Dual Networks, Neumann-Dirichlet

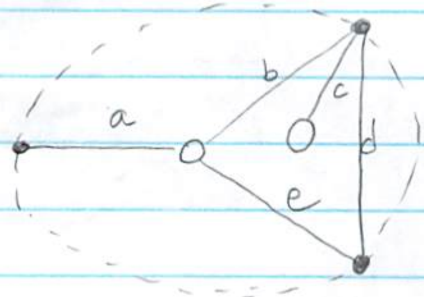
Given: connected circular planar network (with specified embedding in disk.) (V, E, ∂, γ)

Define <u>Dual network</u> $(V^\perp, E^\perp, \partial^\perp, \gamma^\perp)$ by:

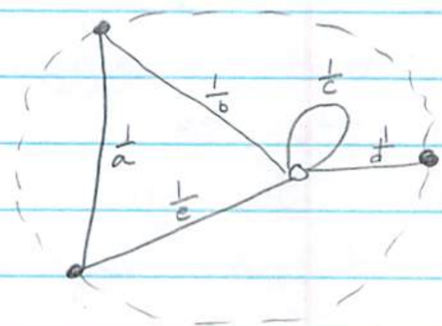
$V^\perp = \{$ connected components of $D \setminus G$, "faces" $\}$

For each edge $e \in E$, an edge $e^\perp$ in $E^\perp$ connects the two faces bordering on e. $\gamma^\perp(e^\perp) = \frac{1}{\gamma(e)}$.

$\partial^\perp \subseteq V^\perp$ is those faces bordering on boundary of disk.



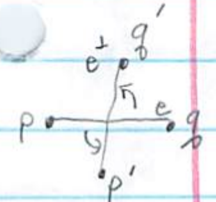
Γ



$\Gamma^\perp$

$\Gamma^\perp$ comes with a circular planar embedding; $(\Gamma^\perp)^\perp \simeq \Gamma$.

Harmonic conjugate: Given γ-harmonic u on Γ, want a $\gamma^\perp$-harmonic v on $\Gamma^\perp$ that satisfies:



$$\gamma(p \xrightarrow{e} q)\,(u(q) - u(p)) = (v(q') - v(p'))$$

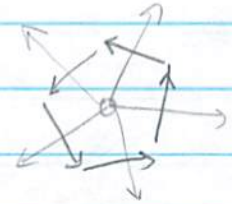
for every edge e.

Does such a v exist? If so, only unique up to overall constant.

Need to check: $\alpha(e^\perp) := \gamma(p \xrightarrow{e} q)(u(q)-u(p))$ satisfies Kirchhoff's voltage law, i.e. $\sum_{e^\perp} \alpha(e^\perp) = 0$.

Only need to check sums around (interior) faces.

But $\displaystyle\sum_{e^\perp \in \partial(F)} \alpha(e^\perp) = \sum_{q,e:\, p \xrightarrow{e} q} \gamma(e)\,(u(q)-u(p))$ where $p^\perp = F$



$= -$(current out of int node p into rest of network)

$= 0$, as desired.

So such a v exists.

Is v $\gamma^\perp$-harmonic?

$-$(Current out of node $p \in V^\perp$) is

$\displaystyle\sum_{q,e^\perp:\, p \xrightarrow{e^\perp} q} \underbrace{\gamma^\perp(e^\perp)}_{=\frac{1}{\gamma(e)}} (v(q)-v(p))$

$\displaystyle= \sum_{\substack{e \in \partial(F) \\ p' \xrightarrow{e} q'}} (u(p')-u(q'))$ where $F^\perp = p$.

$= 0$ by "Kirchhoff's voltage law" as desired.

The Neumann-to-Dirichlet Map II

Given boundary currents ψ, what is a boundary potential ϕ such that $\Lambda\phi = \psi$?

Solution does not exist unless $1^T\psi = 0$, in which case ϕ only defined up to constant.

Suppose we also constrain $1^T\phi = 0$. Then the solution is unique! We can find it:

$$\begin{bmatrix} \Lambda \\ \hdashline 1^T \end{bmatrix} \phi = \begin{bmatrix} \psi \\ \hdashline 0 \end{bmatrix} \qquad \begin{aligned} Ax &= B \\ A^TAx &= A^TB \\ x &= (A^TA)^{-1}A^TB \end{aligned}$$

$$\begin{bmatrix} \Lambda & \vdots & 1 \end{bmatrix} \begin{bmatrix} \Lambda \\ \hdashline 1^T \end{bmatrix} \phi = \begin{bmatrix} \Lambda & \vdots & 1 \end{bmatrix} \begin{bmatrix} \psi \\ 0 \end{bmatrix}$$

$$(\Lambda^2 + 11^T)\phi = \Lambda\psi$$

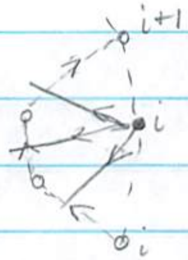
Is $\Lambda^2 + 11^T$ invertible? Yes, pos. def:

$$\begin{aligned} \phi^T(\Lambda^2 + 11^T)\phi = 0 &\Rightarrow \|\Lambda\phi\|^2 + (1^T\phi)^2 = 0 \\ &\Rightarrow \Lambda\phi = 0 \;\&\; 1^T\phi = 0 \\ &\Rightarrow \phi = 0. \text{ since } \phi \text{ must be const.} \end{aligned}$$

$$\phi = (\Lambda^2 + 11^T)^{-1}\Lambda\psi =: H\psi.$$

Relating ϕ on ∂ to $\psi^\perp$ on $\partial^\perp$

Given ϕ on the ∂, we can extend to u, harm. conj. to v, find $\psi^\perp$ (not depending on choice of v).

