<u>Defn</u>: X is n-to-1 if $\exists\ \Lambda$ w/ precisely n distinct cond γ on X w/ $\Lambda_{(X,\gamma)} = \Lambda$.

generally take $n \in \mathbb{N}$

note: don't require <u>all</u> Λ w/ solns to have precisely n

to study, we make use of a version of the <u>star-K xformation</u>:

~~if X is an n-star ($n>2$), then $\exists$ a resp. preserving bij. corr. btw conds on X & conds on~~

First a defn. γ on K_n ($n>2$) satisfies the quad rule if $\forall$ distinct i,j,k,ℓ, $\gamma_{ij}\gamma_{k\ell} = \gamma_{ik}\gamma_{j\ell}$.

~~implies~~ equiv to triangle cond: For distinct i, j, k,

$$\frac{\gamma_{ij}\gamma_{ik}}{\gamma_{jk}} \quad \text{ind. of } j, k.$$

Ti: if i given, $j \neq k$ distinct from i & $m \neq n$ distinct from i, then $\frac{\gamma_{ij}\gamma_{ik}}{\gamma_{jk}} = \frac{\gamma_{im}\gamma_{in}}{\gamma_{mn}}$

pf: ~~[illegible]~~

triangle $\Leftarrow$: $\frac{\gamma_{ij}\gamma_{il}}{\gamma_{jl}} \overset{(*)}{=} \frac{\gamma_{ik}\gamma_{il}}{\gamma_{kl}}$ $\forall$ distinct i, j, k, l

$\Leftrightarrow \gamma_{ij}\gamma_{kl}\cancel{\gamma_{il}} = \gamma_{ik}\gamma_{jl}\cancel{\gamma_{il}}$

(*) shows $\frac{\gamma_{ij}\gamma_{ik}}{\gamma_{jk}} = \frac{\gamma_{in}\gamma_{ik}}{\gamma_{nk}}$ &

$$= \frac{\gamma_{ik}\gamma_{in}}{\gamma_{kn}} = \frac{\gamma_{im}\gamma_{in}}{\gamma_{mn}}$$

WLOG

if $j \neq n$

if $k \neq n$

elsewise

$\square$

if X is an n-star $(n > 2)$ $\exists$ a resp. preserving bij corr btw cond. on X & cond on K_n satisfying the quad rule.

<u>pf.</u> (Russel)

$X \to K_n$, $\gamma \to \mu$

$\gamma_{j,n+1} = \gamma_j$, $\sigma = \sum_{j=1}^{n} \gamma_j$

set $\mu_{ij} = \dfrac{\gamma_i \gamma_j}{\sigma}$ $(i \neq j)$

$\mu_{ij}\mu_{kl} = \dfrac{\gamma_i\gamma_j\gamma_k\gamma_l}{\sigma^2} = \dfrac{\gamma_i\gamma_k\gamma_j\gamma_l}{\sigma^2} = \mu_{ik}\mu_{jl}$

resp pres. as result of Schur comp.

$X \leftarrow K_n$, $\gamma \leftarrow \mu$

$\alpha_i = \sqrt{\dfrac{\mu_{ij}\mu_{ik}}{\mu_{jk}}}$ distinct i,j,k

$\alpha_i\alpha_j = \sqrt{\dfrac{\mu_{ij}\mu_{ik}}{\mu_{jk}}}\sqrt{\dfrac{\mu_{ji}\mu_{jk}}{\mu_{ik}}}$

$= \mu_{ij}$ distinct i,j

$\gamma_j = \alpha_j \sum_k \alpha_k$.

Thus to est claim STS inverses of each other

$\mu \to \gamma \to \mu'$: $\mu'_{ij} = \dfrac{\gamma_i\gamma_j}{\sigma} = \dfrac{(\alpha_i \sum_k \alpha_k)(\alpha_j \sum_k \alpha_k)}{\sum_l(\alpha_l \sum_k \alpha_k)} = \alpha_i\alpha_j = \mu_{ij}$.

$\gamma \to \mu \to \gamma'$: $\gamma'_j = \alpha_j \sum_k \alpha_k = \sum_{k\neq j} \alpha_j\alpha_k + \alpha_j\alpha_j$

$= \sum_{k \neq j} \mu_{jk} + \dfrac{\mu_{ji}\mu_{jl}}{\mu_{il}}$ $(i, l \neq k)$

$= \sum_{k\neq j} \dfrac{\gamma_j\gamma_k}{\sigma} + \dfrac{\frac{\gamma_j\gamma_i}{\sigma}\frac{\gamma_j\gamma_l}{\sigma}}{\gamma_i\gamma_l/\sigma} = \gamma_j\left(\dfrac{\sum_k \gamma_k}{\sigma}\right) = \gamma_j$. $\square$

extends to embedded stars

(obviously bij; resp. pres. as one step in G.E.)

Q's?

Goal today: show $X =$  is 2-to-1.

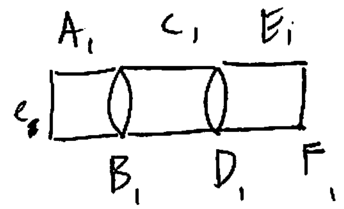
sans motivation;
cf. Ernie

- show $\gamma \leftrightarrow \gamma^{*}$ (quad rule) by repeated use of prev. thm, i.e. $\forall \Lambda$ $\Lambda_{(x,\gamma)} = \Lambda$ iff $\Lambda_{(x^*,\gamma^*)} = \Lambda$.
- by bij, # of γ^* = # of γ.
  w/ qr
  & $\Lambda_{(x^*,\gamma^*)} = \Lambda_{(x,\gamma)}$
- nice to work on X^* as $K_{(x^*,\gamma^*)} = \Lambda_{(x^*,\gamma^*)}$.

- to begin take $\gamma_1 > 0$ & let $\Lambda_1 = \Lambda_{(x,\gamma)}$.

- suppose γ^* on X^* has $\Lambda_{(x^*,\gamma^*)} = \Lambda_1$.
  - some values of γ^* determined

- name consts on graphs



- γ^* must assume some value on e, say X

- can now determine other values
  (>0, & c., as γ^* a cond. by hyp.)
  ↑ denoms & fcns
- name fcns: u_1, v_1, h_1, g_1, f_1.
- note cond. on X.

- conv., if all >0 @ y_0 & cond on X, & let $\gamma^*_{y_0}$ assume values det. by y_0 in above formus., then $\gamma^*_{y_0}$ a cond on X^* sat. quad rule & $\Lambda_{(X^*, \gamma^*_{y_0})} = \Lambda_1$.

- thus, # of q.r. conds on X^* w/ resp $= \Lambda_1$
  $=$ # solns y_0 to $4, + f_1(y_0) = F_1$ in $\{$allfcns$(z) > 0\}$
- same w/ $1 \to j$.

- First, consider the fcns:

- define LFTs
  - note: deriv has "Fixed sign"
  - note: same limiting values @ $\pm\infty$ if singular $\to$ ⋮
- all are LFTs
  - can determine sign of deriv. From graph alone; ind. of j.

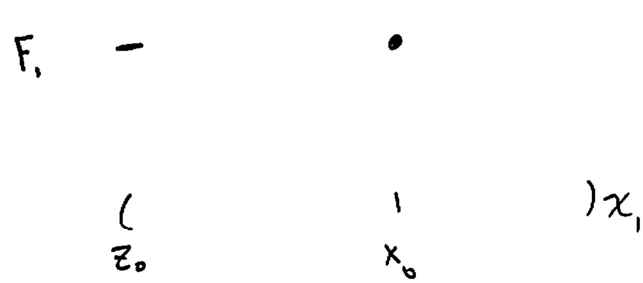
- If f_1 sing, @ most two solns to $x+f_1(x) = F_1$ (ditto j).

- next, let $\mathcal{X}_1 = \{x : x>0, u_1(x)>0, v_1(x)>0, \ldots, f_1(x)>0\}$
  - $\underline{\underline{\infty \neq 0}}$
  - $\mathcal{X}$ an open interval
    - kill off some conds; reduce others.

- thus, WTF # of $x \in \mathcal{X}$ w/ $x+f_1(x) = F_1$.

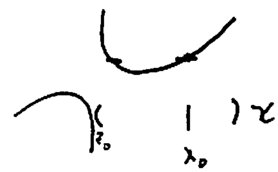
- note: γ_1 induces one such value; call it x_0

$x_0+f_1(x_0)=F_1$



Say z_0 is the LEPO x_1.

One poss. for sum to get two solns is



important. ① as. to $+\infty$ as $x \to z_0^+$
② deriv. @ $x_0 > 0$

Suppose (magically):

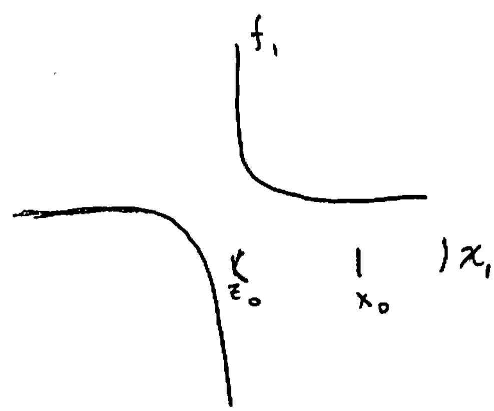
① f_1 sing @ z_0

② $|f_1'(x_0)| < 1$.

