ON THE CYCLICALLY FULLY COMMUTATIVE ELEMENTS OF COXETER GROUPS

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ABSTRACT. Let W be a Coxeter group. We say that $w \in W$ is cyclically fully commutative (CFC) if every cyclic shift of every reduced expression for w is fully commutative (FC). This definition is motivated by the conjugacy problem, because a cyclic shift of $w \in W$ is simply conjugation by the initial generator. In this paper, we classify the CFC-finite groups – those that only contain finitely many CFC elements, and show that they are precisely the FC-finite groups. In these groups, we characterize the CFC elements, enumerate them via a recurrence relation, and determine the conjugacy classes that contain them. In Type A, we show that the CFC elements are precisely the permutations that avoid the patterns 321 and 3412, and we ask how this might generalize to other Coxeter groups.

1. COXETER GROUPS

A *Coxeter group* W is a generalized reflection group, generated by a set $S = \{s_1, \dots, s_n\}$ of involutions. It is defined by its presentation

$$\langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle ,$$

where $m_{ij} := m(s_i, s_j) \in \mathbb{N} \cup \{\infty\}$ are called bond strengths, and $m_{ij} = 1$ if and only if $i = j$. This information can be encoded by a *Coxeter graph* Γ, which has vertex set S and an edge for each pair of non-commutating generators (i.e., $m(s,t) \geq 3$). Additionally, each edge is labeled with the corresponding bond strength, though if $m(s,t) = 3$, it is customary to drop the label. If $m(s,t) = 2$, then we have the *commutation relation* $st = ts$. If $m(s,t) = 3$, then we have $sts = tst$, and with $m(s,t) = 4$ we have $stst = tsts$, and so on. These are called *braid relations*. For any subset $J \subseteq S$, the subgroup $W_J = \langle s_i \mid s_i \in J \rangle$ is called the *standard parabolic subgroup* generated by J.

If S^* is the free monoid over S, then words in S^* correspond to elements of W. A word $w = s_{x_1} s_{x_2} \cdots s_{x_m}$ of minimal length is a *reduced expression* for w of length $\ell(w) = m$, and we say that a word $w = w_1 w_2 \cdots w_m$, where $w_i \in W$, is *reduced* if $\ell(w) = \ell(w_1) + \cdots + \ell(w_m)$. Put an equivalence relation $\approx$ on S^* where two words are equivalent if they differ by a sequence of commutation moves (i.e., $st \longmapsto ts$ provided that $m(s,t) = 2$). The resulting equivalence classes are called *commutation classes*. An element $w \in W$ is said to be *fully commutative* if all of its reduced expressions lie in the same commutation class. We denote the set of fully

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commutative elements by $\mathrm{FC}(W)$. Matsumoto's theorem says any two reduced expressions for w differ by a sequence of commutation relations and braid relations.

The classification of the finite Coxeter groups is well-known, and consists of several infinite families and a few exceptional cases. Perhaps the most common infinite family is A_n for $n \geq 1$, whose Coxeter graph is a line-graph with bond strengths 3. This group is isomorphic to the symmetric group SYM_{n+1}, through the map $s_i \longmapsto (i, i+1)$. It is well-known that the fully commutative elements of A_n are precisely the 321-*avoiding* permutations [**?**]. These are the permutations π such that there is no $i < j < k$ such that $\pi(i) > \pi(j) > \pi(k)$, where $\pi(1)\pi(2)\cdots\pi(n+1)$ is the 1-line notation of π. These are counted by the ubiquitous Catalan numbers [**?**].

A Coxeter group is *FC-finite* if $|\mathrm{FC}(W)| < \infty$. The FC-finite groups were classified by Stembridge [**?**], and they consist of seven infinite families. In a follow-up paper [**?**], Stembridge characterized and enumerated the FC elements in these groups. In this paper, we will do the same for the CFC-finite groups, which coincide with the FC-finite groups. Perhaps surprisingly, enumerating the CFC elements is much easier than the FC elements, because they all satisfy a common characterization and recurrence relation.

2. Cyclically Fully Commutative Elements

In this section, we will introduce two equivalence relations on the elements of a Coxeter groups that are both relevant to conjugacy. One of these relations will lead us directly to the definition of cyclically fully commutative elements, and we will utilize the other later in this paper.