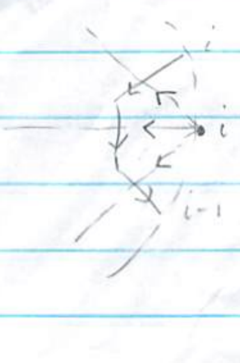


Easier done than said:

$\boxed{\text{Current out of } \partial^\perp \text{ node } i \text{ is } \phi(i+1) - \phi(i).}$

Define $D = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ -1 & & & & 1 \end{pmatrix}$ so that $\psi^\perp = D\phi$.

We also have $\psi = D^T \phi^\perp$:



$\psi(i) = \phi^\perp(i-1) - \phi(i)$

$D^T = \begin{pmatrix} 1 & & & & 1 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix}$

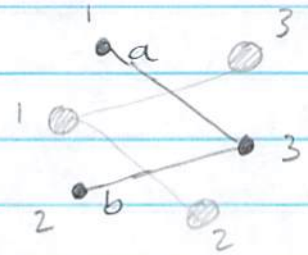
Then

$$\begin{tikzcd} \phi \arrow[r, bend left, "\Lambda"] \arrow[d, "D"'] & \psi \arrow[l, bend left, "H"] \\ \psi^\perp \arrow[r, bend left, "H^\perp"] & \phi^\perp \arrow[l, bend left, "\Lambda^\perp"] \arrow[u, "D^T"'] \end{tikzcd}$$

commutes. (if all vectors are required to Σ to 0)

So $\Lambda^{\perp} = DHD^{T} = D(\Lambda^{2} + 11^{T})^{-1}\Lambda D^{T}$

Small Example:



$$\Lambda = \begin{pmatrix} a & 0 & -a \\ 0 & b & -b \\ -a & -b & a+b \end{pmatrix}$$

$$\phi = H\psi \iff \begin{aligned} &\phi_1 + \phi_2 + \phi_3 = 0, \\ &a(\phi_1 - \phi_3) = \psi_1 \\ &b(\phi_2 - \phi_3) = \psi_2 \end{aligned}$$

$$\iff \begin{aligned} &\phi_1 + \phi_2 + \phi_3 = 0 \\ &\phi_1 = \phi_3 + \frac{\psi_1}{a} \\ &\phi_2 = \phi_3 + \frac{\psi_2}{b} \end{aligned}$$

$$\text{sum} = 3\phi_3 + \psi_1/a + \psi_2/b$$

$$\Rightarrow \phi_3 = -\frac{1}{3}(\psi_1/a + \psi_2/b)$$

Now: given $(\phi_1^{\perp}, \phi_2^{\perp}, \phi_3^{\perp})$, we have

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 1 & & -1 \\ -1 & 1 & \\ & -1 & 1 \end{pmatrix} \begin{pmatrix} \phi_1^{\perp} \\ \phi_2^{\perp} \\ \phi_3^{\perp} \end{pmatrix} = \begin{pmatrix} \phi_1^{\perp} - \phi_3^{\perp} \\ \phi_2^{\perp} - \phi_1^{\perp} \\ \phi_3^{\perp} - \phi_2^{\perp} \end{pmatrix}$$

So
$$\phi_3 = -\frac{1}{3}\left(\frac{\phi_1^\perp - \phi_3^\perp}{a}\right) + \frac{1}{3}\left(\frac{\phi_2^\perp - \phi_1^\perp}{b}\right)$$
$$\phi_2 = -\frac{1}{3}\left(\frac{\phi_1^\perp - \phi_3^\perp}{a}\right) + \frac{2}{3}\left(\frac{\phi_2^\perp - \phi_1^\perp}{b}\right)$$
$$\phi_1 = \frac{2}{3}\left(\frac{\phi_1^\perp - \phi_3^\perp}{a}\right) - \frac{1}{3}\left(\frac{\phi_2^\perp - \phi_1^\perp}{b}\right)$$

$$\begin{pmatrix} \psi_1^\perp \\ \psi_2^\perp \\ \psi_3^\perp \end{pmatrix} = \begin{pmatrix} 1 & -1 & \\ & 1 & -1 \\ -1 & & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \phi_1 - \phi_2 \\ \phi_2 - \phi_3 \\ \phi_3 - \phi_1 \end{pmatrix}$$
$$= \begin{pmatrix} \left(\frac{\phi_1^\perp - \phi_3^\perp}{a}\right) - \left(\frac{\phi_2^\perp - \phi_1^\perp}{b}\right) \\ \left(\frac{\phi_2^\perp - \phi_1^\perp}{b}\right) \\ -\left(\frac{\phi_1^\perp - \phi_3^\perp}{a}\right) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{a}+\frac{1}{b} & \frac{-1}{b} & \frac{-1}{a} \\ \frac{-1}{b} & \frac{1}{b} & 0 \\ \frac{-1}{a} & 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \phi_1^\perp \\ \phi_2^\perp \\ \phi_3^\perp \end{pmatrix}$$

Which is $\Lambda^\perp$, response for