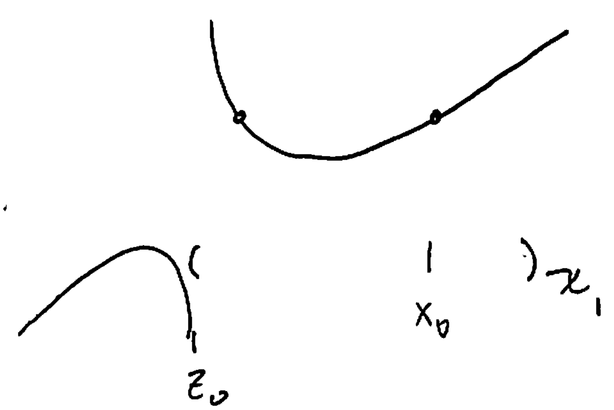
how diff from x?

(bends upward while maintaining deriv @ z_0)

f_1 has following form:



hence $f_1(x)+x$ ~



two solns

end lect 1.

No reason for δ_1 to have these props (@ least, no reason we have given)

idea: change Λ_1 to Λ_j s.t. δ_j has props & x_j is ~~is not too different~~ $\underline{\underline{\text{"desirable"}}}$

many issues can arise ~~~~

throughout, convenient to have $\gamma^*_{x_0}$ satisfy q.r. & corr. to Λ_j (depends on entries in Λ_j)

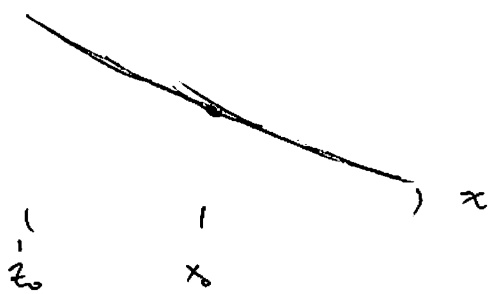
- more essential in higher n

First, singularity: $f_2 = \dfrac{F_2}{D_2 - h_2}$

if we don't change h (i.e., if $h_2 = h_1$) then $f_2^{(x)}$ is sing. when $D_2 = h_2(x)$.

WTMake f_2 sing to the left of x_0 ; won't exactly be @ z_0, but idea still applies

- h_1 dec. on $\mathcal{X}_1$:



- choose a pt $y_0 \in (z_0, x_0)$. Note $h_1(y_0) > h_1(x_0)$.

- Set $D_2 = h_1(y_0)$ & $F_2 = \dfrac{E_1}{h_1(y_0) - h_1(x_0)} + x_0$

(sing) $\quad (\gamma^*_{x_0} \sim \Lambda_2)$

- Check Λ_2 a valid r.m. & $\gamma^*_{x_0} \sim \Lambda_2$

ST Check latter point:

- most unchanged
- $g_2(x_0) = h_1(y_0) - h_1(x_0) > 0$
- $f_2(x_0)$ has same sgn as $g_2(x_0)$.
- $\gamma^*_{x_0} \sim \Lambda_2$ by construction.

What of ~~[illegible]~~ $\mathcal{X}_2$?

(| |) x_1

t_0 y_0 x_0

$x_0 \in \mathcal{X}_2$

g_2 cont on (y_0, x_0) as g_1 cont on (y_0, x_0)

$\Rightarrow$ $g_2 > 0$ on (y_0, x_0) as $g_2(y_0) = 0$

$\Rightarrow$ $f_2 > 0$ on (y_0, x_0) & g_2 s.i

$\Rightarrow$ $\mathcal{X}_2 \not\supseteq (y_0, x_0]$

- plot $x + f_2(x)$ w/ F_2

- Fix deriv @ x_0

- done. ✓

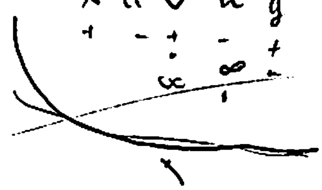
$x \; u \; v \; f \; g \; h$

$h \; g \; f$

$x \quad u \quad v \quad h \quad g \quad f$

$+ \quad - \quad + \quad - \quad + \quad -$

$\infty \quad \infty$



$f: \dfrac{c_1}{c_2 - h}$

x_0

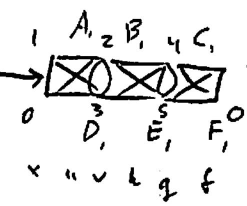
$\tilde{c}_2 : h_1(z_0)$

z_0

> 0

$< \infty$

1/

- Talking abt 2-br-1 graphs, i.e. this graph X.
- Studying via our version of star-K
- Recall what we're calling X^*, the star-K transformation of X.
- 1st reduction: # γ on X w/ $\Lambda_{(X,\gamma)} = \Lambda_1$ = # γ^* on X^* sat. gr. w/ $K_{(X^*,\gamma^*)} = \Lambda_1$
- Note: γ^* on X^* sat. gr. & $K_{(X^*,\gamma^*)} = \Lambda_1$ is determined by its value on edge 

- if value is x, values on remaining edges are

$$u_1(x) = \frac{A_1}{x}$$

$$v_1(x) = D_1 - \frac{A_1}{x}$$

$$h_1(x) = \frac{B_1}{D_1 - \frac{A_1}{x}}$$

$$g_1(x) = E_1 - \frac{B_1}{D_1 - \frac{A_1}{x}}$$

$$f_1(x) = \frac{C_1}{E_1 - \frac{B_1}{D_1 - \frac{A_1}{x}}}$$

where $D_1 = (\Lambda_1)_{3,3}$, $E_1 = (\Lambda_1)_{4,5}$, $F_1 = (\Lambda_1)_{0,1}$

&, e.g.,

$A_1 = (\Lambda_1)_{1,2}(\Lambda_1)_{0,3}$

$B_1 = (\Lambda_1)_{2,4}(\Lambda_1)_{3,5}$

$C_1 = (\Lambda_1)_{1,4}(\Lambda_1)_{0,5}$

- Since $K_{(X^*,\gamma^*)} = \Lambda_1$, necessarily $x + f_1(x) = F_1$.

- Conversely, if $x \in \mathbb{R}^+$ is given and

$u_1(x), v_1(x), h_1(x), g_1(x), f_1(x) > 0$ & $x + f_1(x) = F_1$,

then the fcn γ_x^* assuming the appropriate values on the appropriate edges of X^* is a wd. on X^* which sat g.r. & has $K_{(X^*, \gamma_x^*)} = \Lambda_1$.

explain why in terms of fcns above & how we determined them from Λ_1.

- Thus, if we set $\mathcal{X}_1 = \{x \in \mathbb{R}^+ : u_1(x) > 0, v_1(x) > 0, h_1(x) > 0, g_1(x) > 0, f_1(x) > 0\}$, then we have

- 2^{nd} reduction: # γ^* on X^* sat. g.r. w/ $K_{(X^*, \gamma^*)} = \Lambda_1$

$= \#\ x \in \mathcal{X}_1$ w/ $x + f_1(x) = F_1$.

- Showed: $u_1, v_1, \ldots$ are LFTs

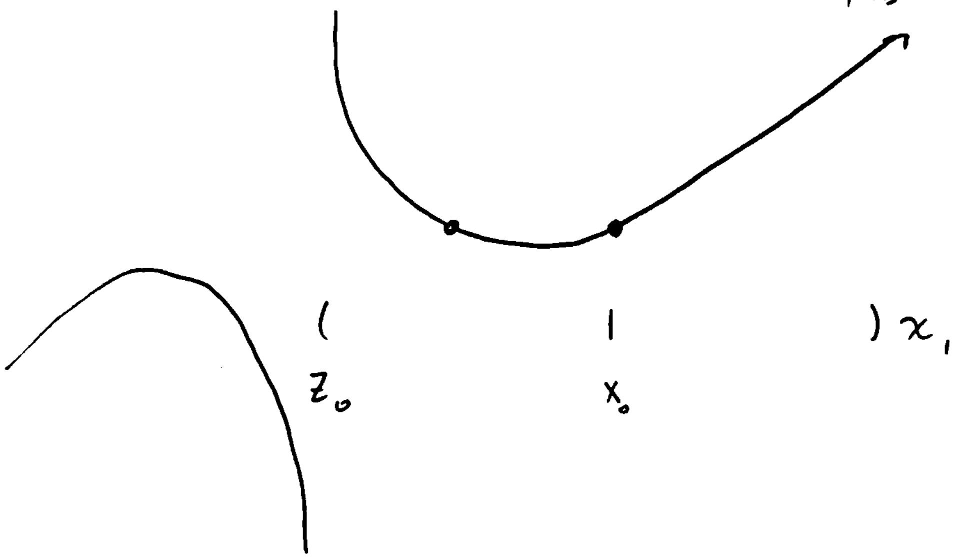
$\mathcal{X}_1$ a cxd open set

if f_1 sing., @ most two x s.t. $x + f_1(x) = F_1$.

(sloppy arg. on my part; will give "better" arg. for a slightly more general result in 3-to-1 case)

3]

Possible "nice" behavior of $x + f_1(x)$:



essential characteristics:

① f_1 sing. @ z_0

② $|f_1'(x_0)| < 1$

③ $x_1 \in (z_0, x_0]$.

goto p. 4

Will achieve by changing Λ_1 (& thus subscripts)

First ①, then ②, while "maintaining" ③

& fact that $\hat{K}_{(X^*, \gamma^*_{x_0})} = \Lambda_j$ $j = 1, 2, 3$.

