Grupo VARIDIS
REU 2010, UW
Notes for the Lecture, June 2010.

An Introduction to Boundary Value Problems on Finite Networks

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We aim here at introducing the basic terminology and results on *self-adjoint boundary value problems on finite networks*. **Firstly we define the discrete analogue of a manifold with boundary, which includes the concept of outer normal field. Then, we prove the Green Identities in order to establish the variational formulation of boundary value problems. Moreover, we prove the discrete version of the Dirichlet Principle.**

1. Green Identities

Throughout these notes we follow the notations and definitions given in the notes *An Introduction to Discrete Vector Calculus on Finite Networks*. From now on we suppose fixed the weighted network (Γ, c, ν) and also the associated inner products on $\mathcal{C}(V)$ and $\mathcal{X}(\Gamma)$.

Given a vertex subset $F \subset V$, we denote by F^c its complement in V and by χ_F its characteristic function. Moreover, we define the sets

$$\overset{\circ}{F} = \{x \in F : \{y \sim x\} \subset F\} \qquad \text{interior of } F$$

$$\delta(F) = \{x \in F^c : \text{ exists } y \in F \text{ such that } y \sim x\} \qquad \text{(vertex) boundary of } F$$

$$\bar{F} = F \cup \delta(F) \qquad \text{closure of } F.$$

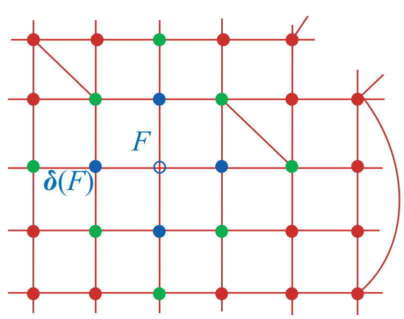

Figure 1. Blue: F, Green: $\delta(F)$, Circle: $\overset{\circ}{F}$

If $F \subset V$ is a proper subset, we say that F is *connected* if for any $x, y \in V$ there exists a path joined x and y whose vertices are all in F. It is easy to prove that $\bar{F}$ is connected when F is. In the sequel we always assume that F is a connected set. Moreover, if $F \subset V$, $\mathcal{C}(F)$ denotes the subspace of $\mathcal{C}(V)$ formed by the functions whose support is contained in F.

We are also interested in the Divergence Theorem and the Green's Identities, that play a fundamental role in the analysis of boundary value problems. These results are given on a finite vertex subset, the discrete equivalent to a compact region, so we need to define the discrete analogous of the exterior normal vector field to the set.

The *normal vector field to* F is defined as $\mathsf{n}_F = -\mathsf{d}\chi_F$. Therefore, the component function of n_F is given by $n_F(x,y) = 1$ when $y \sim x$ and $(x,y) \in \delta(F^c) \times \delta(F)$, $n_F(x,y) = -1$ when $y \sim x$ and $(x,y) \in \delta(F) \times \delta(F^c)$ and $n_F(x,y) = 0$, otherwise. In consequence, $\mathsf{n}_{F^c} = -\mathsf{n}_F$ and $\mathsf{supp}(\mathsf{n}_F) = \delta(F^c) \cup \delta(F)$.

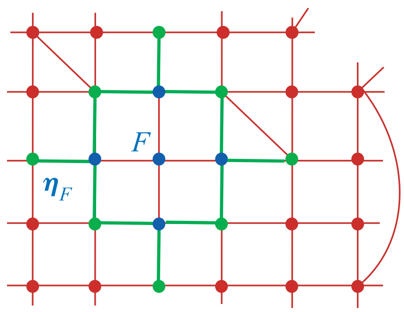

Figure 2. Normal vector field to F

Corollary 1.1. (Divergence Theorem) *For any* $\mathsf{f} \in \mathcal{X}(\Gamma)$, *it is verified that*

$$\int_F \mathsf{div}\, \mathsf{f}\, d\nu = \int_{\delta(F)} \big(\mathsf{f}^a, \mathsf{n}_F\big)\, dx,$$

where $\big(\mathsf{f}, \mathsf{g}\big)(x) = \sum_{y \in V} f(x,y) g(x,y)$, *denotes the standard inner product on* $\mathcal{T}_x(\Gamma)$.