Observe that for any $w = s_{x_1}s_{x_2}\cdots s_{x_m} \in W$, $s_{x_1}ws_{x_1}$ is a cyclic shift of w. Clearly, $\ell(s_{x_1}ws_{x_1}) \leq \ell(w)$. If every cyclic shift of every reduced expression for w has length $|w|$, then we say that w is *cyclically reduced*.

Definition 2.1. An element $w \in W$ is *cyclically fully commutative* (or CFC) if for any reduced expression $s_{x_1}s_{x_2}\cdots s_{x_m}$ of w, every cyclic shift is (i) fully commutative, and (ii) cyclically reduced. If only (i) holds, then we say that w is *weakly CFC*.

We denote the set of CFC elements of W by $\mathrm{CFC}(W)$, and the set of weakly CFC elements by $\mathrm{cfc}(W)$.

Example 2.2. Let $W = A_4$, which is isomorphic to the symmetric group SYM_5. Then $s_3s_2s_1s_4s_3s_2$ is FC, but it is not CFC (or weakly CFC) because there is a braid relation that wraps around the end of the word. In contrast, the word $s_3s_2s_1s_4s_3$ is weakly CFC but not CFC.

We will now define the equivalence relations $\sim$ and $\sim_\kappa$ on W. The former first appeared in a paper on characters of Hecke algebras [**?**], and the latter has been used for the conjugacy problem [**?**], and has also been called *rotation equivalence* [**?**].

Definition 2.3. Let W be a Coxeter group.

(i) Write $w \sim w'$ if there is a sequence $w = w_0, \ldots, w_r = w'$ such that $\ell(w_i) = \ell(w_{i+1})$, $w_{i+1} = x_i w_i x_i^{-1}$, and $\ell(x_i w_i) = \ell(x_i) + \ell(w_i)$ or $\ell(w_i x_i^{-1}) = \ell(w_i) + \ell(x_i^{-1})$ for all x_i and some $x_i \in W$.

(ii) If $w \sim w'$, and furthermore, $|x_i| = 1$ for each i, then we write $w \sim_\kappa w'$.

An alternative way to define κ-equivalence is through the *conjugacy graph*, which has vertex set W, and a directed edge (w, w') is present if $\ell(w) \geq \ell(w')$, and $s_{x_1} w s_{x_1} = w'$, for some reduced expression $s_{x_1} \cdots s_{x_m}$ of w. The strongly connected components are the κ-equivalence classes. Furthermore, define a partial ordering $\preceq_\kappa$, where $w' \preceq_\kappa w$ if there is a directed path from w to w' in the conjugacy graph. The rank function $\rho(w) := \frac{1}{2}\ell(w)$ turns $(W/{\sim_\kappa}, \preceq_\kappa)$ into a graded poset.

The following example shows how two elements can be equivalent without being κ-equivalent.

Example 2.4. Let $W = A_n$, and put $w_0 = s_1$ and $x_0 = s_1 s_2$ Then, $s_2 = x_0 w_0 x_0^{-1}$, and so $s_1 \sim s_2$ despite $s_1 \nsim_\kappa s_2$. By continuing this argument, we see that any two simple generators in A_n are conjugate.

Remark 2.5. The CFC elements are precisely the FC elements that are minimal in $(W/{\sim_\kappa}, \preceq_\kappa)$, and $\mathrm{CFC}(W) \subseteq \mathrm{cfc}(W) \subseteq \mathrm{FC}(W)$.

The following result relates the two types of equivalence.

Theorem 2.6. Let W be a finite Weyl group, C be a conjugacy class of W, and C_{min} the set of elements of C of minimal length. Then
(a) For each $w \in C$, there is a $w' \in C_{min}$ with $w' \preceq_\kappa w$.
(b) If $w, w' \in C_{min}$, then $w \sim w'$.

3. Classification of CFC-finite groups

In [?], John Stembridge gave a complete classification of the Coxeter groups that contain finitely many FC elements, called the *FC-finite groups*. In a similar vein, the *CFC-finite groups* can be defined as the Coxeter groups that contain only finitely many CFC elements. In this section, we will show that a group is CFC-finite if and only if it is FC-finite. Since $\mathrm{CFC}(W) \subseteq \mathrm{FC}(W)$, the CFC-finite groups contain the FC-finite groups, thus it suffices to show that every FC-finite group is also CFC-finite. The following simple lemma is needed.

