Old REU lessons: extensions to non circular planar examples

1) Determinant formula & boundary spike formula

$$\det \Lambda(P,Q) \cdot \det K(I,I) = (-1)^k \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \Big\{ \sum_{\substack{\alpha \\ \tau_\alpha = \tau}} \underbrace{\prod_{e \in E_\alpha}}_{\text{edges in } \alpha} \gamma(e) \underbrace{D_\alpha}_{\det K(I_\alpha, I_\alpha)} \Big\}$$

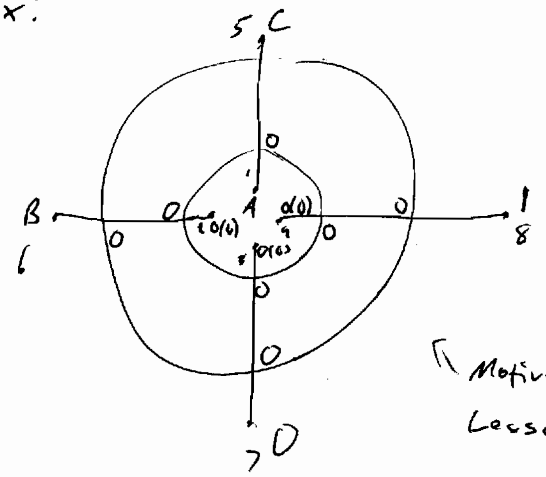
unused interior nodes

$\Rightarrow (-1)^k \det \Lambda(P,Q) \geq 0$ for cp networks

- connection $\Rightarrow$ det is nonzero in cp case, but not in general

Adds wrinkle to recovery algorithms for non cp networks

Ex:



$Q = [2\ 3\ 4]$ zero current conditions

$P = [1\ 6\ 5]$ unknown potentials

$p = 8$ location of potential 1

Motivation: solve for boundary spike

Lesson: Could have used boundary spike formula since contracting spike breaks all P, Q connections and $\Lambda(P,Q)$ invertible

First, solving for $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$

$$\vec{0} = \Lambda(Q, [P\ p]) \begin{bmatrix} A \\ B \\ C \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A \\ B \\ C \end{bmatrix} = -\Lambda(Q,P)^{-1} \Lambda(Q,p)$$

Not enough for connection to exist, need to make sure signs don't cancel to conclude invertibility

Possible connections: $2,3,4 \to 1,6,5$

$\to 6,5,1$ even permutation $\Rightarrow \operatorname{sgn}(\tau)$ stays the same for both τ

$\Rightarrow$ invertible

Connection to boundary spike formula:

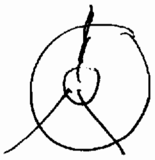
Current at spike is $\Lambda(p, [P\ p])\begin{bmatrix}A\\B\\C\\1\end{bmatrix} = \Lambda(p,p) - \Lambda(p,P)\Lambda(Q,P)^{-1}\Lambda(Q,p)$

potential difference is 1, hence γ = above formula = current

Interesting note: formula $\Rightarrow$ 0 potential at spike's interior node for ~~that current~~ those boundary conditions.

Other comparisons:

non recoverable $\not\Rightarrow$ Y-Δ equiv to series or parallel in non cp case

Ex:  infinite to 1 despite 15 equations + 15 unknowns

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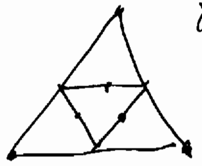
Recently lots of work on networks whose $\gamma \to \Lambda$ maps are n to 1 for n other than 1 or ∞, which is all that can happen in cp case.

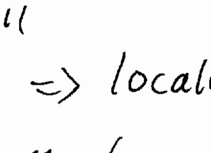
Ex: 

$\gamma \to \Lambda$ map is 2 to 1

Rank of differential of $T: \gamma \to \Lambda$ gives interesting info

- rank = 12 (full) $\Rightarrow$ locally 1-1, corresponds to globally 2-1
- rank = 11 (not full) $\Rightarrow$ not locally invertible, but corresponds to set of γ where T is 1-1

$T|_{f(\gamma)=0}$ is 1-1

Interesting analogy: 

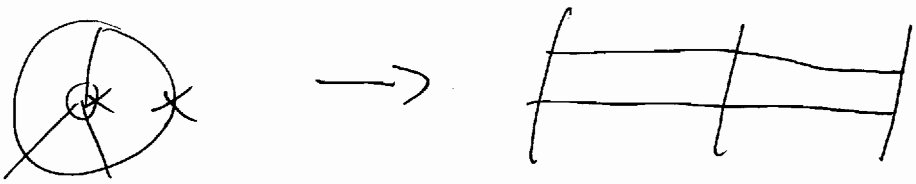
full rank $\Rightarrow$ globally 2-1, locally 1-1
not full rank $\Rightarrow$ globally 1-1, not locally invertible

Also $x \geq 0 \Rightarrow$ 1-1 and it turns out $T|_{f(\gamma) \geq 0}$ is 1-1

Another trick for extending cp analysis to non cp case

Set some γ to 0.