Proof. Taking $u = \chi_F$ in the definiton of div we get

$$\int_F \mathsf{div}\,(\mathsf{f})\, d\nu = \int_V \chi_F \mathsf{div}\,(\mathsf{f})\, d\nu = -\frac{1}{2}\int_V \langle \mathsf{f}^a, \nabla \chi_F \rangle\, dx = \frac{1}{2}\int_V \big(\mathsf{f}^a, \mathsf{n}_F\big)\, dx$$
$$= \frac{1}{2}\int_{\delta(F)} \big(\mathsf{f}^a, \mathsf{n}_F\big)\, dx + \frac{1}{2}\int_{\delta(F^c)} \big(\mathsf{f}^a, \mathsf{n}_F\big)\, dx.$$

The result follows taking into account that

$$\int_{\delta(F^c)} \big(\mathsf{f}^a, \mathsf{n}_F\big)\, dy = \sum_{y \in \delta(F^c)} \sum_{x \in \delta(F)} \mathsf{f}^a(y,x) \mathsf{n}_F(y,x) = \sum_{x \in \delta(F)} \sum_{y \in \delta(F^c)} \mathsf{f}^a(x,y) \mathsf{n}_F(x,y) = \int_{\delta(F)} \big(\mathsf{f}^a, \mathsf{n}_F\big)\, dx. \qquad \square$$

Recall that the *Laplacian* of Γ is the linear operator $\mathcal{L} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \frac{1}{\nu(x)} \int_V c(x,y) \big(u(x) - u(y)\big)\, dy, \quad x \in V. \tag{1}$$

Given $q \in \mathcal{C}(V)$ the *Schrödinger operator* on Γ with *potential* q is the linear operator $\mathcal{L}_q : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$.

For each $u\in\mathcal{C}(\bar F)$ we define the *normal derivative of* u *on* F as the function in $\mathcal{C}(\delta(F))$ given by

$$\left(\frac{\partial u}{\partial \mathsf{n}_F}\right)(x)=\frac{1}{\nu(x)}\,\langle\nabla u,\mathsf{n}_F\rangle(x)=\frac{1}{\nu(x)}\int_F c(x,y)\big(u(x)-u(y)\big)\,dy,\quad\text{for any } x\in\delta(F). \tag{2}$$

The *normal derivative on* F is the operator $\dfrac{\partial}{\partial \mathsf{n}_F}:\mathcal{C}(\bar F)\longrightarrow\mathcal{C}(\delta(F))$ that to any $u\in\mathcal{C}(\bar F)$ assigns its normal derivative on F.

The relation between the values of the Schrödinger operator with potential q on F and the values of the normal derivative at $\delta(F)$ is given by the following identities.

Proposition 1.2. *Consider the function* $c_F=c\cdot\chi_{(\bar F\times\bar F)\setminus(\delta(F)\times\delta(F))}$. *Then, given* $u,v\in\mathcal{C}(\bar F)$ *the following properties hold:*

(i) First Green Identity

$$\int_F v\mathcal{L}_q(u)d\nu=\frac12\int_{\bar F\times\bar F}c_F(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)dxdy+\int_F quvd\nu-\int_{\delta(F)}v\frac{\partial u}{\partial \mathsf{n}_F}d\nu.$$

(ii) Second Green Identity

$$\int_F\big(v\mathcal{L}_q(u)-u\mathcal{L}_q(v)\big)d\nu=\int_{\delta(F)}\left(u\,\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}-v\,\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}}\right)d\nu.$$

(iii) Gauss Theorem

$$\int_F\mathcal{L}(u)d\nu=-\int_{\delta(F)}\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}}d\nu.$$

Proof. Taking into account that for any $x\in F$, $c(x,y)=0$ for each $y\notin\bar F$, we get that