$\gamma^*_{x_0}$ is a cond. on $\hat{X}^*_{\text{sat gr.}}$ w/

(this will, i.p., tell us Λ_j is a valid resp. mult.)

Now $f_1(x) = \dfrac{c_1}{E_1 - h_1(x)}$.

When change j in Λ_j, will define $A_j, B_j, \ldots F_j$ as before w/ Λ_1 replaced by Λ_j, & will define u_j, v_j, h_j, g_j, f_j as before w/ $A_1, \ldots, E_1$ replaced w/ $A_j, \ldots, E_j$. Same conclusions above will hold w/ 1 replaced by j. I.p., once Λ_2 is defined

$$f_2(x) = \frac{c_2}{E_2 - h_2(x)}$$

[Initially said WTMake sing @ z_0: LEPOX$_1$. Can present some issues ~~…~~ since $z_0 \notin X_1$: some fcns (i.p. h_1) could be 0 or undef. @ x_1.

Easier to take $y_0 \in (z_0, x_0)$ & make sing. @ z_0'; if ② & ③$'$: $x_1 \in (y_0, x_0]$ hold, still get conc. (actually, should rep. 1 by j)]

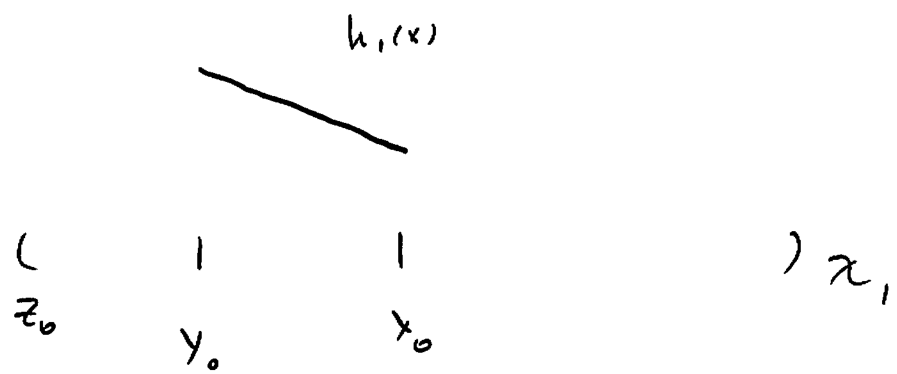
Now $f_2(x)$ sing. pre. when $E_2 = h_2(x)$.

Suppose Λ_1 & Λ_2 agree on entries defining h_1 & h_2

Then $h_1 = h_2$ as fcns of x.

I.p., $f_2(x)$ sing. when $h_1(x) = E_2$.

Now take $y_0 \in (z_0, x_0)$



$- \; h_1(y_0) > h_1(x_0)$

Let Λ_2 agree w/ Λ_1 except that

$$E_2 = h_1(y_0) \quad \& \quad F_2 = x_0 + \frac{c_1}{h_1(y_0) - h_1(x_0)} .$$

6]

UTCheck $\gamma^*_{x_0}$ (w/ values given by u_2, v_2, h_2, g_2, f_2)
is a cond on X^* sat q.r. w/ $K_{(X^*, \gamma^*_{x_0})} = \Lambda_2$.

No change until g_2:

$$g_2(x_0) = E_2 - h_2(x_0) = h_1(y_0) - h_1(x_0) > 0.$$

$$f_2(x_0) = \frac{c_2}{g_2(x_0)} = \frac{c_1}{h_1(y_0) - h_1(x_0)} > 0$$

hence $\gamma^*_{x_0}$ a cond, so q.r. follows

also $$x_0 + f_2(x_0) = x_0 + \frac{c_1}{h_1(y_0) - h_1(x_0)} = F_2$$
by defn.

So Λ_2 "valid"

By constr., f_2 sing @ y_0.

What about x_2?

7) . $x_0 \in \mathcal{X}_2$

so, WTS $(y_0, x_0) \subseteq \mathcal{X}_2$

$x, u_2, v_2, h_2, > 0$ on (y_0, x_0) as $(y_0, x_0) \subseteq \mathcal{X}_1$
& these fcns unchanged.

$g_2 = E_2 - h_2$.

~~thus $g_2(x)$ sing iff $h_2(x)$ sing.~~

$h_2 = h_1$ & h_1 cont on $[y_0, x_0)$
as $[y_0, x_0) \subseteq \mathcal{X}_1$

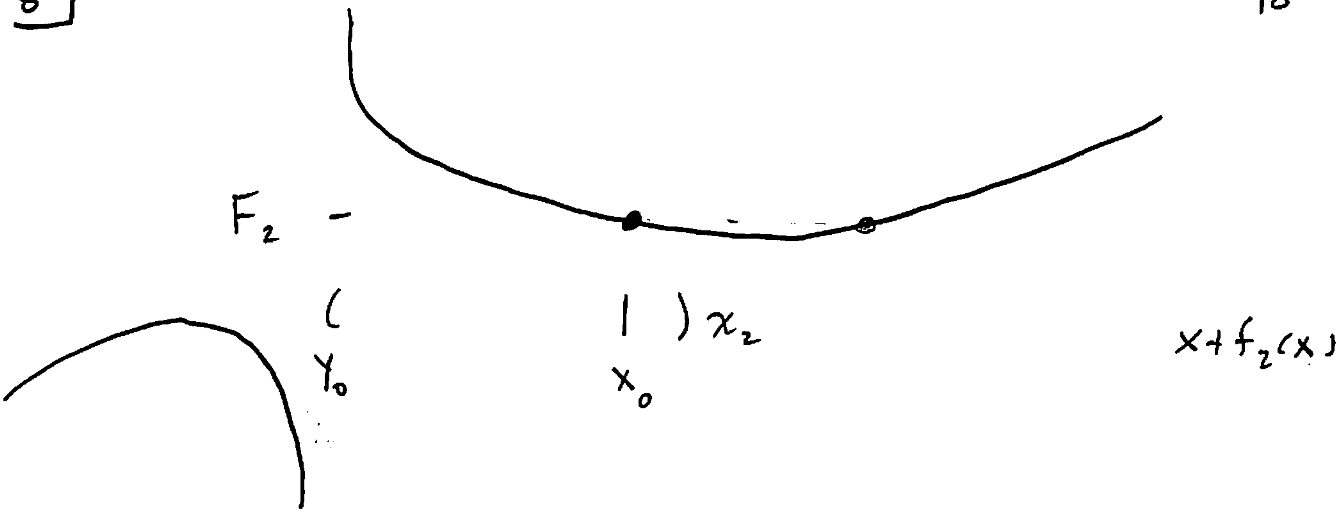
thus g_2 cont on $[y_0, x_0)$

g_2 strictly inc. on any interval where cont. (as LFT w/ + deriv)

as $g_2(y_0) = E_2 - h_2(y_0) = h_1(y_0) - h_1(y_0) = 0$, $g_2 > 0$ on (y_0, x_0)

thus $f_2 > 0$ on (y_0, x_0)

thus $\mathcal{X}_2 \supseteq (y_0, x_0]$.



Remains to enforce $|f_2'(x_0)| < 1$

When Λ_3 defined,

" $f_3 = \dfrac{c_3}{g_3}$ "

$$f_3' = -g_3' \frac{c_3}{g_3^2}$$

$$|f_3'(x_0)| = \frac{g_3'(x_0) c_3}{g_3(x_0)^2} < 1 \qquad \text{if} \qquad c_3 < \frac{g_3(x_0)^2}{g_3'(x_0)}$$

If $g_3 = g_2$, $|g_3(x_0)|$, $|g_3'(x_0)| > 0$ (as $x_0 \in \mathcal{X}_2$)

9] Let

$$c_3 = \frac{1}{2} \cdot \frac{g_2(x_0)^2}{g_2'(x_0)}$$

$$F_3 = x_0 + \frac{c_3}{g_2(x_0)} = x_0 + \frac{1}{2}\frac{g_2(x_0)}{g_2'(x_0)}$$

verify $\gamma^*_{x_0}$.

done.

open set ~~set~~ [illegible] solns

& on-to-1

"thin set"

Today: 3-to-1 & at time, n-to-1

To start, WTS $X :=$  is 3-to-1

May appear ad hoc, but somewhat natural given 'plexer'

- will discuss briefly in n-to-1 case
- introduced in an Ilya paper
  - many good ideas
- dev. in Zhang/Fawe

As in 2-to-1 analysis, assume $\gamma_1 \sim \Lambda_1$ given

To guarantee a 3-to-1 r.m., will change Λ_1 incrementing subscript & checking is r.m. each time.

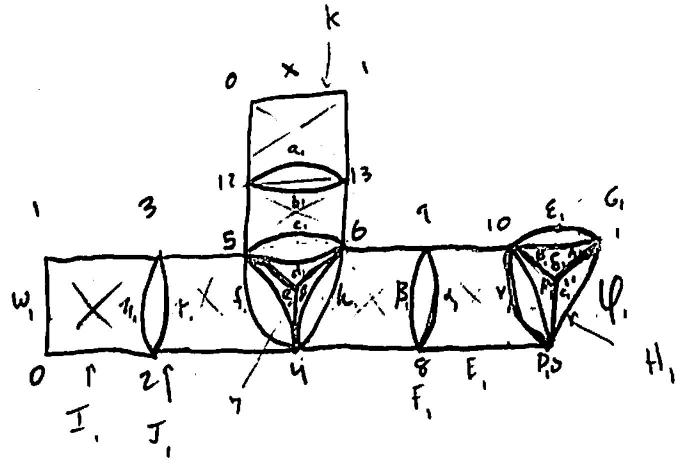
If replace each 4-star in X w/ K_4, obtain X^*: Note K_4s from 3-plexer & inversion

Our version of star-K applies as in the 2-to-1 case,

[illegible] us # of γ on X w/ $\Lambda_{(X,\gamma)} = \Lambda_1$

$=$ # γ^* on X^* w/ q.r. $\wedge$ on each K_4 & $\Lambda_1 = K_{(X^*,\gamma^*)}$.