Ex: if $g(\lambda) = 0$ is relation in response matrix, it still holds if some γ are set to 0



Turns out: same difference of dets is zero in both networks, although each is zero in the cp network.

Problem ideas and some connections to other topics

Edge-node adjacency matrix sometimes useful: A is edges by nodes

$$A_{ek} = \begin{cases} \pm 1 & \text{if edge } e \text{ connected to node } k \\ 0 & \text{otherwise} \end{cases}$$

(arbitrary) Convention about orientation of edges: if e connected to i, j and $i < j$ then $A_{ei} = 1$, $A_{ej} = -1$

(Ting, Sam, George used for some numerical formulations and to help simplify work about rank of differential)

Define C = diagonal matrix with conductances γ_e on diagonal

Then Kirchhoff matrix $K = A^T C A$, direct analogue to $-\nabla \cdot (\sigma \nabla)$

$A \sim \nabla$, $A^T \sim -\text{div}$, $C \sim \sigma$

Any additional insight about Λ?

Index $A = [E \; F]$ according to boundary, interior nodes. $K = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$

$$K = \begin{bmatrix} E^T \\ F^T \end{bmatrix} \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} E & F \end{bmatrix} = \begin{bmatrix} E^T C E & E^T C F \\ F^T C E & F^T C F \end{bmatrix}$$

$$\Lambda = A - B D^{-1} B^T = E^T C E - E^T C F (F^T C F)^{-1} F^T C E$$

$$= E^T C^{\frac{1}{2}} \underbrace{\left(I - C^{\frac{1}{2}} F (F^T C F)^{-1} F^T C^{\frac{1}{2}} \right)}_{P} C^{\frac{1}{2}} E$$

P = orthogonal projection onto $\text{im}(C^{\frac{1}{2}} F)^{\perp} = \text{Ker}(F^T C^{\frac{1}{2}})$

$P = Q Q^T$ where columns of Q form orthonormal basis for $\text{im}(C^{\frac{1}{2}} F)^{\perp}$

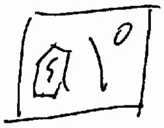
$$\Lambda = E^T C^{\frac{1}{2}} Q Q^T C^{\frac{1}{2}} E = (Q^T C^{\frac{1}{2}} E)^T (Q^T C^{\frac{1}{2}} E)$$

graph interpretation

Power dissipated $= x_B^T \Lambda x_B = \| Q^T C^{\frac{1}{2}} E x_B \|^2$

Using the forward problem for image segmentation

Talk by Leo Grady about this.

Problem: segment  given user input of some labels

Define lattice network with "conductances" such that high conductance between similar intensities and low conductance between dissimilar intensities.

Normalize K by $W^{-\frac{1}{2}} K W^{-\frac{1}{2}}$ where $W = \operatorname{diag}(K)$

Assume K already normalized.

Assign labels to nodes by randomly walking and seeing which label is ~~~~ most likely to be hit first.

Use equivalence between random walks & forward problem.

$$K = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \qquad 1 = -D^{-1}B^T 1 = -D^{-1}B^T(s_1 + \dots + s_k)$$

can treat as probabilities...

$\leftarrow$ k labels, potential 1 on label, 0 elsewhere

$$x_i = \begin{cases} 1 & \text{at int node } i \\ 0 & \text{at other int nodes} \end{cases}$$

Then $\underline{-x_i^T D^{-1} B^T s_j}$ is probability of random walk from x_i hitting s_j first

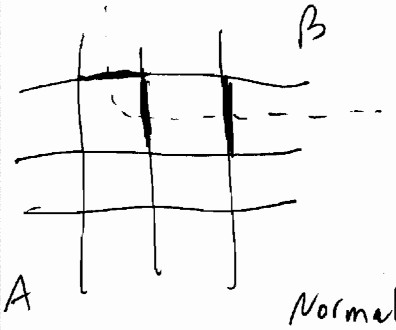
Transition matrix $T = \begin{bmatrix} I & -B \\ 0 & I-D \end{bmatrix}$