Lemma 3.1. If $w^k \in \mathrm{FC}(W)$ for all $k \in \mathbb{N}$, then $w^k \in \mathrm{CFC}(W)$ for all $k \in \mathbb{N}$.

Proof. We prove the contrapositive. If $w^k \notin \mathrm{CFC}(W)$, then there is some subword $stst\cdots$, possibly wrapping around the end of the word. However, this is contained as a subword of w^{k+1}, and thus $w^{k+1} \notin \mathrm{CFC}(W)$. □

Theorem 3.2. The irreducible CFC-finite Coxeter groups are A_n ($n \geq 1$), B_n ($n \geq 2$), D_n ($n \geq 4$), E_n ($n \geq 6$), F_n ($n \geq 4$), H_n ($n \geq 3$), and $I_2(m)$ ($5 \leq m < \infty$).

Proof. Stembridge classified the FC-finite groups by classifying their Coxeter diagrams. In particular, he gave a list of ten forbidden properties that an FC-finite group cannot have. The list of FC-finite groups are precisely those that avoid all ten of these obstructions. The first five conditions are easy to state, and are listed below.

(1) Γ cannot contain a cycle.
(2) Γ cannot contain an edge of weight $m(s,t) = \infty$.

(3) Γ cannot contain more than one edge of weight ≥ 4
(4) Γ cannot have a vertex of degree ≥ 4, or more than one vertex of degree 3.
(5) Γ cannot have both a vertex of degree 3 and an edge of weight ≥ 4.

For each of the ten conditions, including the above five, Stembridge shows that if it fails, one can produce a word $w \in W$ such that $w^k \in \mathrm{FC}(W)$ for all $k \in \mathbb{N}$. This, together with Lemma 3.1, implies that if W is CFC-finite, then it is FC-finite, and the result follows immediately. □

[insert picture]

4. STRUCTURE OF CFC ELEMENTS

Each of the infinite families of the CFC-finite Coxeter graphs consist of a subgraph Γ_0 with a distinguished vertex s_1 called the *joint*, attached to a line-graph $\{s_2, \dots, s_k\}$, called the *branch* of Γ. We call Γ_0 the *center* of Γ.

[insert picture of Γ_0 for the 6 types]

The following result shows that CFC elements have a very restricted form.

Lemma 4.1. If W is CFC-finite, and $w \in \mathrm{CFC}(W)$, then each branch generator can only occur once in a reduced expression of w.

Proof. Assume by contradiction that the statement is false, and that $w \in \mathrm{CFC}(W)$ is a counterexample. Let s_i be the branch generator that occurs most frequently in a reduced expression of w, say c times. If we view w as a cyclic word, between consecutive occurence of s_i must be at least two instances of generators from the set $\{s_{i-1}, s_{i+1}\}$. This eliminates the possiblity that s_i is the endpoint generator, s_k. Moreover, since neither s_{i-1} nor s_{i+1} occur more frequently than s_i, they both must occur exactly c times. Therefore, every branch generator occurs with the same frequency in w. However, between any two instances of the endpoint generator s_k in w must be at least two instances of s_{k-1}, a contradition. □

Lemma 4.1 allows us to enumerate the CFC elements of the CFC-finite groups. Let W_n denote a rank-n CFC-finite group of a fixed type, where $n \geq 3$, and let W_{n-1} be the parabolic subgroup generated by all generators except the final branch generator of W_n.

Theorem 4.2. Let W be a CFC finite group. Suppose that $w \in \mathrm{CFC}(W)$, and some generator $s \in S$ appears in w more than once. Then

(i) $W = H_n$ or $W = I_2(m)$.
(ii) There is a (unique) generator t such that $m(s,t) = 2k+1 > 3$.
(iii) The occurences s and t alternate, occurring with the same frequency, but more than k times each.

Proof. To do. □

Corollary 4.3. If $a_n = |\,\mathrm{CFC}(W_n)|$, then a_n satisfies the recurrence

$$a_n = 3a_{n-1} - a_{n-1}\ . \tag{4.1}$$

	1	2	3	4	5	6	7	8	9	10
$\lvert$ FC $\rvert$	2	5	14	42	132	429	1430	4862	16796	58786
$\lvert$ CFC $\rvert$	2	5	13	34	89	233	610	1597	4181	10946

TABLE 1. The number of FC and CFC elements in Type A.

