$

$u_m^2 \le \frac{1}{L}\|u\|_{L_h^2}^2$

Finally $u_M^2 = u_m^2 + (u_M^2 - u_m^2)$

$\le \frac{1}{L}\|u\|_{L_h^2}^2 + \|u\|_{L_h^2}^2 + \|D^+ u\|_{L_h^2}^2$

$$\boxed{\|u\|_\infty^2 \le \left(1+\frac{1}{L}\right)\|u\|_{H_h^1}^2}$$

On periodic domain we could analogously define discrete Sobolev norms using DFT

$$\hat{u}(k) = \frac{1}{N}\sum_{j=0}^{N-1} u(jh)e^{-2\pi i j k/N}$$

Local Existence example:

$$u_t = \Delta u + \underbrace{|u|^{\gamma-1}u}_{f(u)} \quad , \gamma > 1 \qquad \Omega = [0,L] \subset \mathbb{R}$$

$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = u_0$$

$f(u)$ makes existence nontrivial.
Possible to choose u_0 so solution blows up in finite time.

Strategy for proving local existence: contraction mapping argument

Duhamel: $\Phi(u)(t) = S(t)u_0 + \int_0^t S(t-s)\, f(u(s))\,ds$

Solution is fixed point of Φ, so show Φ is contraction mapping from $C([0,T); H_0^1)$ to itself (actually make in smaller ball)

Take $u \in C([0,T); H_0^1) \cap C^1((0,T); L^2(\Omega))$

$u_0 \in H_0^1$

Key requirement: $f: H_0^1 \to L^2$ must be locally lipschitz

Can show $\|f(u) - f(v)\| = \| |u|^{\gamma-1}u - |v|^{\gamma-1}v \|_{L^2}$

$$\leq C(\|u\|_\infty^{\gamma-1} + \|v\|_\infty^{\gamma-1})\|u-v\|_{L^2}$$
$$\leq C(\|u\|_{H^1}^{\gamma-1} + \|v\|_{H^1}^{\gamma-1})\|u-v\|_{H^1} \quad \text{by Sobolev embedding}$$

(Note: we know what $S(t)$ is
$S(t)u_0 = \frac{1}{\sqrt{4\pi t}} \int e^{(x-y)^2/4t} u_0(y)\,dy$ solution to heat equation)

Semi-Discrete Approximation:

$$\partial_t u_j = \underbrace{u_{j+1} - 2u_j + u_{j-1}}_{h^2} + |u_j|^{\gamma-1} u_j$$

$\Omega = [0, L]$ mesh size $h = \frac{L}{N-1}$ $u_j = u(jh)$ $j = 0, \dots, N-1$

$u_0 = u_{N-1} = 0$

Consider solutions u in $C([0,T); \mathbb{R}^{N-2})$

First order ODE system: $\partial_t u = Au + f(u) = G(u)$

linear operator on finite dim space $\Rightarrow$ bounded

f locally Lipschitz $\Rightarrow$ so is G

$$A = \begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & 1 & \\ 0 & & \ddots & \end{bmatrix}$$

Could use Picard's Theorem to get existence on $[0,T)$

$$\Phi(u)(t) = u_0 + \int_0^t G(u(s))\,ds$$

$$\|\Phi u - \Phi v\| \le Lip \int_0^t \|u - v\|\,ds \le (T \cdot Lip) \cdot \|u-v\|_{C([0,T) \to \mathbb{R}^{N-2})}$$

choose T so < 1

Choice of T appears to depend on h.

Don't want $T \to 0$ as $h \to 0$.

Although all norms on $\mathbb{R}^n$ equivalent, discrete Sobolev norms can be used by analogy to continuous arguments to show T can be chosen independent of h (if u_0 smooth enough)

New space X, $\|u\|_X = \|\sqrt{-A}\,u\|_{L_h^2}$

ok since $-A$ is pos. semi. def matrix

There exists $T > 0$ and unique soln $u \in C([0,T); X)$ with $u(0) = u_0$

Pf: let $B_\epsilon = \{u \in X : \|u - u_0\|_X < \epsilon\}$

Define $\Phi(u)(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(u(s))\,ds$

$$C_\epsilon = (2e)^{-\frac{1}{2}} \gamma (1+\tfrac{1}{2})^{\gamma - \frac{1}{2}} \left(2\left(\epsilon + \|u_0\|_{H_h^2}\right)^{\gamma-1}\right)$$

Pick any $T < \min\left(\dfrac{\epsilon^2}{4\|u_0\|_{H_h^2}^2}, \dfrac{\epsilon}{2\beta}, \dfrac{1}{4C_\epsilon^2}\right)$