As in 2-to-1 case, such γ^* determined by value on one edge



x
$\uparrow$
Y

$A_1 : (\Lambda_1)_{1,11}$

$B_1 := (\Lambda_1)_{10,11}$

$C_1 := (\Lambda_1)_{0,11}$

$D_1 := (\Lambda_1)_{0,10}$

$E_1 : (\Lambda_1)_{0,8} (\Lambda_1)_{9,10}$

$F_1 : (\Lambda_1)_{8,9}$

$G_1 : (\Lambda_1)_{1,10}$

$H_1 : (\Lambda_1)_{0,1}$

$I_1 : (\Lambda_1)_{0,2} (\Lambda_1)_{1,3}$

$J_1 \cdot (\Lambda_1)_{2,3}$

$$a_1(x) : \frac{(\Lambda_1)_{0,12} (\Lambda_1)_{1,13}}{x}$$

$$b_1(x) : (\Lambda_1)_{12,13} - a_1(x)$$

$$c_1(x) \cdot \frac{(\Lambda_1)_{5,12} (\Lambda_1)_{6,13}}{b_1(x)}$$

$$d_1(x) : (\Lambda_1)_{5,6} - c_1(x)$$

$$e_1(x) \cdot (\Lambda_1)_{6,7} : d_1(x) \cdot (\Lambda_1)_{4,7}$$

$$\Rightarrow e_1(x) : \frac{(\Lambda_1)_{4,7}\, d_1(x)}{(\Lambda_1)_{6,7}}$$

LHS: as before

RHS: as in 3-plexer

exp.: $\delta_1(x) : \frac{A_1}{C_1} \mu_1(x)$

$\varphi_1(x) : \frac{A_1}{B_1} \mu_1(x)$

If γ^* a cond on X^* sat. q.r. w/ $K_{(X^*,\gamma^*)} = \Lambda_1$ & value $x>0$ on edge K, then all fcns$_1(x) > 0$ & $x + \varphi_1(x) + w_1(x) = H_1$.

Conv., if $x \in \mathbb{R}^+$, all fcns$_1(x) > 0$, & $x + \varphi_1(x) + w_1(x) = H_1$, then cond. γ^*_x on X^* taking fcn$_1(x)$ values sat. q.r. & has $K_{(X^*,\gamma^*_x)} = \Lambda_1$.

Thus, # γ^* on X^* w/ q.r. & $K_{(X^*,\gamma^*)} = \Lambda_1$

$= \#\ x \in \mathbb{R}^+$ w/ all fcns$_1(x) > 0$ & $H_1 = x + \varphi_1(x) + w_1(x)$.

~~Let $X_1 = \{x$ all fcns$_1(x) > 0\}$.~~

As before, fcns are LFTs:

- ~~x is LFT~~
- ~~each~~ ~~det from~~ prev.
- ~~two~~ three 'seqs' $\{f^i\}, \{g^i\}, \{h^i\}$
- x begins each; is lft.
- $f^i: \begin{cases} c^{i-1} - f^{i-1} \\ c^{i-1} f^{i-1} \\ c^{i-1} \\ f^{i-1} \end{cases}$
- easy to show if f^{i-1} an LFT, then f^i an LFT. $\square$

<u>Get signs of fcns</u>

Hence signs of derivs fixed; can read off from X^* (b).

~~As in 2-to-1 case, $\mathcal{X}$ is a cxd open set.~~

Let $\mathcal{X}_1^* = \{x : \text{all fcns}_1(x) > 0\}$.

As in 2-to-1 case, $\mathcal{X}$ a cxd open set

pf. Hence seqs $\{f^i\}_1^n$, $\{g^j\}_1^m$, $\{h^i\}_1^e$, will show $\{x : f^j(x) > 0 \ \forall_j\}$ is cxd, open. then, so is $\{x : g^j(x) > 0 \ \forall_j\}$, & hence so is their int ($= \{x : \text{all fcns}_1(x) > 0\}$).

As before, $f^i = \begin{cases} c f^{i-1} \\ c - f^{i-1} \\ \frac{c}{f^{i-1}} \end{cases}$ some $c > 0$ (dep. on j)

$f' = g' = x$

By defn, $x \in \mathcal{X}_1$ iff
$$f^1(x) > 0$$
$$f^2(x) > 0$$
$$\vdots$$
$$f^{n-1}(x) > 0$$
$$f^n(x) > 0$$

Idea ~~"basis"~~ assuming $f^1(x) > 0, \ldots, f^{n-1}(x) > 0$, reduce $f^n(x) > 0$ to condition on $f^{n-1}(x)$. (strict ineq.)

"induction step": have reduced conditions on $f^{n-l+1}, f^{n-l+2}, \ldots, f^n$ to conditions on f^{n-l}, ~~reduce~~ conditions on f^{n-l} to conditions on f^{n-l-1} in presence of assumptions $f^1(x) > 0, \ldots, f^{n-2}(x) > 0$

pf: ~~$x \in \mathcal{X}_1$ iff~~ f^n iff iff iff iff

Claim: $\forall n$, (finitely many) strict ineqs on $f^1, \ldots, f^n$ including $f^j(x) > 0 \; \forall j$

are equiv to strict (finitely many) ineqs on f^1 alone.

pf ind: $n=1$: triv.

if holds for $n-1$, holds for n:

$\boxed{f^n(x) > a}$ (do for $>a$ alone)

$\cdot\ f^{n-1}(x) > 0$

$\cdot\ C$: {conditions on f^k, $k \le n-1$}

iff $\begin{cases} c f^{n-1}(x) > a \\ c - f^{n-1}(x) > a \\ c / f^{n-1}(x) > a \end{cases}$

$-\ f^{n-1}(x) > 0$

C

iff $-\ f^{n-1}(x) > a/c$ $\}$

$-\ f^{n-1}(x) < c - a$

$-\ c > a f^{n-1}(x)$ iff $\begin{cases} a = 0 & c > 0 \\ a > 0 & f^{n-1}(x) < c/a \\ a < 0 & f^{n-1}(x) > c/a \end{cases}$

$-\ f^{n-1}(x) > 0$

$-\ C$

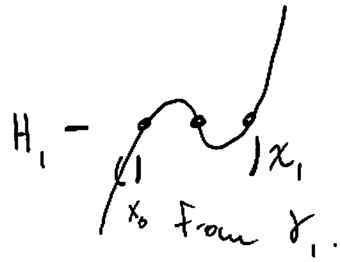
done $\square$

5

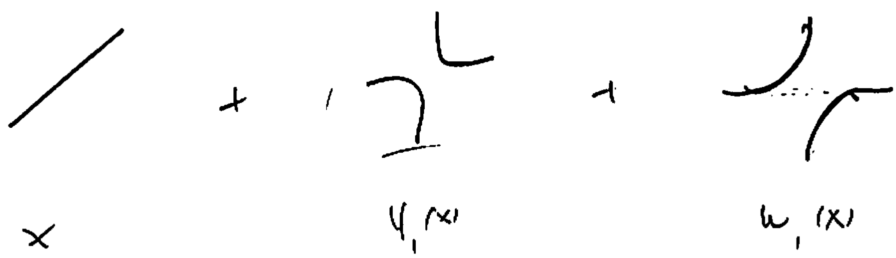
so, want exactly 3 solns to $x+\varphi_1(x)+w_1(x) = H_1$ in some int x_1.

Can check this is impossible unless φ_1 & w_1 sing.
And that if φ_1, w_1 sing, then @ most 3 solns (cubic = 0).

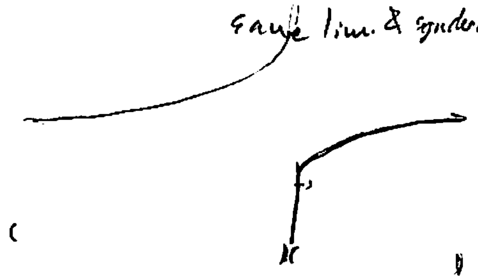
One poss is



How can we achieve this w/.



(Note: LFT ing')
if lin., ✓
if —, noted
same lim. & standardized



let $\sigma(x) := x + \varphi_1(x)$

props:

① $|\varphi_1'(x_0)|<1$

② sing to right of x_0

③ $x_1 \ni [x_0, x_1]$.

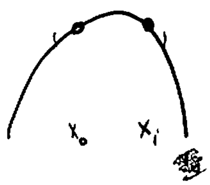
$\to \sigma_1'(x_0), \sigma_1'(x_1) \neq 0$

Note: ~~$(x+\varphi_1(x))$~~ ~~$\neq 0$~~ @ x_1, use $\varphi_1' \sim (x+1)^2$

imp to note: $\varphi_1(x)$ ev neg as $x \to$ ~~[illegible]~~, so x_1 its sing.

can't extend to ~~[illegible]~~; makes arg a bit more comp. its sing (cf. 2-to-1 case & below)

Assume we have ①-③; what could w_1 look like to give desired behavior?