Proof. Every CFC element in W_n is also CFC in W_{n-1}, and there are a_{n-1} of these. Let $s = s_k$, the final branch generator of W_n, and consider the CFC elements that contain s. Since each generator appears at most once, every element can be written as sw or ws, thus we need to compute the cardinality of

$$\{sw \mid w \in \mathrm{CFC}(W_n)\} \cup \{ws \mid w \in \mathrm{CFC}(W_n)\} .$$

Each of these two sets has size a_{n-1}, and $sw = ws$ iff $s_{k-1} \notin w$, thus the intersection has size $|\,\mathrm{CFC}(W_{n-1})| = a_{n-2}$. By inclusion-exclusion, the union has size $2a_{n-1} - a_{n-2}$. In summary, there are $2a_{n-1} - a_{n-2}$ CFC elements that contain s, and a_{n-1} CFC elements that don't, thus $a_n = 3a_{n-1} - a_{n-2}$. □

5. ENUMERATION OF THE CFC ELEMENTS

By Proposition 4.3, to enumerate the CFC elements in W_n for each type, we just need to count them in the smallest groups of that family. In this section, we will denote the number of CFC elements in a type by the corresponding lowercase letter, e.g., $b_n = |\,\mathrm{CFC}(B_n)|$.

5.1. **Type** A**.** Both elements of $A_1 = \{1, s\}$ have order 2, and there are five CFC elements in $A_2 = I_2(3) = \{1, s, t, st, ts, sts\}$, thus $a_1 = 2$ and $a_2 = 5$. The odd-index Fibonacci numbers satisfy the recurrence in (4.1) as well as the initial seeds (see [**?**]). Therefore, $a_n = F_{2n-1}$. By induction, the CFC elements in A_n are precisely those that have no repeat generators. In the language of [**?**], these are the Boolean permutations, and are characterized by avoiding the patterns 321 and 3412. (A permutation π avoids 3412 if there is no set $\{i, j, k, l\}$ with $i < j < k < \ell$ and $\pi(k) < \pi(\ell) < \pi(i) < \pi(j)$). The following result is immediate.

Corollary 5.1. An element $w \in A_n$ is CFC if and only if w is 321-avoiding and 3412-avoiding.

It is worth noting that F_{2n-1} also counts the 1324-avoiding *circular* permutations on $[n+1]$ (see [**?**]). However, these are set-wise not the same as the CFC elements in $A_n = \mathrm{S{\small YM}}_{n+1}$. As a simple example, the permutation $(2, 3) \in A_3$ does not avoid 1324 but is clearly CFC. Also, the element $s_2s_3s_1s_2s_4s_3 \in A_4$ (or $(1, 3, 5, 2, 4)$ in cycle notation) has no (circular) occurence of 1324, but is not CFC.

5.2. **Type** B**.** Both elements of B_1 have order 2. In $B_2 = I_2(4)$, the elements $sts = tst$ are weakly CFC but not cyclically reduced. All remaining element other than the longest element are CFC, so we have $b_1 = 2$ and $b_2 = 5$.

5.3. **Type** D**.** The group D_1 consists of two commuting generators, and D_2 is isomorphic to A_3. Therefore, $d_1 = 4$ and $d_2 = 13$.

	1	2	3	4	5	6	7	8	9	10
| FC |	2	7	24	83	293	1055	3860	14299	53481	201551
| CFC |	2	5	13	34	89	233	610	1597	4181	10946

TABLE 2. The number of FC and CFC elements in Type B.

	2	3	4	5	6	7	8	9	10
| FC |	4	14	48	167	593	2144	7864	29171	109173
| CFC |	4	13	35	92	241	631	1652	4325	11323

TABLE 3. The number of FC and CFC elements in Type D.

	3	4	5	6	7	8	9	10
| FC |	10	42	167	662	2670	10846	44199	180438
| CFC |	10	34	92	242	634	1660	4346	11378

TABLE 4. The number of FC and CFC elements in Type E.

	1	2	3	4	5	6	7	8	9	10
| FC |	2	5	24	106	464	2003	8560	36333	153584	647775
| CFC |	2	5	13	34	89	233	610	1597	4181	10946

TABLE 5. The number of FC and CFC elements in Type F.

5.4. **Type** E**.** The groups E_4 and E_5 are isomorphic to A_4 and D_5, respectively, and so $e_4 = 34$ and $e_5 = 92$. We note that if we define E_3 by removing the joint vertex from the Coxeter graph, leaving an edge and singleton vertex, then is is readily checked that $e_3 = 10$, and so $e_5 = 3e_4 - e_3$.