$$\begin{aligned}\int_F v\mathcal{L}(u)\,d\nu&=\int_F\int_V c(x,y)v(x)\big(u(x)-u(y)\big)dydx=\int_F\int_{\bar F}c(x,y)v(x)\big(u(x)-u(y)\big)dydx\\&=\int_{\bar F}\int_{\bar F}c(x,y)v(x)\big(u(x)-u(y)\big)dydx-\int_{\delta(F)}\int_{\bar F}c(x,y)v(x)\big(u(x)-u(y)\big)dydx\\&=\int_{\bar F}\int_{\bar F}c(x,y)v(x)\big(u(x)-u(y)\big)dydx-\int_{\delta(F)}\int_{\delta(F)}c(x,y)v(x)\big(u(x)-u(y)\big)dydxx\\&-\int_{\delta(F)}v\frac{\partial u}{\partial \mathsf{n}_F}\,d\nu=\int_{(\bar F\times\bar F)\setminus(\delta(F)\times\delta(F))}c(x,y)v(x)\big(u(x)-u(y)\big)dydx-\int_{\delta(F)}v\frac{\partial u}{\partial \mathsf{n}_F}\,d\nu\\&=\int_{\bar F\times\bar F}c_F(x,y)v(x)\big(u(x)-u(y)\big)dydx-\int_{\delta(F)}v\frac{\partial u}{\partial \mathsf{n}_F}\,d\nu\\&=\frac12\int_{\bar F\times\bar F}c_F(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)dydx-\int_{\delta(F)}v\frac{\partial u}{\partial \mathsf{n}_F}\,d\nu\end{aligned}$$

and the First Green Identity follows. The proof of the Second Green Identity and the Gauss Theorem are straightforward consequence of (i). □

2. Self-adjoint boundary value problems

Given $\delta(F)=H_1\cup H_2$ a partition of $\delta(F)$ and functions $q\in\mathcal{C}(F)$, $p\in\mathcal{C}(H_1)$, $g\in\mathcal{C}(F)$, $g_1\in\mathcal{C}(H_1)$, $g_2\in\mathcal{C}(H_2)$, a *boundary value problem on* F consists in finding $u\in\mathcal{C}(\bar F)$ such that

$$\mathcal{L}_q(u)=g\quad\text{on}\quad F,\qquad\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}}+pu=g_1\quad\text{on}\quad H_1\qquad\text{and}\qquad u=g_2\quad\text{on}\quad H_2. \tag{3}$$

The associated homogeneous boundary value problem consists in finding $u \in \mathcal{C}(\bar{F})$ such that

$$\mathcal{L}_q(u) = 0 \quad \text{on} \quad F, \qquad \frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + pu = 0 \quad \text{on} \quad H_1 \qquad \text{and} \qquad u = 0 \quad \text{on} \quad H_2. \tag{4}$$

It is clear that the set of solutions of the homogeneous boundary value problem is a vector subspace of $\mathcal{C}(F \cup H_1)$ that we denote by $\mathcal{V}$. Moreover if Problem (3) has solution and u is a particular one, then $u + \mathcal{V}$ describes the set of all its solutions.

Problem (3) is generically known as a *mixed Dirichlet-Robin problem*, specially when $p \neq 0$, and $H_1, H_2 \neq \emptyset$, and summarizes the different boundary value problems that appear in the literature with the following proper names:

(i) *Dirichlet problem*: $\emptyset \neq H_2 = \delta(F)$ and hence $H_1 = \emptyset$.
(ii) *Robin problem*: $p \neq 0$, $\emptyset \neq H_1 = \delta(F)$ and hence $H_2 = \emptyset$.
(iii) *Neumann problem*: $p = 0$, $\emptyset \neq H_1 = \delta(F)$ and hence $H_2 = \emptyset$.
(iv) *Mixed Dirichlet-Neumann problem*: $p = 0$ and $H_1, H_2 \neq \emptyset$.
(v) *Poisson equation* on V: $H_1 = H_2 = \emptyset$ and hence $F = V$.

Applying the Second Green Identity, we can show that the raised boundary value problem has some sort of symmetry. In addition, we obtain the conditions that assure the existence and uniqueness of solutions of the boundary value problem (3).

PROPOSITION 2.1. *The boundary value problem* (3) *is self–adjoint, that is, for any* $u, v \in \mathcal{C}(F \cup H_1)$ *such that* $\dfrac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + pu = \dfrac{\partial v}{\partial \mathsf{n}_{\mathsf{F}}} + pv = 0$ *it is satisfied that*

$$\int_F v \mathcal{L}_q(u) d\nu = \int_F u \mathcal{L}_q(v) d\nu.$$

PROPOSITION 2.2. (Fredholm Alternative) *The boundary value problem* (3) *has solution iff*

$$\int_F gv \, d\nu + \int_{H_1} g_1 v \, d\nu = \int_{H_2} g_2 \frac{\partial v}{\partial \mathsf{n}_F} \, d\nu, \qquad \textit{for each } v \in \mathcal{V}.$$

In addition, when the above condition holds, then there exists a unique solution $u \in \mathcal{C}(\bar{F})$*, such that* $\displaystyle\int_{\bar{F}} uv \, d\nu = 0$*, for any* $v \in \mathcal{V}$.