~~$|\varphi_1'(x_0)| \ldots$~~

$\sigma_1'(x_0) \neq 0 \neq \sigma_1'(x_1)$

so local cont. $\Rightarrow$ $\exists$ nbhds N_0 of x_0 & N_1 of x_1

s.t. ~~$\sigma' \ldots$~~ nonzero on N_0, N_1

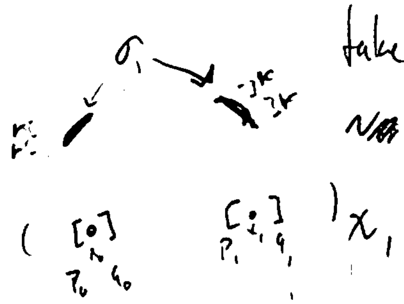
i.e. const sign.

$|\sigma_1(x_j) - \sigma_1(p_j)|, |\sigma_1(x_j) - \sigma_1(q_j)| \geq r > 0$.

&

take N_0, N_1 ~~[illegible]~~ WLOG

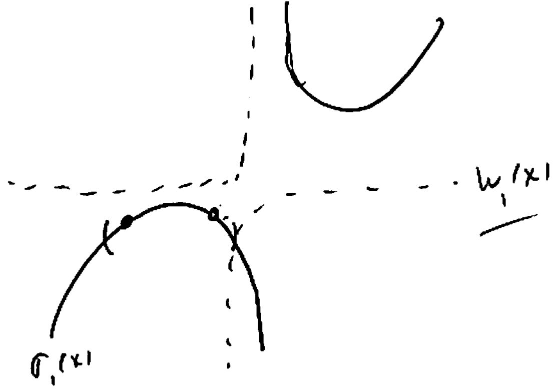
$N_0 = [p_0, q_0]$, $N_1 = [p_1, q_1]$



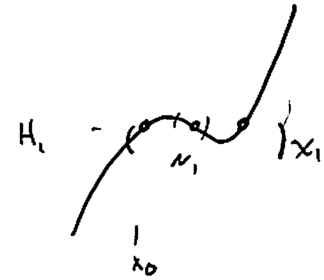
$p_0 < q_0 < p_1 < q_1$

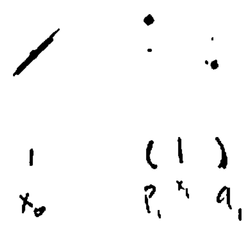
Now, suppose w_1 ④ sing @ REPBX$_1$, & $|w_1'(q_1)| < \frac{K}{q_1 - p_0}$ ⑤

s.t. $w_1' < \frac{K}{q_1 - p_0}$ on $[p_0, q_1]$ as $\sim (cx+d)^{-2}$

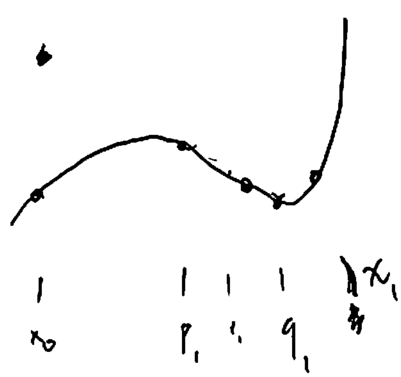


$\sim$





- One soln is x_0



- $\sigma_1(p_1) > \sigma_1(x_0)$
  $w_1(p_1) > w_1(x_0)$

$\sigma_1 + w_1 \big|_{p_1} > \sigma_1 + w_1 \big|_{x_0}$

- $\sigma_1(q_1) = \sigma_1(x_0) - r$
- $w_1(q_1) - w_1(x_0) = \int_{x_0}^{q_1} w_1'(x)\,dx$

$= \int_{x_0}^{q_1} \frac{r}{q_1 - p_0}\,dx$

$= r \cdot \frac{q_1 - x_0}{q_1 - p_0}$

$< r \cdot 1 = r$

$w_1(q_1) < r + w_1(x_0)$

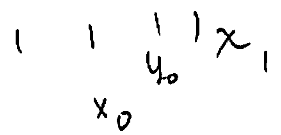
- $\sigma_1(q_1) + w_1(q_1) < r + w_1(x_0) + \sigma_1(x_0) - r$
  $= w_1(x_0) + \sigma_1(x_0)$

(end)

review ~~recent things~~;
esp. det. of how
φ_1, ω_1 should work.

$\boxed{8}$

First, ① - ③.



Pick any y_0 ~~x_0~~ btw. x_0 & REPOX$_1$.

$$\varphi_2(x) = \frac{A_2}{B_2}\mu_2(x) = \frac{A_2}{B_2}\left(D_2 - \frac{E_2}{F_2 - \beta_2(x)}\right)$$

WTMake $\boxed{F_2 = \beta_2'(y_0) \atop = \beta_2(y_0)}$ ~~$\Leftarrow \beta(y_0)$~~

issue: could inc. $v_2(x_0)$ & make other fns neg @ x_0.

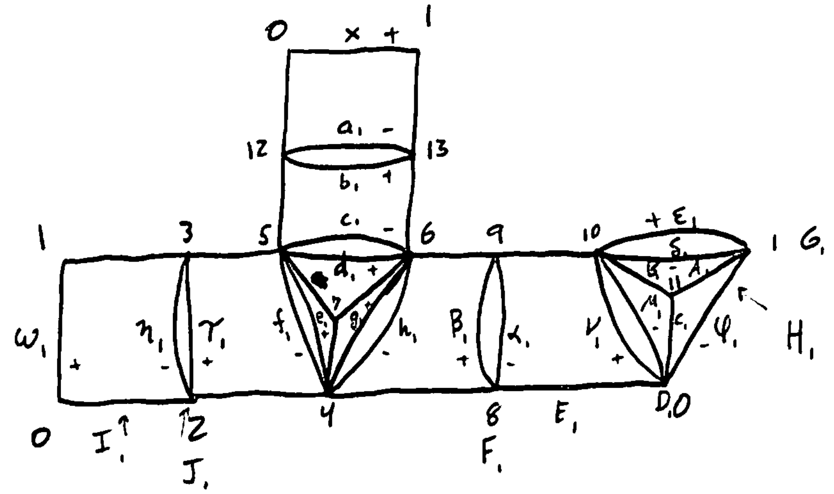
~~sol'n: $D_2 = D_1 + \frac{E_1}{\beta_1(x_0) - \beta_1(y_0)} = \frac{E_1}{F_1 - \beta_1(y_0)}$.~~

many sol'ns, e.g. make $v_1(x_0) = v_2(x_0)$:

$$v_1(x_0) = \frac{E_1}{F_1 - \beta_1(x_0)} \overset{\text{WTHave}}{=} \frac{E_2}{\beta_1(y_0) - \beta_1(x_0)} = v_2(x_0)$$

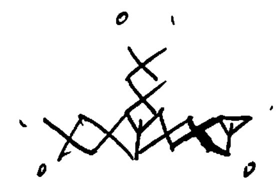
$$\Rightarrow \boxed{E_2 = \frac{\beta_1(y_0) - \beta_1(x_0)}{F_1 - \beta_1(x_0)} \cdot E_1.}$$

$\boxed{0}$



$= X^*$

$X :$



$A_1 : (\Lambda_1)_{1,11}$

$B_1 : (\Lambda_1)_{10,11}$

$C_1 : (\Lambda_1)_{0,11}$

$D_1 : (\Lambda_1)_{0,10}$

$E_1 : (\Lambda_1)_{0,8} (\Lambda_1)_{9,10}$

$F_1 : (\Lambda_1)_{8,9}$

$G_1 : (\Lambda_1)_{1,10}$

$H_1 : (\Lambda_1)_{0,1}$

$I_1 : (\Lambda_1)_{0,2} (\Lambda_1)_{1,3}$

$J_1 : (\Lambda_1)_{2,3}$

Talking a/b 3-to-1 graphs

I.p., WTS X is 3-to-1

Recall X^*; will study to show X is 3-to-1

As before, let γ_1 on X be given, $\Lambda_1 := \Lambda_{(X,\gamma_1)}$.

Our version of the star-K transformation implies

$\#\gamma$ on X w/ $\Lambda_{(X,\gamma)} = \Lambda_1$

$= \#\gamma^*$ on X^* w/ $K_{(X^*,\gamma^*)} = \Lambda_1$ which sat. q.r.

Recall such a γ^* is fully determined by value on one edge: if value is x then values on remaining parallel edges determined (in terms of x) by q.r. & condition $K_{(X^*,\gamma^*)} = \Lambda_1$.

For such an x, all fcns$_1(x) > 0$ & $x + \varphi_1(x) + w_1(x) = H_1$.

If, conv., $x \in \mathbb{R}^+$, all fcns$_1(x) > 0$, & $x + \varphi_1(x) + w_1(x) = H_1$, ~~then x is the value some such γ^* assumes on this edge. We call this cond. γ^*_x~~ and we let γ^*_x be the fcn whose value on this edge is x, whose values on each remaining parallel edge is the value of the corresponding fcn evaluated @ x, & whose value on 'single edges' is the appropriate entry in Λ_1, then γ^*_x is a cond. on X^* w/ q.r. & $K_{(X^*,\gamma^*_x)} = \Lambda_1$.