(where f is bounded by β on B_ϵ)

1) Show $\Phi(u)(t)$ maps $C([0,T); B_\epsilon)$ to itself

$$\|\Phi(u)(t) - u_0\|_X \le \underbrace{\|e^{tA}u_0 - u_0\|_X}_{= \|\sqrt{-A}\,(e^{tA}u_0 - u_0)\|_{L_h^2}} + \left\|\int_0^t e^{(t-s)A} f(u(s))\,ds\right\|_X$$

Now can use discrete sine transform, which diagonalizes A and satisfies boundary conditions

$$u_k = \frac{2}{N-1}\sum_{n=1}^{N-1} u_n \sin\left[\frac{\pi}{N-1}(n)(k+1)\right] \quad \text{denote by } \tilde{u}$$

$$\underline{\|Au\|_{L_h^2} = \sqrt{\tfrac{N-1}{2}}\,\|\tilde{A}\tilde{u}\|_{L_h^2}}, \quad \tilde{A} = \left(2\cos\left(\tfrac{\pi}{N-1}(k+1)\right) - 2\right)$$

$$\|e^{tA}u_0 - u_0\|_X = \sqrt{\tfrac{N-1}{2}}\,\|\sqrt{-\tilde{A}}\,(e^{t\tilde{A}} - 1)\,\hat{u}_0\|_{L^2_h}$$

$$\le \sqrt{t}\,\sqrt{\tfrac{N-1}{2}}\,\|\tilde{A}\,\hat{u}_0\|_{L^2_h}$$

$$= \sqrt{t}\,\|A u_0\|_{L^2_h} \le \sqrt{t}\,\|u_0\|_{H^2_h}$$

Also $\|\|e^{tA}\|\| \le 1$

More estimates $\Rightarrow$ maps B_ϵ to B_ϵ

2) Show $\|\phi(u) - \phi(v)\|_{C([0,T];X)} < \|u-v\|_{C([0,T];X)}$

can show

$$\|e^{tA}u\|_X \le (2et)^{-\frac{1}{2}}\|u\|_{L^2_h}$$

So $\|\phi(u) - \phi(v)\|_X \le 2e^{-\frac{1}{2}}\int_0^t (t-s)^{-\frac{1}{2}}\|f(u) - f(v)\|_{L^2_h}\,ds$

$$\le C_\epsilon \int_0^t (t-s)^{-\frac{1}{2}}\|u-v\|_X$$

Take sup, integrate

$\Rightarrow \|\phi(u) - \phi(v)\|_{C([0,T];X)} \le 2C_\epsilon\sqrt{T}\,\|u-v\|_{C([0,T];X)}$

< 1 by choice of T

T only depended on h through $\|u_0\|_{H^2_h}$, which can be controlled if u_0 is, say, piecewise smooth.

(DS) Also used to estimate convergence rates for finite difference schemes

Ref Jovanović

Compare $u_t^h - D^+D^- u^h = f^h$ could be trouble if f not nice

to $u_t - u_{xx} = f$

Use averaging operators $T^+ f = \frac{1}{h}\int_0^1 f(x+hs_2, t)\, ds_2$

$$T^- f = \frac{1}{h}\int_0^1 f(x-hs_1, t)\, ds_1$$

Coarea
$$\int_\Omega f(x)\,|\nabla u(x)|\,dx = \int_{-\infty}^{\infty} \int_{u^{-1}(t)} f(x)\, ds(x)\, dt$$

$y = s_2 - s_1$

$$T^+T^- f = \frac{1}{h^2}\int_0^1\int_0^1 f(x+h(s_2-s_1), t)\, ds_2\, ds_1$$

$$= \frac{1}{h^2}\int_{-1}^{1} \frac{1}{\sqrt{2}}\sqrt{2}(1-|y|)\, f(x+hy)\, dy$$

$\boxed{\text{use } T^+T^- u_{xx} = D^+D^- u^h}$

Replace f^h by $T^+T^- f$ evaluated at grid

Then $u_t - \cancel{u_t^h} - (D^+D^- u - D^+D^- u^h) = u_t - T^+T^- u_t$

Error $e = u - v \Rightarrow e_t - D^+D^- e = (I - T^+T^-) u_t$

Estimate $\underline{\|e\|_{H_h^2} \le c h^2 \|u\|_{H^4}}$ for $f \in H^2$ by regularity

take sup over $t \in [0,T]$ (need at least $f \in L^2$)