Proof. First, observe that problem (3) is equivalent to the boundary value problem

$$\mathcal{L}_q(u) = g - \mathcal{L}(g_2) \quad \text{on} \quad F, \quad \frac{\partial u}{\partial \mathsf{n}_F} + pu = g_1 - \frac{\partial g_2}{\partial \mathsf{n}_F} \quad \text{on} \quad H_1 \quad \text{and} \quad u = 0 \quad \text{on} \quad H_2$$

in the sense that u is a solution of this problem iff $u + g_2$ is a solution of (3).

Consider now the linear operator $\mathcal{F}\colon \mathcal{C}(F \cup H_1) \longrightarrow \mathcal{C}(F \cup H_1)$ defined as

$$\mathcal{F}(u) = \begin{cases} \mathcal{L}(u) + qu, & \text{on} \quad F, \\ \dfrac{\partial u}{\partial \mathsf{n}_F} + pu, & \text{on} \quad H_1. \end{cases}$$

Then, $\ker \mathcal{F} = \mathcal{V}$ and moreover, by applying Proposition 2.1 for any $u, v \in \mathcal{C}(F \cup H_1)$ it is verified that

$$\begin{aligned} \int_{F \cup H_1} v \mathcal{F}(u) \, d\nu &= \int_F v \mathcal{L}_q(u) \, d\nu + \int_{\delta(F)} v \left(\frac{\partial u}{\partial \mathsf{n}_F} + pu \right) d\nu \\ &= \int_F u \mathcal{L}_q(v) \, d\nu + \int_{\delta(F)} u \left(\frac{\partial v}{\partial \mathsf{n}_F} + pv \right) d\nu = \int_{F \cup H_1} u \mathcal{F}(v) \, d\nu. \end{aligned}$$

Therefore the operators $\mathcal{F}$ is self-adjoint with respect to the inner product induced in $\mathcal{C}(F\cup H_1)$ by the weight ν and hence Img$\mathcal{F}=\mathcal{V}^\perp$ by applying the classical Fredholm Alternative. Consequently problem (3) has a solution iff function $\tilde{g}\in\mathcal{C}(F\cup H_1)$ given by $\tilde{g}=g-\mathcal{L}(g_2)$ on F and $\tilde{g}=g_1-\dfrac{\partial g_2}{\partial \mathsf{n}_F}$ on H_1 verifies that

$$\begin{aligned}0=\int_{F\cup H_1}\tilde{g}v\,d\nu &= \int_F gv\,d\nu+\int_{H_1} g_1 v\,d\nu-\int_F v\mathcal{L}(g_2)\,d\nu-\int_{H_1} v\,\frac{\partial g_2}{\partial \mathsf{n}_F}\,d\nu\\ &=\int_F gv\,d\nu+\int_{H_1} g_1 v\,d\nu-\int_F g_2\mathcal{L}(v)\,d\nu-\int_{\delta(F)} g_2\frac{\partial v}{\partial \mathsf{n}_F}d\nu\\ &=\int_F gv\,d\nu+\int_{H_1} g_1 v\,d\nu-\int_{H_2} g_2\frac{\partial v}{\partial \mathsf{n}_F}d\nu,\end{aligned}$$

for any $v\in\mathcal{V}$.

Finally, when the necessary and sufficient condition is attained there exists a unique $w\in\mathcal{V}^\perp$ such that $\mathcal{F}(w)=\tilde{g}$. Therefore, $u=w+g_2$ is the unique solution of Problem (3) such that for any $v\in\mathcal{V}$

$$\int_{\bar{F}} uv\,d\nu=\int_{F\cup H_1} uv\,d\nu=\int_{F\cup H_1} wv\,d\nu=0,$$

since $v=0$ on H_2 and $g_2=0$ on $F\cup H_1$. ∎

Observe that as a by-product of the above proof, we obtain that uniqueness is equivalent to existence for any data.