Same prob is $+\begin{bmatrix} s_j^T & 0 \end{bmatrix} \lim_{n\to\infty} T^n \begin{bmatrix} 0 \\ x_i \end{bmatrix}$ $\qquad T^n = \begin{bmatrix} I & -B(I + (I-D) + (I-D)^2 + \dots \\ 0 & (I-D)^n \end{bmatrix}$

Neumann series $\rightarrow \begin{bmatrix} I & -BD^{-1} \\ 0 & (I-D)^n \end{bmatrix}$ if $\deg \geq 2$ for int nodes

$$= -s_j^T B D^{-1} x_i = \underline{-x_i^T D^{-1} B^T s_j}$$

~~Very~~ ~~Similar~~ Related to normalized cuts: Shi-Malik



$$\text{cut}(A,B) = \sum_{u \in A,\, v \in B} w(u,v)$$

find partition that minimizes

maybe some constraints on sizes of A, B ...

Normalized version: $\dfrac{\text{cut}(A,B)}{|A|} + \dfrac{\text{cut}(A,B)}{|B|}$ $\quad |\cdot|$ = total weights in partition

Unfortunately: NP hard

Fortunately: Similar to $\min \dfrac{\sum_{ij} (v_i - v_j)^2 w_{ij}}{\sum_i w_i v_i^2}$ st $v \perp 1$

$v = W^{-\frac{1}{2}} z$

$$\Rightarrow \min \frac{z^T W^{-\frac{1}{2}} K W^{-\frac{1}{2}} z}{z^T z} \quad \text{st} \quad z \perp W^{\frac{1}{2}} 1$$

Raleigh quotient

eigenvector for smallest eigenvalue, 0, of normalized graph lap

minimizer is eigenvector for second smallest eigenvalue

$\Rightarrow$ 2nd smallest generalized evec for $Kv = \lambda W v$

diagonal weights

For the numerics of segmentation using Dirichlet prob; the evecs for the smallest eigs are most important

Krylov methods for evec iteration ...

Numerics for recovering γ in electrical networks:

continue work by Sam, Ting, George 2006

Many different numerical approaches

Ex! They had good success minimizing $\|\Lambda(\gamma) - \Lambda\|^2$ via Levenberg-Marquardt
(somewhere between Gauss-Newton & gradient descent, w/ regularization)

Still questions!

How to regularize γ? piecewise constant $\Rightarrow \sum_{\text{int}} \sum_{j\sim i,\, k\sim i} (\gamma_{ij} - \gamma_{ik})^2$

known values $\Rightarrow$ round to correct

Something else? Maybe power dissipated $x_B^T \Lambda(q) x_B$

Can use extra measurements:

$$G(q) = \sum_i \| \phi_{B_i}(q) - \phi_{B_i} \|^2$$ for associated imposed x_{B_i}

$\phi_B = \Lambda x_B = \underset{E^TCE}{A} x_B + \underset{E^TCF}{B} x_I$ rewrite ...

Slick trick for differentiating $\Lambda(q)$ wrt q

Different trick for simplifying derivative wrt q is to add interior potentials as extra variables

$$G(q, x_I) = \sum_{i=1}^{N} \| \phi_{B_i}(q, x_{I_i}) - \phi_{B_i} \|^2 \quad \text{constrain } \phi_{I_i}(q, x_{I_i}) = 0$$

$i = 1, \dots, N$

No inverses required to compute derivs wrt q or x_I.

But many more variables ...

L-M: $f = \frac{1}{2} \sum F_i^2 \Rightarrow f' = J^T F$, F is vectorized $(\Lambda(q) - \Lambda)$

$J^T J$ approximates Hessian

$(J^T J + \mu_k I) d_k = -J^T F$ solves for search direction

reg

$f(\underbrace{x_k + \alpha d_k}_{\text{update}}) < f(x_k)$ for some α

ODD problems:

- Find Network
  $\gamma = 0$ or 1

- See Strang talk or paper for interesting max flow / min cut problems

Interesting because of dual approaches

Ex: Max λ st $Ax = \lambda b$, $|x_i| \le c_i$

equivalent to

$$\text{Min} \quad \sum_i |(Au)_i| c_i \quad \text{st} \quad ub = 1$$

$\sim \sum |u_j - u_i| c_{ij}$ in node notation