Thus, $\#\gamma^*$ on X^* w/ q.r. & $K_{(X^*,\gamma^*)} = \Lambda_1$

$= \# x \in \mathbb{R}^+$ w/ all fcns$_1(x) > 0$ & $x + \varphi_1(x) + w_1(x) = H_1$

Taking $\mathcal{X}_1 = \{x : \text{all fcns}_1(x) > 0\}$, we have

$\#\ x \in \mathbb{R}^+$ w/ all fcns$_1(x) > 0$ & $x + \varphi_1(x) + w_1(x) = H_1$
$= \#\ x \in \mathcal{X}_1$ w/ $x + \varphi_1(x) + w_1(x) = H_1$

For convenience, define $\sigma_1(x) := x + \varphi_1(x)$,
$S_1(x) := x + \varphi_1(x) + w_1(x)$.

Note: If x_0 is the value of x determined by γ_1 & the star-K xformation, then $x_0 \in \mathcal{X}_1$ & $S_1(x_0) = H_1$

Last time, we showed:

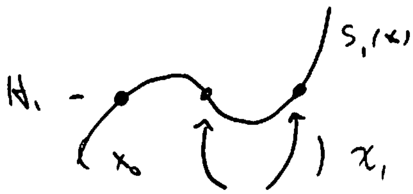
- fcns are LFTs
- signs of derivs can be read off from X^* (i.e. don't depend on resp. matrix)
- $\mathcal{X}_1$ cxd, open
- @ most 3 solns to $\cancel{x + \varphi_1(x) + w_1(x)}$ $S_1(x) = H_1$ if φ_1, w_1 sing. (also noted φ_1, w_1 sing. is necc. to have ~~3 soln~~ exactly 3 solns).

Noted one possible "form" of S_1 to give 3 solns is

3)

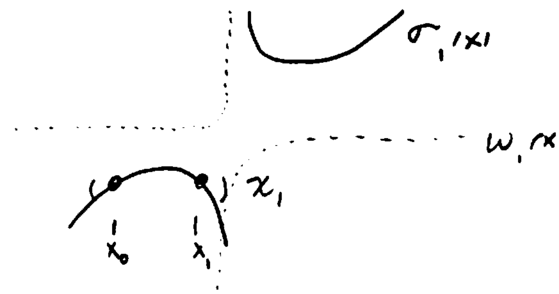
Idea: use sing. of φ_1 to make $\sigma_1(x)\ (= x + \varphi_1(x))$ "bend down":

Then use sing. of w_1 @ right endpoint of $\mathcal{X}_1$ to make $s_1(x)$ "bend up", while "preserving" the downward bend introduced by φ_1:



two addtl. solns.

=



For the picture of $\sigma_1(x)$ to be accurate, we need:

① φ_1 sing. to the right of x_0

② $|\varphi_1'(x_0)| < 1$

③ $\mathcal{X}_1 \supseteq [x_0, x_1]$

Note: $\mathcal{X}_1$ can't extend to sing of φ_1, as φ_1 is eventually negative!

Note ② => $\sigma_1'(x_0) > 0$, & from this, it is easy to show that $\sigma_1'(x_1) < 0$.

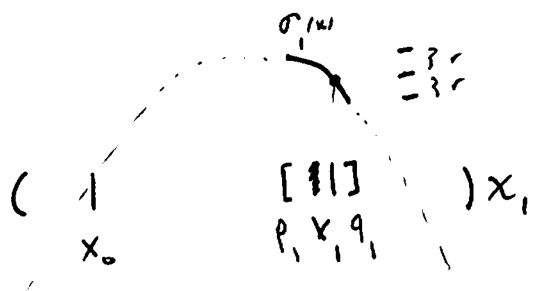
Thus, as σ_1' is cont @ x_1, $\exists\, r>0$ & a nbhd $N_1 = [p_1, q_1]$ of x_1 s.t.

- $N_1 \subseteq \mathcal{X}_1$
- $\sigma_1'(x) < 0$ on N_1
- $\sigma_1(p_1) - \sigma_1(x_1) = \sigma_1(x_1) - \sigma_1(q_1) = r$

(some redundancy)

Note $\sigma_1'(x) < 0 \Rightarrow x_0 \notin N_1$

Sketch:



Now, for picture of $S_1(x)$ to be accurate, it will suffice to have

(4) w_1 sing @ right endpoint of $\mathcal{X}_1$

(5) ~~$w_1(q_1) < w_1(x_0) + r$~~. $w_1'(q_1) < \dfrac{r}{q_1 - x_0}$.

(new 5!)

To see that ⑤ $\Rightarrow$ w_1 is 'flat enough',

note that ④ $\Rightarrow$ $w_1'(x) \le w_1'(q_1)$ for $x \in [x_0, q_1]$, as mag of deriv. inc. as get closer to sing. Thus, as w_1 is int of deriv. on $[x_0, q_1]$, we have

$$w_1(q_1) - w_1(x_0) = \int_{x_0}^{q_1} w_1'(x)\,dx \le \int_{x_0}^{q_1} w_1'(q_1)\,dx$$

$$= w_1'(q_1) \cdot (q_1 - x_0)$$

$$< \frac{r}{q_1 - x_0} \cdot q_1 - x_0$$

$$= r.$$

Thus, $w_1(q_1) < w_1(x_0) + r$.

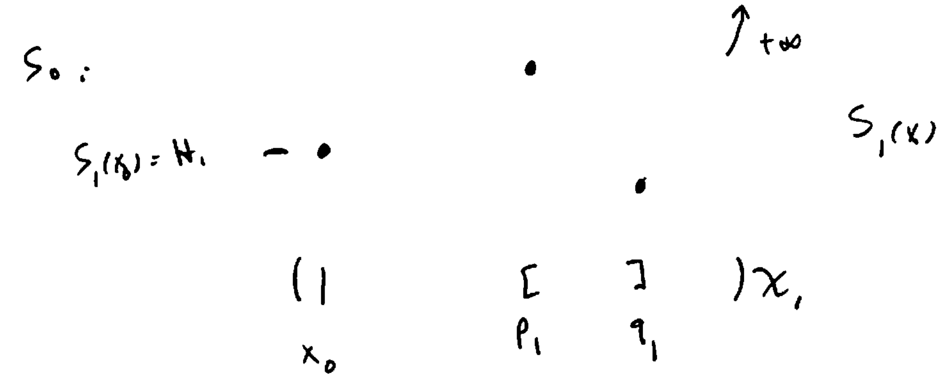
As w_1 is inc. on $[x_0, q_1]$, we have

$$\begin{aligned} S_1(p_1) &= \sigma_1(p_1) + w_1(p_1) \\ &> \sigma_1(x_0) + r + w_1(x_0) \\ &= S_1(x_0) + r \\ &> S_1(x_0) \end{aligned}$$

& by the above est.,

$$\begin{aligned} S_1(q_1) &= \sigma_1(q_1) + w_1(q_1) \\ &< \sigma_1(x_0) - r + w_1(x_0) + r \\ &= S_1(x_0). \end{aligned}$$

6] S_0:



Two apps of Intermediate Value Thm $\Rightarrow$ $\exists$ a soln in N_1 & a soln btwn q_1 & rightendpoint of x_1.

So, we will do (1)-(5).

Note theme of placing eqn. & deriv @ pt. (w/ x being "large enough"); suffices for n-to-1 arg.

Note: (1) & (5) basically as in 2-to-1 case; same "end. struct." & related conditions (rightendpoint instead of left endpoint as inc fcn instead of dec. fcn.)

(1)-(2) (& imps for (3)) rel. "new" and difficult; we do them first.

① - ③ "rel. hard"; ④ - ⑤ sim to 2-6-1 case.

First, ① - ③. Once we define Λ_2, we will have

$$\varphi_2 = \frac{A_2}{B_2}\mu_2 = \frac{A_2}{B_2}(D_2 - \nu_2) = \frac{A_2}{B_2}\left(D_2 - \frac{E_2}{\alpha_2}\right)$$

$$= \frac{A_2}{B_2}\left(D_2 - \frac{E_2}{F_2 - \beta_2}\right)$$

Thus $\varphi_2(x)$ sing. when $F_2 = \beta_2(x)$.

(finish 3-to-1; write n-to-1
- local behavior
- plexers
  - sum of n fns
  - each adds 'wiggle'
- n-plexer vs. n-2 3-plexers)

Choose any y_0 btw x_0 & REPOx_1.

If we do not change β (i.e., if $\beta_1 = \beta_2$)
then φ_2 will be sing @ y_0 if
we set $\underline{\underline{F_2 = \beta_1(y_0)}}$. ~~For now, leave others alone.~~

Note β_1 inc. on $\mathcal{X}_1$, so $\beta_1(y_0) > \beta_1(x_0)$.

Thus, w/ F_2 as defined, $\alpha_2(x_0) = \beta_1(y_0) - \beta_1(x_0) > 0$,
and hence $\nu_2(x_0) = \dfrac{E_2}{\alpha_2(x_0)} > 0$ (taking $E_1 = E_2$).

$\boxed{7}$

There is, however, a potential issue w/ $\mu_2(x_0) = D_2 - V_2(x_0)$ if we do not change D_2.

To see what might go wrong, first note that
$F_1 = \beta_1(x_0) + \alpha_1(x_0) > \beta_1(x_0)$.

Moreover, $\varphi_1(x)$ is sing when $F_1 = \beta_1(x)$. As $[x_0, y_0] \subseteq \mathcal{X}_1$, we have $F_1 \neq \beta_1(x)$ $\forall x \in [x_0, y_0]$, & hence $F_1 > \beta_1(y_0) = F_2$.