Next, we establish the variational formulation of the boundary value problem (3), that represents the discrete version of the weak formulation for boundary value problems. Prior to describe the claimed formulation, we give some useful definitions. *The bilinear form* associated with the boundary value problem (3) is $\mathcal{B}\colon\mathcal{C}(\bar{F})\times\mathcal{C}(\bar{F})\longrightarrow\mathbb{R}$ given by

$$\mathcal{B}(u,v)=\int_F v\mathcal{L}_q(u)\,d\nu+\int_{\delta(F)} v\frac{\partial u}{\partial \mathsf{n}_F}\,d\nu+\int_{H_1} p\,uv\,d\nu, \tag{5}$$

and hence, from the Second Green Identity, $\mathcal{B}(u,v)=\mathcal{B}(v,u)$ for any $u,v\in\mathcal{C}(\bar{F})$, that is $\mathcal{B}$ is symmetric. In addition by applying the First Green Identity, we obtain that

$$\mathcal{B}(u,v)=\frac{1}{2}\int_{\bar{F}\times\bar{F}} c_F(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,dxdy+\int_F q\,uv\,d\nu+\int_{H_1} p\,uv\,d\nu. \tag{6}$$

Associated with any pair of functions $g\in\mathcal{C}(F)$ and $g_1\in\mathcal{C}(H_1)$ we define the linear functional $\ell\colon\mathcal{C}(\bar{F})\longrightarrow\mathbb{R}$ as $\ell(v)=\displaystyle\int_F gv\,d\nu+\int_{H_1} g_1 v\,d\nu$, whereas for any function $g_2\in\mathcal{C}(H_2)$ we consider the convex set $K_{g_2}=g_2+\mathcal{C}(F\cup H_1)$.

Proposition 2.3. (Variational Formulation) *Given $g\in\mathcal{C}(F)$, $g_1\in\mathcal{C}(H_1)$ and $g_2\in\mathcal{C}(H_2)$, then $u\in K_{g_2}$ is a solution of Problem (3) iff*

$$\mathcal{B}(u,v)=\ell(v),\qquad \textit{for any}\;\; v\in\mathcal{C}(F\cup H_1)$$

and in this case, the set $u+\Big\{w\in\mathcal{C}(F\cup H_1):\mathcal{B}(w,v)=0,\ \text{for any } v\in\mathcal{C}(F\cup H_1)\Big\}$ describes all solutions of (3).

Proof. A function $u\in K_{g_2}$ satisfies that $\mathcal{B}(u,v)=\ell(v)$ for any $v\in\mathcal{C}(F\cup H_1)$ iff

$$\int_F v(\mathcal{L}_q(u)-g)\,d\nu+\int_{H_1} v\left(\frac{\partial u}{\partial \mathsf{n}_F}+pu-g_1\right)d\nu=0.$$

Then, the first result follows by taking $v=\varepsilon_x$, $x\in F\cup H_1$. Finally, $u^*\in K_{g_2}$ is another solution of (3) iff $\mathcal{B}(u^*,v)=\ell(v)$ for any $v\in\mathcal{C}(F\cup H_1)$ and hence iff $\mathcal{B}(u-u^*,v)=0$ for any $v\in\mathcal{C}(F\cup H_1)$. ∎

Observe that the equality $\mathcal{B}(u,v)=\ell(v)$ for any $v\in\mathcal{C}(F\cup H_1)$ assures that the condition of existence of solution given by the Fredholm Alternative holds, since for any $v\in\mathcal{C}(\bar F)$ it is verified that

$$\int_F gv\,d\nu+\int_{H_1} g_1 v\,d\nu=\mathcal{B}(u,v)=\mathcal{B}(v,u)=\int_F u\mathcal{L}_q(v)\,d\nu+\int_{\delta(F)} u\frac{\partial v}{\partial \mathsf{n}_F}\,d\nu+\int_{H_1} p\,uv\,d\nu.$$

In particular if $v\in\mathcal{V}$ we get that

$$\int_F gv\,d\nu+\int_{H_1} g_1 v\,d\nu=\int_{H_2} g_2\frac{\partial v}{\partial \mathsf{n}_F}\,d\nu.$$

On the other hand, we note that the vector subspace

$$\Big\{w\in\mathcal{C}(F\cup H_1):\mathcal{B}(w,v)=0,\ \text{ for any } v\in\mathcal{C}(F\cup H_1)\Big\}$$

is precisely the set of solutions of the homogeneous boundary value problem associated with (3). So, Problem (3) has solution for any data g, g_1 and g_2 iff it has a unique solution and this occurs iff $w=0$ is the unique function in $\mathcal{C}(F\cup H_1)$ such that $\mathcal{B}(w,v)=0$, for any $v\in\mathcal{C}(F\cup H_1)$. Therefore, to assure the existence (and hence the uniqueness) of solutions of Problem (3) for any data it suffices to provide conditions under which $\mathcal{B}(w,w)=0$ with $w\in\mathcal{C}(F\cup H_1)$, implies that $w=0$. In particular, this occurs when $\mathcal{B}$ is positive definite on $\mathcal{C}(F\cup H_1)$.