5.5. **Type** F**.** The groups F_2 and F_3 are isomorphic to A_2 and B_3, respectively, and so $f_2 = 5$ and $f_3 = 13$. Similar to Type E, if we define F_1 as having a singleton Coxeter graph, then $f_1 = 2$, and $f_3 = 3f_2 - f_1$. Thus, these are also counted by the odd-indexed Fibonacci numbers with a "shifted" seed, yielding $f_n = F_{2n+1}$.

5.6. **Type** H**.** The group H_1 has order 2, and in $H_2 = I_2(5)$, the elements sts and tst are weakly CFC but not cyclically reduced. All other elements except the longest element are CFC, so $h_1 = 2$ and $h_2 = 7$.

Let's give a slightly better characterization in Type H. (These are the only ones where CFC is not equivalent to containing every generator at most once).

6. CONJUGACY CLASSES OF CFC ELEMENTS

In this section, we will investigate when two CFC elements in a CFC-finite group are conjugate. For a word $w \in W$, let $\Gamma[w]$ be the *subgraph induced by* w. Explictly, it has vertex

	1	2	3	4	5	6	7	8	9	10
| FC |	2	9	44	195	804	3185	12368	47607	182720	701349
| CFC |	2	7	21	56	147	385	1008	2639	6909	18088

TABLE 6. The number of FC and CFC elements in Type H.

set $\mathrm{supp}(w)$ and an edge $\{s,t\}$ with label $m(s,t)$ is present for each $s,t \in \mathrm{supp}(w)$ such that $m(s,t) \geq 3$.

It turns out the conjugacy of the CFC elements is "usually" determined by the isomorphism class of the induced subgraph $\Gamma[w]$. As we will see, this always holds for simply-laced case. One problem arises due to the even bond strengths in types B and F, and another problem occurs because H_n is not a Weyl group.

As an example, we will first consider the two extreme cases. Suppose W is CFC-finite, but not type H or $I_2(m)$. The CFC elements are the Coxeter elements of the standard parabolic subgroups. First, consider the length-1 CFC elements – the simple generators. It is well-known that s and t are conjugate if and only if they are connected by a path in Γ traversing only edges of odd bond strength. Next, consider the length-n CFC elements – the Coxeter elements. Since the Coxeter graphs of the CFC finite groups are all trees, $C(W)$ is contained in a single conjugacy class [**?**].

Lemma 6.1. Let W be a CFC-finite, and suppose that $w, w' \in \mathrm{CFC}(W)$ are conjugate. Then $\Gamma[w]$ and $\Gamma[w']$ are isomorphic.

Proof. To do.

It may be helpful to use the fact that if w and w' are CFC, then they are in C_{min}, and so by part (b) of Theorem 2.6, $w \sim w'$, implying that they have the same length. However, this only works (as far as we know) for the finite Weyl groups (A_n, B_n, D_n, E_6, E_7, E_8, F_4, $I_2(m)$). □

The converse of Lemma 6.1 obviously fails, because for $s \in S$, $\Gamma[s]$ is a singleton vertex, and not all of the generators in types B or F are conjugate (due to the edge of even bond strength).

Lemma 6.2. Let W be a simply-laced CFC-finite group (i.e., type A, D, or E). If $w, w' \in \mathrm{CFC}(W)$ and $\Gamma[w]$ and $\Gamma[w']$ are isomorphic, then w and w' are conjugate.

Proof. To do. □

I think this will hold in type H and $I_2(m)$ if we also account for the number of repeats of the generators on the "strong" edge.

Finally, we need to finish the type B and F case. The former should be easier because all of the groups are finite (and thus we can try some computations for the small cases).

7. Future Research

The notion of patterns, and pattern avoidance, can be generalized from the symmetric group (type A) to arbitrary Coxeter groups [**?**]. Since the CFC elements in type A have a simple characterization via pattern avoidance, it would be interesting to investigate if this a just a special case of a more general phenonomon in other Coxeter groups. This may be unrealistic, because being fully commutative does not generalize in this manner. However, we've seen how passing from FC to CFC surprisingly simplifies the combinatorics in the FC-finite groups.

Also, are there any nice analogs any other of Stembridge's results on FC-elements?

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