Thus $\alpha_2(x_0) = F_2 - \beta_1(x_0) < F_1 - \beta_1(x_0) = \alpha_1(x_0)$,

so $\dfrac{1}{\alpha_2(x)} > \dfrac{1}{\alpha_1(x_0)}$

so $V_2(x_0) = \dfrac{E_2}{\alpha_2(x_0)} = \dfrac{E_1}{\alpha_2(x_0)} > \dfrac{E_1}{\alpha_1(x_0)} = V_1(x_0)$

Thus $D_1 - V_2(x_0) > 0$ does not immediately follow from $D_1 - V_1(x_0) > 0$, so we may wish to change D_2, e.g. $D_2 = D_1 + V_2(x_0) - V_1(x_0)$.

Then $\mu_2(x_0) = D_2 - V_2(x_0) = D_1 - V_1(x_0) = \mu_1(x_0) > 0$.

Thus, if we fix remaining constants, then remaining fns are pos. @ x_0 (they assume same values for Λ_1 as for Λ_2, as their values @ x_0 depend only on consts which we haven't changed & $\underset{=\mu_1(x_0)}{\mu_2(x_0)}$).

$\boxed{8}$

Thus, $x_0 \in \mathcal{X}_2$, & $\gamma^*_{x_0} \sim \Lambda_2$, as $x_0 + \varphi_2(x_0) + \omega_2(x_0) = x_0 + \varphi_1(x_0) + \omega_1(x_0) = H_1 = H_2$.

Note: $\alpha_2(x), V_2(x) > 0$ on $[x_0, y_0)$.

We will delay further investigation of $\mathcal{X}_2$ until later.

So, we have ①.

Next, ②: once Λ_3 det.,

$$\varphi_3(x) = \frac{A_3}{B_3}\mu_3(x),$$

$$\text{so } \varphi_3' = \frac{A_3}{B_3}\mu_3'$$

$$\text{Thus } |\varphi_3'(x_0)| = \left|\frac{A_3}{B_3}\mu_3'(x_0)\right| < 1$$

$$\text{if } B_3 > A_3|\mu_3'(x_0)|.$$

If fix all but H_3 & B_3, then can take, e.g.,

$$B_2 = 2 A_2 |\mu_2'(x_0)| \quad (= 2 A_3 \cdot |\mu_3'(x_0)|).$$

Note, still have ①.

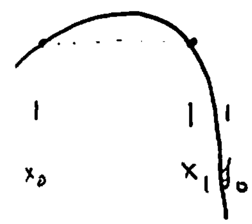
Note $\mathcal{X}_3 = \mathcal{X}_2$; only fcn changed is φ_3, which is a pos. mult $\left(\frac{B_2}{B_3}\right)$ of φ_2. I.p., $x_0 \in \mathcal{X}_3$.

If take $H_3 = x_0 + \omega_2(x_0) + \frac{B_2}{B_3}\varphi_2(x_0)$, then $\gamma^*_{x_0} \sim \Lambda_3$.
($\mathcal{X}_3$ doesn't depend H_3, so can change H_3 w/o altering $\mathcal{X}_3$).

Now $\alpha_3 = \alpha_2$, $\nu_3 = \nu_2$, so terms 'left' of μ_3 are pos on $[x_0, y_0)$ (as $\in \mathcal{X}_1$, & α_3, ν_3 verified sep.)
left stuff unmodified,

Sketch of $x + \varphi_3^{(x)}$:

$\sigma_3(x)$



$\mathcal{X}_3 = ?$

WTMake changes s.t. ~~$[x_0, x_1]$~~ $[x_0, x_1] \subseteq \mathcal{X}_4$

Primary concern will be $\mu_4, \delta_4, \varphi_4, \& \ \varepsilon_4$

Note $\mu_4, \delta_4, \varphi_4$ have same sign, so will start by making μ_4 pos. on $[x_0, x_1]$.

As $[x_0, x_1] \subset\subset [x_0, y_0)$ & ν_3 ~~cont, pos~~ pos. (hence cont.) on $[x_0, y_0)$, $M = \sup_{x \in [x_0, x_1]} \nu_3(x) < \infty$.

Let $D_4 = M + 1$.

Then $\mu_4(x) > 0$ on $[x_0, x_1]$.
Note: φ_4 still sing @ y_0 & $|\varphi_4'(x_0)| < 1$
Thus $\delta_4(x) > 0$ (hence cont.) on $[x_0, x_1]$, so $m = \sup_{x \in [x_0, x_1]} \delta_4(x) < \infty$
Let $G_4 = m + 1$, s.t. $\varepsilon_4(x) > 0$ on $[x_0, x_1]$.

$\boxed{10}$ Thus $\mathcal{X}_4 \supseteq [x_0, x_1]$.

Finally, $H_4 = x_0 + w_3(x_0) + \frac{A_3}{B_3}(M+1 - V_3(x_0))$.

$x_0 + w_4(x_0) + \varphi_4(x_0) = S_4(x_0)$.

Thus $\gamma^*_{x_0}$ a cond on X^* w/ q.r. ∋ $K(x^*, \gamma^*) = \Lambda_4$.

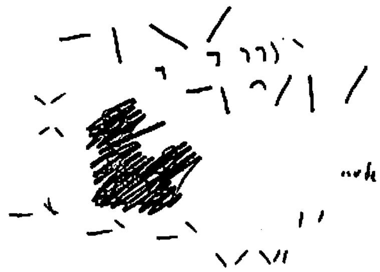
We thus have ① - ③.

(rest is easy) < very st. sing / very st. clos works

fix: only one ublu, & w/ ... pt. $[\gamma_1 : x_1 : \lambda_1]$ N_1

point out that sing, den @ pt, & χ

any base rays.



note: don't expect n-fold Λ closure; probably (n-m-fold) Λ have nonempty int. $\forall$ m ≤ n.

$\boxed{111}$

Today we're going to be talking a/b n-to-1 graphs for general positive integers n

It would be nice if we could apply the ideas developed in the 2-to-1 & 3-to-1 cases to obtain n-to-1 graphs

To begin w/, it would be nice if some of the reductions we made in the 2-to-1 & 3-to-1 cases could be applied in a more general setting.

So, suppose X is any graph & γ on Λ is given. We wish to consider what hypotheses on X are appropriate to give us similar reductions, with the goal being to find an n-to-1 graph X.

Among the first things we established was a version of the star-K xformation, which told us that, if X is the 2-to-1 or 3-to-1 graph, then $\exists$ a bij. corr.

$$\begin{array}{lcl} \tilde{\gamma} \text{ on } X \text{ w/} & & \gamma^* \text{ on } X^* \text{ sat q.r} \\ \Lambda_{(X,\tilde{\gamma})} = \Lambda & \longleftrightarrow & \text{w/ } K_{(X,\gamma)} = \Lambda \end{array}$$

Repeated apps of star-K tell us this holds whenever X has no int-int edges when we interpret 'sat q.r.' to mean sat q.r. on each complete subgraph of X^* resulting from an app. of star-K.

Thus, we might hope to have

— X has no int.-int. edges,

so that the above corr. holds (where X^* has the obvious meaning).

To actually count the number of γ^* on X^* sat q.r. w/ $K_{(x^*,\gamma^*)} = \Lambda$, we made use of the following bij:

γ^* on X^* sat q.r. w/ $K_{(x^*,\gamma^*)} = \Lambda$ $\longleftrightarrow$ $x \in \mathbb{R}^+$ s.t. some nicely related lfts$(x) > 0$ & $x + \sum_{n-1 \text{ lfts}} f_i(x) = \Lambda_{0,1}$

or, w/ $\mathcal{X} = \{x \in \mathbb{R}^+ : \text{fcns}(x) > 0\}$,

γ^* on X^* sat q.r. w/ $K_{(x^*,\gamma^*)} = \Lambda$ $\longleftrightarrow$ $x \in \mathcal{X}$ w/ $x + \sum_{n-1 \text{ lfts}} f_j(x) = \Lambda_{0,1}$.

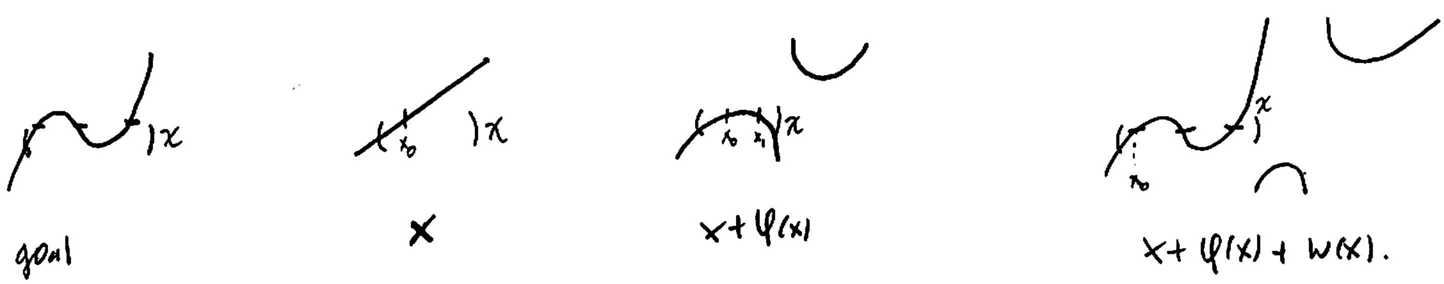
For this, we made use of the following facts:

- X^* has precisely n edges btw a pair of nodes (0 & 1 WLOG)
- if we know the value x of such a γ^* on (a fixed) one of these n edges, we can determine the value of that γ^* in ~~[illegible]~~ each edge in X^* in terms of x & Λ.
- the (nonconstant) values are LFTs of x.
  - For $\mathcal{X}$ to be open & cxd, we want each lft other than x to be obtained from a "previous" one f as cf, c/f, or $c-f$, some $c > 0$
- $x \in \mathcal{X} \Rightarrow \gamma_x^*$ sat q.r. & has $K_{(x^*,\gamma_x^*)} = \Lambda$ except possibly @ entries 0,0, 0,1, 1,0, 1,1.