The quadratic form associated with the boundary value problem (3) is the function $\mathcal{Q}\colon\mathcal{C}(\bar F)\longrightarrow\mathbb{R}$ given by $\mathcal{Q}(u)=\mathcal{B}(u,u)$; that is,

$$\mathcal{Q}(u)=\frac{1}{2}\int_{\bar F\times\bar F} c_F(x,y)\big(u(x)-u(y)\big)^2dxdy+\int_F q\,u^2d\nu+\int_{H_1} p\,u^2d\nu. \tag{7}$$

Next we establishes an easy sufficient condition to assure that $\mathcal{B}$ is positive semi-definite.

Proposition 2.4. *Assume that $q\geq 0$ on F and $p\geq 0$ on H_1. Then $\mathcal{Q}(u)\geq 0$ for any $u\in\mathcal{C}(F\cup H_1)$. Moreover, $\mathcal{Q}(u)>0$ for $u\neq 0$ except when $q=0$ on F, $p=0$ on H_1 and $H_2=\emptyset$ simultaneously, in which case $\mathcal{Q}(u)=0$ iff u is constant on $\bar F$.*

Proof. From Identity (7) we get that $\mathcal{Q}(u)\geq 0$ for any $u\in\mathcal{C}(F\cup H_1)$. Moreover, $\mathcal{Q}(u)=0$ iff

$$0=\frac{1}{2}\int_{\bar F\times\bar F} c_F(x,y)\big(u(x)-u(y)\big)^2dxdy=\int_F q\,u^2d\nu=\int_{H_1} p\,u^2d\nu.$$

The first equality implies that u is constant, since $\bar F$ is connected. So $u=0$ except when $q=0$ on F, $p=0$ on H_1 and $H_2=\emptyset$, simultaneously. □

Corollary 2.5. (Dirichlet Principle) *Assume that $\mathcal{Q}$ is positive semi-definite. Let $g\in\mathcal{C}(F)$, $g_1\in\mathcal{C}(H_1)$, $g_2\in\mathcal{C}(H_2)$ and consider the quadratic functional $\mathcal{J}\colon\mathcal{C}(\bar F)\longrightarrow\mathbb{R}$ given by*

$$\mathcal{J}(u)=\mathcal{Q}(u)-2\ell(u).$$

Then $u\in K_{g_2}$ is a solution of problem (3) iff it minimizes $\mathcal{J}$ on K_{g_2}.

Proof. Firstly note that when $u\in K_{g_2}$, then $K_{g_2}=u+\mathcal{C}(F\cup H_1)$.

If u is a minimum of $\mathcal{J}$ on K_{g_2} then for any $v\in\mathcal{C}(F\cup H_1)$ the function $\varphi_v\colon\mathbb{R}\longrightarrow\mathbb{R}$ given by

$$\varphi_v(t)=\mathcal{J}(u+tv)=\mathcal{J}(u)+t^2\mathcal{Q}(v)+2t[\mathcal{B}(u,v)-\ell(v)]$$

attains a minimum value at $t=0$ and hence $0=\varphi_v'(0)=\mathcal{B}(u,v)-\ell(v)$. Therefore, from Proposition 2.3, u is a solution of Problem (3). Conversely if $u\in K_{g_2}$ is a solution of Problem (3), then $\mathcal{B}(u,v)=\ell(v)$ for any $v\in\mathcal{C}(F\cup H_1)$ and hence we get that

$$\mathcal{J}(u+v)=\mathcal{J}(u)+\mathcal{Q}(v)+\mathcal{B}(u,v)-\ell(v)=\mathcal{J}(u)+\mathcal{Q}(v)\geq\mathcal{J}(u);$$

since $\mathcal{Q}$ is positive semi-definite; that is u is a minimum of $\mathcal{J}$ on K_{g_2}. □