Again, these are props. we might like a "general" n-to-1 graph to possess.

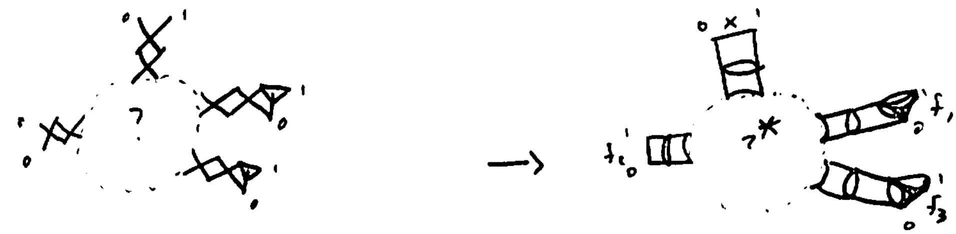
Once we had these reductions when $n = 2, 3$, we used the structure of X^* near its n-fold edge to get precisely n values of $x \in X$ w/ $x + \sum_{n-1 \text{ lfts}} f_i(x) = \Lambda_{g_1}$.

Recall in the 3-to-1 case we proceeded roughly as follows: think of "starting" w/ x & "iteratively" adding the other LFTs to introduce "bends" in the sum, taking care @ each step not to disturb "prior" bends:



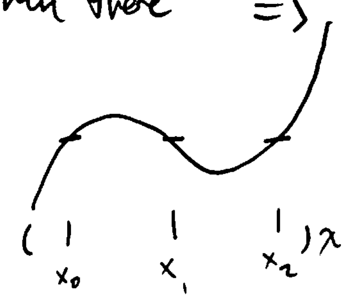
Structure of X^* used to "place" sing. & bound derivs.; only "local" structure was used.

Suppose that, when $n=4$, our candidate n-to-1 graph has the following form:

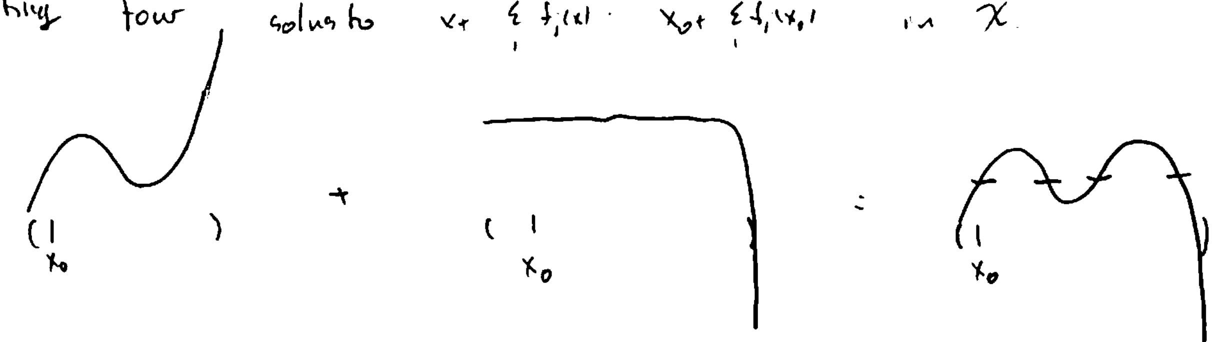


where the blob is s.t. X satisfies our "desired properties" above, & values of "nice" γ^* parametrizable in terms of x above. Assuming signs of derivs are as in the 3-to-1 case (f_1^-, f_3^-, f_2^+), the eq. given there $\Rightarrow$

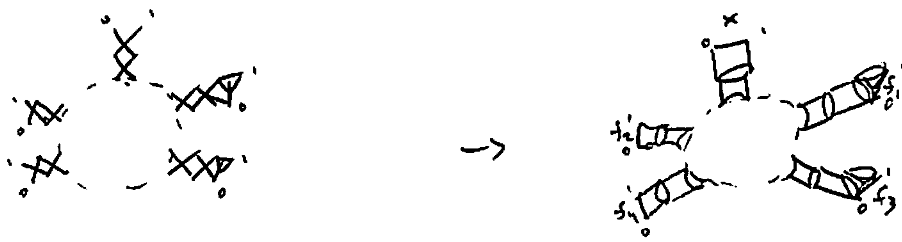
$x+f_1(x)+f_2(x) =$



For an appropriate cond. $\gamma \sim \Lambda$ on X.
Derivs nonzero (# solns, lest "local" poly have >3 roots. $\varepsilon=0, \varepsilon'=0 \Rightarrow p=0, p'=0$.
As in the 3-to-1 case, modifications to γ (which we will still call γ) can be made to guarantee that adding $f_3(x)$ to $x+f_1(x)+f_2(x)$ "bends" the graph downward to the right of x_2, creating four solns to $x+\sum_1^3 f_i(x) = x_0+\sum_1^3 f_i(x_0)$ in X.

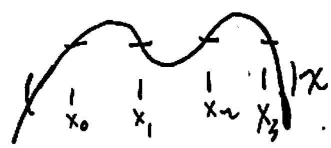


Similarly, when $n=5$, it our graphs of the Form



w/ "same derivatives", these $n=4$ case =>

$x+\sum_1^3 f_i(x)$:



is possible.

Deriva nonzero as above.

To get 5 solns after adding f_4, modify γ to get

- f_4 sing btw x_3 & repo x
- f_4 Flat on appropriate superset of $[x_0, x_3]$

then

(Do as in 3-to-1 case.)

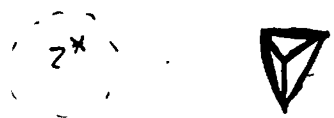
~~Note: soln & deriv 0 => poly has double root~~

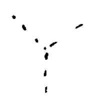
Thus, we are essentially reduced to Finding something to put in For

to give X nice props.

As above, the case $n=3$ can provide some ideas.

There, we have 



First, note that once we know  (value on) we can use the q.r. & our knowledge of  (from Λ_0) to get 

Moreover, if all we know is  then we cannot determine any of  using the q.r.

<u>Defn</u>: A p-plexer is a ~~together w/~~ complete graph K_n together w/ a partition $E = A \cup Z$ of the edge set E of K_n s.t. $|X| = p$ & if γ is a conn. on K_n sat. the q.r., then

~~the values of γ on els of A are given & the values of γ on els of Z are unknown, the values of γ on els of X cannot be determined (using the q.r.)~~

~~, the value~~

- there is no $z \in Z$ for which $\gamma(z)$ can be determined from $\gamma|_A$ (using the q.r.)

6] - if, for any $z \in Z$, $\gamma(z)$ is given, then using the values of $\gamma|_A$ & the q.r. we can determine $\gamma(e)$ $\forall e \in E$.

e.g., 3-plexer on K_4:

observed props above.

e.g., 2-plexer on K_4

~~[illegible]~~

Note: ~~sometimes say 2 p-plexer 2 when~~ if p-plexer on K_n, may say p-plexer when referring to n-star, idea being that star-K will be applied

~~Roughly First plexer cond. ensures no superfluous "extra" restrictions on~~

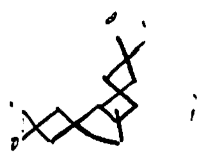
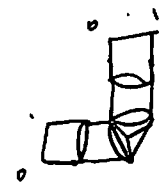
~~General idea arrange p-plexer in~~

plexers in 3-to-1 graph: idea to arrange s.t. "unknown edges" (els of Z) are in parallel w/ other edges in X^*, while "known edges" (els of A) are not, so their values can be read off from A

7

then, when we begin parametrizing values of γ_x^* in terms of x, we "reach" an el. of Z, from which we can (i.e., determine as an LFT of x) determine value of γ_x^* on other els of Z using q.r. & values on A.

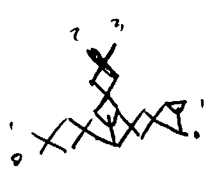
the condition that els of Z not be determined value on by values on els of A "ensures" that we do not introduce add'l conditions like lft(x) = const corr. to some edge of Z

non e.g. :  ; ~ 

"misused plexer" ; can read el of Z off from Λ_0

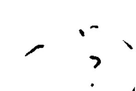
since all els of A also known from Λ_0, can determine values on entire K_4 & proceed outward

i.e., γ^* w/ q.r. & $K_{(x^*,\gamma^*)} \cdot \Lambda_0$ uniquely determined (if exists)

also :  also recoverable.

can be diff plexers on same K_4, but no "proper subplexer"

plexer
polar
flow
also between

Idea now is to place an n-plexer in the n-to-1 graph, w/ n "end stretches" For an 

8] produces edges "to "unknowns" in X^*. ex'l to then introduce param, reach one unknown, whence others, then cont to "endfcns" as before.

what to take as n-plexes?

K_n

&c.

signs good.

also repeated use of 2&3-plexes; creates sign issues